The p-rank of curves and covers of curves

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These notes give an overview of definitions and properties of the p-rank of a curve. We focus on the relation between the p-rank and covers of the curve. Let X be a projective smooth connected curve over an algebraically closed field of characteristic p. We would like to get some information on $\pi_1(X)$. The prime to p part of $\pi_1(X)$ is known; it is the same as the prime to p part of the fundamental group of a genus g curve over $\mathbb{C}$. What about the p-part? The first step to consider is $\mathrm{Hom}(\pi_1(X), \mathbb{Z}/p)$. The dimension over $\mathbb{F}_p$ of this space is called the p-rank, denoted by $\sigma(X)$. In fact, the p-part of $\pi_1(X)$ is a free pro-p group, hence a finite p-group P is a quotients of $\pi_1(X)$ iff it can be generated by $\sigma(X)$ elements. Now suppose that G is an extension of a prime to p group H by a p-group P. When is G a quotient of $\pi_1(X)$? Fix a surjective map $\alpha\colon \pi_1(X) \to H$, we want to lift it to a surjective map $\beta\colon \pi_1(X) \to G$. (This is called an embedding problem, see Section 2.) Let $Y \to X$ be the H-cover corresponding to α. A first condition is that P should be a quotient of $\pi_1(Y)$. In other words, P should be generated by at most $\sigma(Y)$ elements. This leads us to the question of the p-rank of a cover of X. We will study this question by considering deformation of covers of semistable curves. Therefore we start in Section 1 by defining the p-rank for semistable curves, and proving that the p-rank goes up under deformation.

1 The p-rank of a semistable curve

Let X be a semistable curve over an algebraically closed field k of characteristic $p > 0$. By F we denote the absolute Frobenius morphism. Let $J = J(X)$ be the generalized Jacobian of X.

Definition 1.1 The *p-rank* $\sigma(X)$ of X is defined as $\dim_{\mathbb{F}_p} H^1(X, \mathcal{O}_X)^F$.

Equivalently, it is equal to the dimension over k of the largest subspace of $H^1(X, \mathcal{O}_X)$ on which F is a bijection. This follows from the fact that there exist a splitting $H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X)^s \oplus H^1(X, \mathcal{O}_X)^n$, where F is a bijection on $H^1(X, \mathcal{O}_X)^s$ and nilpotent on $H^1(X, \mathcal{O}_X)^n$, [21].

Since the curve X is projective, Artin–Schreier theory implies $H^1(X, \mathbb{Z}/p) \simeq H^1(X, \mathcal{O}_X)^F$. Therefore we can also define the p-rank as $\dim_{\mathbb{F}_p} H^1(X, \mathbb{Z}/p)$. Furthermore,

$$H^1(X, \mathbb{Z}/p) \simeq \mathrm{Hom}(\pi_1(X), \mathbb{Z}/p).$$

This gives an important characterization of the p-rank, namely in terms of the number of étale p-cyclic covers of X. There is a canonical isomorphism of k-vector spaces, $H^1(X, \mathbb{Z}/p) \simeq \mathrm{Hom}(\mu_p, J(X))$, [13, Prop. 6.2.3] (where 'Hom' is regarded in the category of group schemes.

We see from the definition of the p-rank that $0 \le \sigma(X) \le p_a(X)$.

Definition 1.2 A curve X for which $\sigma(X) = p_a(X)$ is called *ordinary*.

In the case that X is smooth over k, we can compute the p-rank in terms of the Hasse–Witt matrix A, this is the matrix of the Frobenius morphism on $H^1(X, \mathcal{O}_X)$. The p-rank is the rank of $A \cdot A^{(p)} \cdots A^{(p^{g-1})}$, here $A^{(p)}$ denotes the matrix obtained by raising all coefficients of A to the pth power, [7]. One can compute the Hasse–Witt matrix of a curve for instance if one has an equation for the curve. For example, for hyperelliptic curves one gets a fairly explicit description of the Hasse–Witt matrix, and one can in principle compute the p-rank, see e.g. [27]. If the curve X is defined over a finite field $\mathbb{F}_q$, then one can compute the p-rank using the Zeta function of X. More precisely, the p-rank of X can be deduced from the reduction mod p of the characteristic polynomial of the Frobenius endomorphism $x \mapsto x^q$ on the Jacobian, [7]. This characteristic polynomial can be deduced from the Zeta function of X.

The Hasse–Witt matrix can also be computed using the Cartier operator on $H^0(X, \Omega_X)$. The spaces $H^0(X, \Omega_X)$ and $H^1(X, \mathcal{O}_X)$ are dual, and the Cartier operator is the transpose of the Frobenius, [21].

In case $J(X)$ is an abelian variety, for example if X is smooth, $\mathrm{Hom}(\mu_p, J(X))$ is dual to $\mathrm{Hom}(\mathbb{Z}/p, J(X))$. Hence the p-rank is equal to the dimension over $\mathbb{F}_p$ of $J(X)[p](k)$, the p-torsion points of $J(X)$.

Let Γ_X be the dual graph of X, i.e. the vertices correspond to the irreducible components of X and the edges correspond to the singular points. Let $\pi : \tilde{X} \to X$ be the normalization of X and $X_1, \ldots, X_n$ the irreducible components of $\tilde{X}$. Let m be the number of singular points of X.

Lemma 1.3

$$\sigma(X) = \sum_{i=1}^{n} \sigma(X_i) + b_1(\Gamma_X),$$

where $b_1(\Gamma_X) = \dim_{\mathbb{Z}} H^1(\Gamma_X, \mathbb{Z}) = 1 - n + m$.

Proof. [28]. Let $Q = \pi_*(\mathcal{O}_{\tilde{X}})/\mathcal{O}_X$. The exact sequence

$$0 \to \mathcal{O}_X \to \pi_*(\mathcal{O}_{\tilde{X}}) \to Q \to 0$$

gives rise to the long exact cohomology sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \pi_*(\mathcal{O}_{\tilde{X}})) \to H^0(X, Q) \to$$
$$H^1(X, \mathcal{O}_X) \to H^1(X, \pi_*(\mathcal{O}_{\tilde{X}})) \to 0.$$

The lemma follows, since $H^1(X, \pi_*(\mathcal{O}_{\tilde{X}})) \simeq \oplus_i H^1(X_i, \mathcal{O}_{X_i})$ and F is a bijection on $H^0(X, \cdot)$. $\square$

Recall that there is a similar formula for the genus: $p_a(X) = \sum g(X_i) + b_1(\Gamma)$. It follows that X is ordinary iff the normalization of all its irreducible components are ordinary. In particular, if all irreducible components of X have genus zero, then X is ordinary. The following example illustrates that a loop in the dual graph of X contributes to the p-rank of X.

Example 1.4 Let X be a semistable curve of genus 1, consisting of two $\mathbb{P}^1$'s meeting in two points. Then X has a p-cyclic étale cover $Y \to X$. Here Y is semistable genus one curve, consisting of a cycle of $2p$ copies of $\mathbb{P}^1$. It follows that X is ordinary.

Proposition 1.5 *Let R be a complete discrete valuation ring in equal characteristic p. Suppose that the residue field k of R is algebraically closed. Let $\mathcal{X} \to \mathrm{Spec}(R)$ be a semistable curve, whose geometric generic fiber $\mathcal{X}_{\bar{\eta}}$ is smooth. Then*

$$\sigma(\mathcal{X}_s) \leq \sigma(\mathcal{X}_{\bar{\eta}}).$$

Proof. See e.g. [16]. The proposition follows from the fact that every étale p-cyclic cover of $\mathcal{X}_s$ can be lifted to a *unique* étale p-cyclic cover of $\mathcal{X}$. This implies that

$$\mathrm{Hom}(\pi_1(\mathcal{X}_s), \mathbb{Z}/p) \hookrightarrow \mathrm{Hom}(\pi_1(\mathcal{X}_{\bar{\eta}}), \mathbb{Z}/p).$$

The proposition follows, since $\sigma(X) = \dim_{\mathbb{F}_p} \mathrm{Hom}(\pi_1(X), \mathbb{Z}/p)$. $\square$

Let X_{gen} be the generic curve of genus g, i.e. the curve corresponding to the geometric generic point of the moduli space of genus g curves $\mathcal{M}_g \otimes \mathbb{F}_p$. There exist a semistable curve $\mathcal{X} \to \operatorname{Spec} R$ whose geometric generic fiber is isomorphic to X_{gen} and which is a Mumford curve. Here R is some complete discrete valuation ring. Since the normalization of each irreducible component of $\mathcal{X}_s$ has genus zero, $\mathcal{X}_s$ is ordinary and hence also X_{gen} is ordinary. In fact, the locus of ordinary curves is open and dense in $\mathcal{M}_g \otimes \mathbb{F}_p$, [11]

2 The p-part and pro-p Sylow groups of the fundamental group

From now on X will be a smooth projective irreducible curve over an algebraically closed field k of genus g. Let S be a nonempty set of k-points and $U = X - S$. In this section we study the largest pro-p quotients and the pro-p Sylow group of the fundamental groups $\pi_1(X)$ and $\pi_1(U)$.

Let Γ be a profinite group. We denote by Γ^p the *largest pro-p quotient* of Γ. We will refer to it as the p-part of Γ. For fundamental groups, the p-part is denoted by $\pi_1^p(X)$ etc. A *pro-p Sylow group* Γ_p is a pro-p subgroup of Γ such that $[\Gamma : \Gamma_p]$ is prime to p. As in the case of finite groups, pro-p Sylow groups exist, they are all conjugates, and every pro-p subgroup of Γ is contained in some pro-p Sylow group, [23, Section 1.4].

Proposition 2.1 *Let X be a projective curve of genus $g \geq 2$. Then $\pi_1^p(X)$ is a free pro-p group on $\sigma(X)$ generators. The pro-p Sylow group $\pi_1(X)_p$ is a free pro-p groups on infinitely many generators.*

Proof. In Lecture 14 it is shown that $\pi_p(X)$ and $\pi_1(X)_p$ are free pro-p groups. The number of generators of $\pi_1^p(X)$ is equal to $\dim_{\mathbb{F}_p} \operatorname{Hom}(\pi_1^p(X), \mathbb{Z}/p) = \sigma(X)$.

The fact that the number of generators of $\pi_1(X)_p$ is infinite follows from a result of Raynaud [14, Th. 4.3.1]. The results states that if ℓ is a sufficiently large prime and X is a curve of genus $g \geq 2$, then there exists an étale ℓ-cyclic cover $f_\ell : Y_\ell \to X$ such that $\sigma(Y_\ell) - \sigma(X) = g(Y_\ell) - g(X) = (\ell - 1)(g(X) - 1)$. This condition means the following; ϕ_ℓ induces a morphism of Jacobians $J(X) \to J(Y)$ with finite kernel. The condition on the p-rank can be restated as: the new part of the Jacobian $J(Y)/\mathrm{im}(J(X))$ is ordinary.

Put $n_\ell = \sigma(Y_\ell)$. There exist an étale $(\mathbb{Z}/p)^{n_\ell}$-cover $Z \to Y$ and $Z \to X$ is Galois, since $Z \to Y$ is the maximal elementary abelian p-cover of Y. Since n_ℓ can be made arbitrary large by letting ℓ vary, it follows that the pro-p Sylow group $\pi_1(X)_p$ is not finitely generated. □

Proposition 2.2 *Let U be an affine curve over an algebraically closed field k of characteristic p. Then the number of generators of $\pi_1^p(U)$ is equal to the cardinality of k.*

Proof. In Lecture 14 it has been proved that $\pi_1^p(U)$ is a free pro-p group. First suppose that $U = \mathbb{A}^1$. In this case it has been proved in Lecture 14, using Artin–Schreier theory, that $\operatorname{Hom}(\pi_1(U), \mathbb{Z}/p)$ is a k-vector space with basis t^i, where i runs through the positive integers prime to p. This proves the proposition for $\mathbb{A}^1$. The general case is proved analogously. □

An *embedding problem* for a profinite group Γ is a diagram

$$\begin{array}{ccccccccc} & & & & & & \Gamma & & \\ & & & & & & \downarrow \alpha & & \\ 1 & \to & N & \to & G & \to & H & \to & 1, \end{array}$$

where α is surjective and the sequence is exact. A *(weak) solution* to an embedding problem is an arrow $\beta \colon \Gamma \to G$ such that the diagram above commutes. The solution is called *proper* if β is surjective. If Γ is $\pi_1(X)$ or $\pi_1(U)$ then it was shown in Lecture 14 that $\mathrm{cd}_p(\pi_1(X)) \leq 1$ (resp. $\mathrm{cd}_p(\pi_1(U)) \leq 1$). This implies that every embedding problem for $\pi_1(X)$ and $\pi_1(U)$ with kernel a p-group, has a weak solution, [23, Prop. I.16]. One would like to know which embedding problems have proper solutions. In the geometric setting, proper solutions correspond to quotients of the fundamental group.

For projective curves, one can show that an embedding problem

$$\begin{array}{ccccccccc} & & & & & & \pi_1(X) & & \\ & & & & & & \downarrow \alpha & & \\ 1 & \to & P & \to & G & \to & H & \to & 1 \end{array}$$

with p-group kernel P and H prime to p, has a proper solution iff the quotient embedding problem one gets by replacing P by its largest elementary abelian quotient $P/P^p[P,P]$ has a proper solution, [12]. In case X is projective, not every embedding problem with kernel an elementary abelian p-group can be solved. The embedding problem above with P elementary abelian has a solution iff P is a submodule of $H^1(Y,\mathcal{O}_Y)^F$ as $\mathbb{F}_p[H]$-module, [12]. Here $Y \to X$ is the H-cover corresponding to α. It is known that $H^1(Y,\mathcal{O}_Y)$ as an $k[H]$-module is isomorphic to $k[H]^{g-1} \times k$, [2]. If Y is ordinary, then the $\mathbb{F}_p[H]$-module structure of $H^1(Y,\mathcal{O}_Y)^F$ is $\mathbb{F}_p[H]^{g-1} \times \mathbb{F}_p$, and we know which embedding problems can be solved. We come back to this question in Section 4.

For affine curves one can show that every embedding problem with p-group kernel has a strong solution. Versions of this result are used in the proof of Abhyankar's Conjecture, [22] and [15, Section 4]. For generalizations see [5]. As in the case of covers of a projective curve, we can restrict to the case where the kernel is an elementary abelian p-group. Let $V \to U$ be an H-cover. In contrast to the projective case, $\mathrm{Hom}(\pi_1(V),\mathbb{Z}/p)$ is not finitely generated. In fact, one can show that every irreducible $\mathbb{F}_p[H]$-character occurs infinitely often in $\mathrm{Hom}(\pi_1(V),\mathbb{Z}/p)$, [22].

3 Deuring–Shafarevich formula

Notations are as in the previous section, in particular X and Y are supposed to be smooth, projective curves over an algebraically closed field k of characteristic p. Here is an application of the results of the previous section. Suppose $Y \to X$ is an étale Galois cover, whose Galois group is a p-group P. Then

$$1 \to \pi_1^p(Y) \to \pi_1^p(X) \to P \to 1$$

is exact. The group $\pi_1^p(X)$ is a free pro-p group on $\sigma(X)$ generators, and $\pi_1^p(Y)$ is free on $\sigma(Y)$ generators. Exercise 4 of [23, Section 1.4.2] states that since $\pi_1^p(X)$ is free,

$$\sigma(Y) - 1 = |P|(\sigma(X) - 1),$$

for all covers whose Galois group is a p-group. The proof does not go through for affine curves. But still, the formula can be generalized.

Theorem 3.1 (Deuring–Shafarevich formula) *Let $f\colon Y \to X$ be a (possibly ramified) Galois cover, whose Galois group is a p-group P. Then*

$$\sigma(Y) - 1 = |P|(\sigma(X) - 1) + \sum_{y \in Y} (e_y - 1).$$

Here e_y is the ramification index at y.

The proof we sketch here can be found in [3]. Let S be the branch locus of f and $U = X - S$; put $V = f^{-1}(U)$. Then $f: V \to U$ is étale. By $H_c^i(U,\mathcal{F})$ we denote étale cohomology with compact support, see e.g. [8, p. 93]. Let $\chi_c(U,\mathcal{F})$ be the Euler–Poincaré characteristic. Note that $\chi_c(X,\mathbb{F}_p) = \chi(X,\mathbb{F}_p) = 1 - \sigma(X)$ and $\chi_c(Y,\mathbb{F}_p) = 1 - \sigma(Y)$, since X and Y are projective.

We have the following exact sequences (excision, [8, III.1.30])

$$\cdots \to H_c^i(U,\mathbb{F}_p) \to H_c^i(X,\mathbb{F}_p) \to H_c^i(S,\mathbb{F}_p) \to \cdots$$

and

$$\cdots \to H^i_c(V, \mathbb{F}_p) \to H^i_c(Y, \mathbb{F}_p) \to H^i_c(f^{-1}(S), \mathbb{F}_p) \to \cdots.$$

Therefore

$$\chi_c(U, \mathbb{F}_p) + |S| = \chi_c(X, \mathbb{F}_p) = 1 - \sigma(X) \tag{1}$$

and

$$\chi_c(V, \mathbb{F}_p) + |f^{-1}(S)| = \chi_c(Y, \mathbb{F}_p) = 1 - \sigma(Y). \tag{2}$$

Lemma 3.2 *Suppose that $f\colon V \to U$ is an étale Galois cover whose Galois group is a p-group P. Then*

$$\chi_c(V, \mathbb{F}_p) = |P|\, \chi_c(U, \mathbb{F}_p).$$

Proof. Using induction, we may reduce to the case that P is cyclic of order p. Let $\mathcal{F}$ (resp. $\mathcal{G}$) be the constant sheaf $\mathbb{F}_p$ on V (resp. on U). Then $H^i(V, \mathcal{F}) = H^i(U, f_*(\mathcal{F}))$. The group P acts on $f_*(\mathcal{F})$. We can reduce to the case that $f_*(\mathcal{F})$ is indecomposable under the action of P. Since the only irreducible character of P in characteristic p is the trivial character, the sheaf $f_*(\mathcal{F})$ is an extension of sheaves on which P acts trivially. Therefore $\chi_c(V, \mathbb{F}_p) = |P|\, \chi_c(U, \mathbb{F}_p)$. □

Proof of the theorem. We combine the lemma with (1) and (2). This gives

$$\begin{aligned}
\sigma(Y) - 1 &= |P|\,(\sigma(X) - 1 + |S|) - |f^{-1}(S)| \\
&= |P|\,(\sigma(X) - 1) + \sum_{x \in X} \frac{|P|}{e_x}(e_x - 1) \\
&= |P|\,(\sigma(X) - 1) + \sum_{y \in Y} (e_y - 1).
\end{aligned}$$

□

There exist many proofs of the Deuring–Shafarevich formula in the literature. Below I give a list of the (published) proofs I have found. The theorem was first proved by Deuring [4] in the ramified case, but the proof contains a mistake. It was proved by Shafarevich [19] in the étale case; he used it to show that $\pi_1^p(X)$ is free. Subrao [25] extended the proof of Shafarevich to the ramified case. He used the Deuring–Shafarevich formula to show that there exist ordinary curves which violate the Hurwitz bound on the order of the automorphism group in characteristic p. Madan [6] extended and corrected the proof of Deuring. Richter [18] gave a variant on this approach. Nakajima [10] gave a proof using the Galois module structure of cohomology groups. This is a slightly stronger results than the result stated above. A proof of the result of Nakajima using the method of Crew is given by Suwa, [26]. The Deuring–Shafarevich formula is also known under the name Crew formula.

Note that in the Deuring–Shafarevich formula there is no term depending on the conductor of a ramification point. This is in contrast to the Hurwitz genus formula. The Deuring–Shafarevich formula states that étale covers of an ordinary curve with Galois group a p-group, are ordinary. But if $Y \to X$ is a ramified p-cover, then Y will hardly ever be ordinary. The reason being that the contribution of a wild ramification point to the genus is, in general, larger than the contribution to the p-rank.

4 Ordinary covers

Suppose that $f\colon Y \to X$ is a G-Galois cover, where G is not a p-group. One might ask whether there exists a formula relating $\sigma(Y)$ and $\sigma(X)$ as in the case of p-group covers. It is easy to see that there cannot be a formula for $\sigma(Y)$ in terms of $\sigma(X)$, the order of G and the ramification indices in general. For example, suppose $p \neq 2$ and consider 2-cyclic covers $E_\lambda \to \mathbb{P}^1$, branched at four points $0, 1, \infty, \lambda$. Then $\sigma(E_\lambda) = 1$ except for finitely many λ. Similarly, one can show that the p-rank of étale 2-cyclic covers of a genus two curve X varies with X if $p \neq 2$. Hence the p-rank in a family of covers varies with the curve X and the branch points of the cover. A first question to ask is therefore: what happens if X and the branch points are *generic*? For simplicity, we restrict to the case of étale covers of curves of genus $g \geq 2$. For the ramified case, see [1]. Let X_{gen} be the generic curve of genus g, i.e. X_{gen} is the curve corresponding to the geometric generic point of the moduli space $\mathcal{M}_g \otimes \mathbb{F}_p$.

Question 4.1 What is the p-rank of the étale covers of the generic curve X_{gen}?

It turns out that there are many groups G such that some or all the G-covers are ordinary. But for every g there is a nonordinary Galois cover of X_{gen}. It seems that even for the covers of the generic curve there is no easy formula for the p-rank of a cover. In the rest of this section we give examples of ordinary covers of the generic curve. In Lecture 19 the nonordinary covers will be discussed. Showing that a G-cover of the generic curve is ordinary can be translated into solving an embedding problem for $\pi_1(X_{\mathrm{gen}})$ with kernel an elementary abelian p-group, as was explained in Section 2.

Let G be a finite group. Consider a semistable curve $\mathcal{X} \to \operatorname{Spec} R$, with geometric generic fiber isomorphic to X_{gen}, over some complete discrete valuation ring R. We want to construct ordinary covers of X_{gen} by constructing ordinary covers of the special fiber $\mathcal{X}_s$ which can be lifted to covers $\mathcal{Y} \to \mathcal{X}$. By Proposition 1.5, this implies that the cover $\mathcal{Y}_{\bar\eta} \to \mathcal{X}_{\bar\eta}$ is ordinary. One can construct such covers as follows. Suppose that $\mathcal{X}$ is a Mumford curve. Construct a cover $f_s : \mathcal{Y}_s \to \mathcal{X}_s$ by taking a cover of the dual graph of $\mathcal{X}_s$. The cover constructed in Example 1.4 is a particular case of this construction. Then the normalization of any irreducible component of $\mathcal{Y}_s$ has genus zero, so $\mathcal{Y}_s$ is ordinary. Moreover, $\mathcal{Y}_s \to \mathcal{X}_s$ is étale, so it can be lifted to a cover $f : \mathcal{Y} \to \mathcal{X}$ and $\mathcal{Y}_s$ is ordinary. The following two propositions use a variation on this idea.

Proposition 4.2 *Let $f : Y \to X_{\mathrm{gen}}$ be an étale cover whose Galois group G is abelian. Then Y is ordinary.*

Proof. [9], [28], [20]. We sketch the proof. An étale cover of an ordinary curve with Galois group a p-group is ordinary, by Deuring–Shafarevich. Therefore, we may suppose that $p \nmid |G|$. Let n be an integer such that every element of G is killed by n. Since G is a quotient of $\pi_1^{\mathrm{ab}}(X_{\mathrm{gen}})$, it follows that G is a quotient of $H := (\mathbb{Z}/n)^{2g}$. Since a quotient of an ordinary curve is ordinary, it suffices to prove the proposition for $G = H$.

Let X_0 be a chain of g ordinary elliptic curves. There exists a semistable curve $\mathcal{X} \to \operatorname{Spec}(R)$ whose generic fiber is isomorphic to X_{gen} and whose special fiber is isomorphic to X_0. Here R is some equal characteristic p complete discrete valuation ring with algebraically closed residue field.

Let $Y \to X_{\mathrm{gen}}$ be an étale H-cover. Since the dual graph of X_0 is a tree, $\operatorname{Pic}^0_{\mathcal{X}/R}$ is an abelian variety (Lecture 3). In other words, the Jacobian of X_{gen} has potentially good reduction. This implies that $J(\mathcal{X})[n]$ is finite and étale over $\operatorname{Spec}(R)$. Extend K if necessary, to get the n-torsion defined over R. Then $J(\mathcal{X}_{\bar\eta})[n]$ can be identified with $J(\mathcal{X}_s)[n] = \prod_i E_i[n]$. By Kummer theory, we see that the H-cover $Y \to X_{\mathrm{gen}}$ reduces to an *étale* cover of $Y_0 \to \mathcal{X}_0$. The restriction of this cover to E_i consist of copies of $E_i \xrightarrow{\cdot n} E_i$. In particular, every irreducible component of Y_0 is ordinary, so Y_0 is ordinary. Proposition 1.5 implies that therefore also Y is ordinary. □

Proposition 4.3 *Let G be a group generated by g elements. Then there exists a G-cover of the generic curve X_{gen} of genus g which is ordinary.*

Proof. [24]. Let $\mathcal{X} \to \operatorname{Spec}(R)$ be as in the proof of the previous proposition. The special fiber of $\mathcal{X}$ consists of a chain of ordinary elliptic curves, call them $E_1, \ldots, E_g$. The fundamental group $\pi_1(E_i)$ is known; it is isomorphic to $\mathbb{Z}_p \times \prod_{\ell \neq p} \mathbb{Z}_\ell^2$. The fundamental group $\pi_1(\mathcal{X}_s)$ is an amalgamated product of the $\pi_1(E_i)$. Therefore $\pi_1(\mathcal{X}_s)$ has a free profinite group on g generators as a quotient.

In particular, there exists an étale G-cover $Y_0 \to \mathcal{X}_s$. Every irreducible component of Y_0 is an étale cover of one of the E_i, so Y_0 is ordinary. We can deform $Y_0 \to \mathcal{X}_s$ to a G-cover $\mathcal{Y} \to \mathcal{X}$. Proposition 1.5 implies that $\mathcal{Y}_{\bar{\eta}}$ is ordinary. □

Note the difference between the two previous propositions. In case G is generated by g elements we have only constructed *some* cover of X_{gen} which is ordinary. The result of Proposition 4.3 is up to a certain extend generalized by the following proposition of Raynaud [17]. In this result G is supposed to be a central extension of two abelian groups. Using representation theory, one is reduced to show that the covers of X_{gen} whose Galois group H_n is a central extension of $(\mathbb{Z}/n)^{2g}$ by $\mathbb{Z}/n$ are ordinary. With a suitable assumption on n, one can show, using an argument similar to the one we have seen in the proof of Proposition 4.3, that there exists some H_n-cover of $Z \to X_{\mathrm{gen}}$ which is ordinary. One concludes the proof by showing that any H_n-cover of X_{gen} is conjugated by the monodromy to the cover $Z \to X_{\mathrm{gen}}$.

Theorem 4.4 *Let $Y \to X_{\mathrm{gen}}$ be an étale G-Galois cover, where G is a central extension of two abelian groups. Suppose that $\gcd(|G|, g!) = 1$. Then Y is ordinary.*

Proof. [17]

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