THE DIRICHLET PROBLEM FOR GRAPHS OF PRESCRIBED ANISOTROPIC MEAN CURVATURE IN $\mathbb{R}^{n+1}$

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Abstract

We consider the Dirichlet problem for graphs of prescribed mean curvature in $\mathbb{R}^{n+1}$ where the prescribed mean curvature function $H = H(X, N)$ may depend on the point X in space and on the normal N of the graph as well. In some special cases this Dirichlet problem arises as the Euler equation of a generalised nonparametric area funcional.

Introduction

In this paper we study and solve the Dirichlet problem for n-dimensional graphs of prescribed anisotropic mean curvature in $\mathbb{R}^{n+1}$: Given a $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^n$ and Dirichlet boundary values $g \in C^{2+\alpha}(\partial\Omega, \mathbb{R})$ we want to find a solution $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ of

$$\begin{aligned} &\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = nH(x,\zeta,N) \quad \text{in } \Omega\,, \\ &N(x) = \frac{1}{\sqrt{1+|\nabla\zeta|^2}}(-\nabla\zeta, 1) \quad \text{in } \Omega\,, \\ &\zeta = g \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1}$$

Here, $N = N(x)$ denotes the upper unit normal vector of the graph ζ. The function $H : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$, which is called the prescribed mean curvature, depends on the point $(x, \zeta(x))$ in space and on normal $N(x)$ as well. At each point $x \in \Omega$ the geometric mean curvature of the graph ζ, defined as the average of the principal curvatures, is equal to the value $H(x, \zeta(x), N(x))$, thus a solution ζ is also called a graph of prescribed mean curvature H.

For the minimal surface case, i.e. $H \equiv 0$, the first existence result in n dimensions and mean convex domains is given [8]. Later, the case $H = H(x)$ was treated where several existence results based on different methods have been proved. In [6], Gilbarg and Trudinger solve the Dirichlet problem in that case with the Leray-Schauder method. Furthermore, since the Dirichlet problem is in that case given by the Euler equation of the functional

$$A(\eta) := \int\limits_{\Omega} \Big(\sqrt{1+|\nabla\eta|^2} + nH(x)\eta\Big)\,dxdy\;,$$

direct methods from the calculus of variations can be applied to obtain a solution (see e.g. [5]; for the minimal surface case [7]).

Next, for the case of prescribed mean curvature $H = H(x,z)$ weak solutions were constructed by Miranda in [9]. Using suitable barriers those weak solutions were shown to be continuous up to the boundary under certain assumptions on the domain and the boundary values (see Schulz and Williams [14]). In those papers the monotonocity assumption $H_z \geq 0$ plays an important role, as it guarantees uniqueness of solutions via the maximum principle. In two dimensions graphs of prescribed mean curvature $H = H(x,y,z)$ were constructed in [11] by Sauvigny where a stable parametric solution from the corresponding Plateau problem is taken and shown to be a graph over the x,y-plane. Approaches to solve the Dirichlet problem via a continuity method can be found in [12] and [13, §9 of chapter XII], where again the assumption $H_z \geq 0$ is needed.

We will now study and solve the Dirichlet problem for graphs of prescribed mean curvature $H = H(x,z,N)$ depending on the point (x,z) in space and on the normal N as well. In general this problem does not necessarily arise as the Euler equation of some geometric functional and hence direct methods from the calculus of variations are not applicable. In special situations however, namely when H depends linearly on N, such a Dirichlet problem is obtained as the Euler equation of certain generalized nonparametric area functionals such as

$$A(\eta) := \int_{\Omega} \Big(a(x,\eta)\sqrt{1+|\nabla\eta|^2} + b(x,\eta) \Big) dx$$

with functions $a : \overline{\Omega} \times \mathbb{R} \to (0,+\infty)$ and $b : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$. The Euler equation is then given by

$$\operatorname{div} \frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = \frac{\nabla a(x,\zeta)\cdot N + b_z(x,\zeta)}{a(x,\zeta)} =: nH(x,\zeta,N) \quad \text{in } \Omega\,.$$

When $b \equiv 0$ minimizers of this nonparametric functional A were constructed by Tausch in [15] and for $n = 2$ the Plateau problem for the parametric version of this functional was studied in [2]. The functional A then measures area of the graph ζ in the Riemannian space

$$(\mathbb{R}^{n+1}, ds^2)\ ,\ ds^2 = a(x_1,\ldots,x_{n+1})(dx_1^2 + \cdots + dx_{n+1}^2)$$

and solutions of the Euler equation are minimal graphs in this Riemannian space. The existence theorem we will prove in this paper will also apply to the case that $b \neq 0$. With an appropriate choice of the function b this enables us also to construct graphs of constant or prescribed mean curvature within this Riemannian space $(\mathbb{R}^{n+1}, ds^2)$.

Finally, we want to point out that in the context of prescribed mean curvature depending on both the point in space and on the normal as well we are able to solve the Dirichlet boundary value problem under a slightly weaker assumption than $H_z \geq 0$ needed in the papers already quoted, see assumption (A) in section 1. This especially enables us to solve the Dirichlet problem in situations when the uniquess is no longer guaranteed.

This paper is organized as follows: In section 1 we prove a differential equation for the normal (Theorem 1) and use it to derive an interior gradient bound of the graph in terms of its boundary gradient (Corollary 1). In section 2 we then solve the Dirichlet problem (see Theorem 2) using the Leray-Schauder method from [6]. In section 3 we then apply the existence theorem to the functional A and construct critical points and minimizers of this functional (see Theorem 3).

1. An interior gradient estimate in terms of the boundary gradient

For a given a solution $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ of (1) we define the graph parametrisation

$$X = X(x_1, \ldots, x_n) := (x_1, \ldots, x_n, \zeta(x_1, \ldots, x_n)) \quad \text{for } x = (x_1, \ldots, x_n) \in \overline{\Omega}$$

as well as the upper unit normal

$$N = N(x_1, \ldots, x_n) := \frac{1}{\sqrt{1 + |\nabla\zeta|^2}}(-\nabla\zeta, 1) \ .$$

Note that the normal N satisfies $N \cdot e_{n+1} = N_{n+1} > 0$ with $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. This also means that the image of the normal mapping is included in the upper hemisphere

$$S^n_+ := \{x \in \mathbb{R}^{n+1} \ : \ |x| = 1 \ , \ x_{n+1} > 0\} \ .$$

Now for the case a prescribed mean curvature $H = H(X)$ only depending point X in space, the following differential equation for the normal

$$\triangle N + (\mathrm{tr}(S^2) - n\nabla_X H \cdot N)N = -n\nabla_X H$$

was derived in [3, Corollary 4.3]. The differential equation proven there looks slightly different due to a different definition of the mean curvature there as the sum of the principal curvatures. For the special case $n = 2$ and conformal parameters the same equation was proven in [11, Satz 1]. Here, $\triangle$ denotes the Laplace-Beltrami operator and $\mathrm{tr}(S^2) = \sum_{i=1}^n \kappa_i^2$ where κ_i are the principal curvatures of the surface. We now give a generalisation of this equation for the case of prescribed mean curvature $H = H(X, N)$ which will be used later to provide a lower bound of N_{n+1}.

Theorem 1*: Let $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ be a solution of (1) for the prescribed mean curvature $H \in C^{1+\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1}, \mathbb{R})$. Then the normal $N = N(x)$ of the surface satisfies the differential equation*

$$\triangle N + \sum_{i=1}^n a_i(x) N_{x_i} + (tr(S^2) - n\nabla_X H \cdot N)N = -n\nabla_X H \quad in\ \Omega \tag{2}$$

with certain coefficients $a_i \in C^0(\overline{\Omega}, \mathbb{R})$ satisfying the estimate

$$\Big| \sum_{i=1}^n a_i(x)\zeta_{x_i} \Big| \leq n(n+1) \sup_{(x,z,N) \in \Omega \times \mathbb{R} \times S^n_+} |\nabla_N H(x, z, N)| \ . \tag{3}$$

Here we have set $\nabla H = (\nabla_X H, \nabla_N H) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

Proof:
We start with the parameter invariant differential equation

$$\triangle N + \mathrm{tr}(S^2)N = -nDX(\mathrm{grad}H) \tag{4}$$

derived in [3, Theorem 1.1] (see also the remark after the proof of [3, Theorem 1.1]). There, the expression $DX(\mathrm{grad}H)$ is defined by

$$DX(\mathrm{grad}H) = g^{kj}\partial_k H \partial_j X$$

using the sum convention. The matrix with the entries g^{kj} is defined to be the inverse of the metric $g_{ij} := \partial_i X \cdot \partial_j X$. Now, given any vector $\gamma \in \mathbb{R}^{n+1}$, we claim that

$$g^{kj}(\gamma \cdot \partial_k X)\partial_j X = \gamma - (\gamma \cdot N)N \ . \tag{5}$$

To verify that this equation holds, we multiply the left side by $\partial_l X$ for each l and calculate

$$\begin{aligned}\{g^{kj}(\gamma\cdot\partial_k X)\partial_j X\}\cdot\partial_l X &= g^{kj}g_{jl}(\gamma\cdot\partial_k X)=\delta_{kl}(\gamma\cdot\partial_k X)\\ &= \gamma\cdot\partial_l X=\{\gamma-(\gamma\cdot N)N\}\cdot\partial_l X\end{aligned}$$

This holds for all $l=1,\dots,n$ proving the claimed equation. In a similar way, the following equation can be proved

$$g^{kj}(\beta\cdot\partial_k N)\partial_j X=g^{kj}(\beta\cdot\partial_k X)\partial_j N \tag{6}$$

for all vectors $\beta\in\mathbb{R}^{n+1}$. We now calculate

$$\begin{aligned}DX(\text{grad}H) &= g^{kj}\partial_k\{H(X,N)\}\partial_j X\\ &= g^{kj}(\nabla_X H\cdot\partial_k X)\partial_j X+g^{kj}(\nabla_N H\cdot\partial_k N)\partial_j X\\ &= \nabla_X H-(\nabla_X H\cdot N)N+g^{kj}(\nabla_N H\cdot\partial_k X)\partial_j N\end{aligned}$$

using (5) for $\gamma=\nabla_X H$ and (6) for $\beta=\nabla_N H$. Putting this result into (4), we obtain

$$\triangle N+\sum_{i=1}^{n}a_i(x)N_{x_i}+(\text{tr}(S^2)-n\nabla_X H\cdot N)N=-n\nabla_X H\quad\text{in }\Omega$$

with the coefficients a_i defined by

$$a_i(x):=n\sum_{k=1}^{n}g^{ki}(\nabla_N H\cdot\partial_k X)=n\nabla_N H\cdot\Big(\sum_{k=1}^{n}g^{ki}\partial_k X\Big)$$

Finally, to show the claimed estimate (3) we note

$$\sum_{i=1}^{n}a_i(x)\zeta_{x_i}=n\nabla_N H\cdot\Big(\sum_{i,k=1}^{n}g^{ki}\zeta_{x_i}\partial_k X\Big)=n\nabla_N H\cdot v\,.$$

with the vector

$$v:=\sum_{i,k=1}^{n}g^{ki}\zeta_{x_i}\partial_k X\in\mathbb{R}^{n+1}\,.$$

Due to the special parametrisation as a graph $X(x_1,\dots,x_n)=(x_1,\dots,x_n,\zeta(x_1,\dots,x_n))$ the metric is given by

$$g_{ij}=\delta_{ij}+\zeta_{x_i}\zeta_{x_j}\quad\text{and}\quad g^{kj}=\delta_{kj}-\frac{1}{1+|\nabla\zeta|^2}\zeta_{x_k}\zeta_{x_j}\,.$$

Let $v_l=v\cdot e_l$ be the l-th component of the vector v. For $l=1,\dots,n$ we have

$$v_l=\sum_{i,k=1}^{n}g^{ki}\zeta_{x_i}(\partial_k X\cdot e_l)=\sum_{i,k=1}^{n}\Big(\delta_{ki}-\frac{\zeta_{x_i}^2}{1+|\nabla\zeta|^2}\Big)\zeta_{x_k}\delta_{kl}=\Big(1-\frac{|\nabla\zeta|^2}{1+|\nabla\zeta|^2}\Big)\zeta_{x_l}=\frac{\zeta_{x_l}}{1+|\nabla\zeta|^2}$$

and we obtain the estimate $|v_l|\le 1$. For the $n+1$-th compoment of v we have

$$\begin{aligned}v_{n+1} &= \sum_{i,k=1}^{n}g^{ki}\zeta_{x_i}(\partial_k X\cdot e_{n+1})=\sum_{i,k=1}^{n}g^{ki}\zeta_{x_i}\zeta_{x_k}=\sum_{i,k=1}^{n}\Big(\delta_{ki}\zeta_{x_k}\zeta_{x_i}-\frac{\zeta_{x_k}^2\zeta_{x_i}^2}{1+|\nabla\zeta|^2}\Big)\\ &= |\nabla\zeta|^2-\frac{|\nabla\zeta|^4}{1+|\nabla\zeta|^2}=\frac{|\nabla\zeta|^2}{1+|\nabla\zeta|^2}\end{aligned}$$

which gives $|v_{n+1}| \le 1$. Combining these estimate we obtain $|v| \le n+1$ proving the estimate (3). $\Box$

Remark: In case that the prescribed mean curvature $H = H(X)$ does not depend on the normal N, we have $a_i(x) \equiv 0$ and obtain the original differential equation from [3, Corollary 4.3].

We now want to derive a lower bound for the function $\xi(x) := N_{n+1}(x) = N(x) \cdot e_{n+1} > 0$ in Ω in terms of the boundary values of ξ. We note that by Theorem 1 the function ξ satifies

$$\triangle\xi + \sum_{i=1}^{n} a_i(x)\xi_{x_i} + (\mathrm{tr}(S^2) - n\nabla_X H \cdot N)\xi = -nH_z(X, N) \ .$$

where we use the notation $x_{n+1} \equiv z$. We first impose the following structural condition on the prescribed mean curvature:

Assumption (A):
The function H has to satisfy the structure condition

$$H(x, z, N) = H_1(x, z, N) + H_2(x, z, N)\, N_{n+1} \quad \text{for } (x, z, N) \in \Omega \times \mathbb{R} \times S^n_+$$

with two functions $H_1, H_2 \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1}, \mathbb{R})$, where H_1 is monotone in the z-variable, i.e.

$$\frac{\partial}{\partial z} H_1(x, z, N) \ge 0 \quad \text{for } (x, z, N) \in \Omega \times \mathbb{R} \times S^n_+ \ .$$

Remarks:

1.) If we assume $H_z \ge 0$ then assumption (A) is satisfied with $H_1 \equiv H$ and $H_2 \equiv 0$. From $H_z \ge 0$ one can also deduce the uniqeness of solutions to the Dirichlet problem using the maximum principle for quasilinear elliptic equations (see [6, Theorem 10.2]). For a prescribed mean curvature $H = H(x, z)$ only depending on the point in space, assumption (A) is actually equivalent to $H_z \ge 0$. Under this assumption and for $n = 2$ graphs of prescribed mean curvature $H = H(x, z)$ were constructed in [11].

2.) In general assumption (A) is weaker then $H_z \ge 0$. In fact, it does not imply uniqueness for the Dirichlet problem. To see this, consider $n = 2$ and the disc $\Omega := \{x \in \mathbb{R}^2 \ : \ |x|^2 < \frac{3}{4}\}$ and the prescribed mean curvature

$$H(x, z, N) := -2(x_1 N_1 + x_2 N_2 + z N_3) + 1$$

satifying assumption (A). It is easy to see that the two graphs

$$\zeta_1(x) := \frac{1}{2} \quad \text{and} \quad \zeta_2(x) := \sqrt{1 - |x|^2} \quad \text{for } x \in \overline{\Omega}$$

are graphs of prescribed mean curvature H. They also have the same boundary values $\zeta_1 = \zeta_2 = \frac{1}{2}$ on $\partial\Omega$.

Using assumption (A) together with the inequality $\mathrm{tr}(S^2) \ge 0$, we now obtain for $\xi(x) = N_{n+1}(x)$ the differential inequality

$$\triangle\xi + \sum_{i=1}^{n} a_i(x)\xi_{x_i} + c\xi \le 0 \quad \text{in } \Omega \tag{7}$$

with the coefficients a_i from Theorem 1 and

$$c(x) := -n(\nabla_X H \cdot N) - n(H_2)_z \ . \tag{8}$$

We now want to give an interior lower bound for ξ in terms of its boundary values. If c had the right sign, in our case $c \geq 0$, then by the maximum principle ξ would achieve its minimum on $\partial\Omega$. Since however $c \geq 0$ does not hold in general, we use the following product ansatz

$$\tilde{\xi}(x) := \frac{\xi(x)}{\psi(x)} \quad \text{for } x \in \overline{\Omega}$$

for some positive function $\psi \in C^2(\overline{\Omega}, (0, +\infty))$ to be choosen later. Then by putting $\xi(x) = \tilde{\xi}(x)\psi(x)$ into (7) we obtain for $\tilde{\xi}$ the differential inequality

$$\psi \triangle \tilde{\xi} + \sum_{i=1}^{n} \tilde{a}_i(x) \tilde{\xi}_{x_i} + \tilde{c}(x) \tilde{\xi} \leq 0 \tag{9}$$

for some coefficients $\tilde{a}_i$ and $\tilde{c}$ where in particular

$$\tilde{c} = \triangle \psi + \sum_{i=1}^{n} a_i(x) \psi_{x_i} + c(x) \psi \; .$$

To have a minimum principle for $\tilde{\xi}$ we need to choose ψ such that $\tilde{c} \geq 0$. Therefore, we show

Proposition 1: *Let $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ be a solution of (1). Let the prescribed mean curvature $H \in C^{1+\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1}, \mathbb{R})$ be given satifying assumption (A) and the estimates*

$$|H| \leq h \quad , \quad |\nabla H| \leq h_1 \quad in\ \Omega \times \mathbb{R} \times S^n_+$$

with constants $h, h_1 \geq 0$. Then there exists some constant $\lambda = \lambda(h, h_1)$ such that the function $\psi(x) := e^{\lambda \zeta(x)}$ for $x \in \overline{\Omega}$ satisfies the differential inequality

$$\triangle \psi + \sum_{i=1}^{n} a_i(x) \psi_{x_i} + c(x) \psi \geq 0 \quad in\ \widehat{\Omega} \tag{10}$$

where $\widehat{\Omega} := \{x \in \Omega \,:\, |\nabla \zeta(x)| \geq 1\}$.

Proof:
We first note $\nabla \psi = \lambda \psi \nabla \zeta$. Using the metric $g = (g_{ij})$ with its inverse $g^{-1} = g^{kj}$ we compute the Laplace-Betrami operator of $\psi = e^{\lambda \zeta}$ as follows

$$\begin{aligned}
\triangle \psi &= \frac{1}{\sqrt{\det g}} \operatorname{div}\{\sqrt{\det g}\, g^{-1} \nabla \psi\} = \frac{\lambda}{\sqrt{\det g}} \operatorname{div}\{\psi (\sqrt{\det g}\, g^{-1} \nabla \zeta)\} \\
&= \frac{\lambda \psi}{\sqrt{\det g}} \operatorname{div}\{\sqrt{\det g} g^{-1} \nabla \zeta\} + \lambda (g^{-1} \nabla \zeta) \cdot \nabla \psi = \lambda \psi \Big(\triangle \zeta + \lambda (g^{-1} \nabla \zeta) \cdot \nabla \zeta \Big) \\
&= \lambda \psi \Big(\frac{nH(X,N)}{\sqrt{1 + |\nabla \zeta|^2}} + \lambda \frac{|\nabla \zeta|^2}{1 + |\nabla \zeta|^2} \Big) \; .
\end{aligned}$$

In the last step we have used two things: The equation $\triangle \zeta = nH(1 + |\nabla \zeta|^2)^{-\frac{1}{2}}$, being the $n+1$-th component of the prescribed mean curvature equation $\triangle X = nHN$, as well as $g^{-1} \nabla \zeta = \frac{\nabla \zeta}{1 + |\nabla \zeta|^2}$. Using (3) we now obtain at each point $x \in \widehat{\Omega}$ the following estimate

$$\begin{aligned}
&\triangle \psi + \sum_{i=1}^{n} a_i(x) \psi_{x_i} + c(x) \psi = \psi \Big(\lambda^2 \frac{|\nabla \zeta|^2}{1 + |\nabla \zeta|^2} + \lambda \frac{nH}{\sqrt{1 + |\nabla \zeta|^2}} + \lambda \sum_{i=1}^{n} a_i(x) \zeta_{x_i} + c(x) \Big) \\
&\geq \psi \Big(\frac{1}{2} \lambda^2 - nh\lambda - n(n+1) h_1 \lambda - n(2h + h_1) \Big)
\end{aligned}$$

By solving a quadratic inequality we may choose $\lambda = \lambda(h, h_1)$ large enough such that

$$\triangle\psi + \sum_{i=1}^{n} a_i(x)\psi_{x_i} + c(x)\psi \geq 0 \quad \text{in } \widehat{\Omega}$$

holds. $\square$

As a consequence we obtain the following gradient estimate.

Corollary 1: *Given a $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^n$ let $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ be a solution of (1) satisfying $|\zeta(x)| \leq M$ for some constant M. Let the prescribed mean curvature $H \in C^{1+\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1}, \mathbb{R})$ be given satisfying assumption (A) and*

$$|H| \leq h \quad \textit{and} \quad |\nabla H| \leq h_1 \quad \textit{in}\, \Omega \times \mathbb{R} \times S^n_+ \,.$$

Then there exists a constant $C = C(M, h, h_1) \geq 1$ such that

$$\sup_{x\in\Omega} |\nabla\zeta(x)| \leq C\Big(2 + \sup_{x\in\partial\Omega} |\nabla\zeta(x)|\Big) \,.$$

Proof:
Given a solution ζ of (1) we consider the last component of the normal

$$\xi(x) := N(x) \cdot e_{n+1} = \frac{1}{\sqrt{1 + |\nabla\zeta(x)|^2}} > 0 \,.$$

By (9) together with Proposition 1 there exists some constant $\lambda = \lambda(h, h_1) > 0$ such that for $\tilde{\xi}(x) = \xi(x)e^{-\lambda\zeta(x)} > 0$ we have the differential inequality

$$\psi\triangle\tilde{\xi} + \sum_{i=1}^{n} \tilde{a}_i(x)\tilde{\xi}_{x_i} \leq 0 \quad \text{within } \widehat{\Omega} := \{x \in \Omega \,:\, |\zeta(x)| \geq 1\} \,.$$

By the maximum principle $\tilde{\xi}|_{\widehat{\Omega}}$ must achieve its minimum on $\partial\widehat{\Omega}$. We now choose a point $x_0 \in \overline{\Omega}$ where $|\nabla\zeta|$ achieves its maximum within $\overline{\Omega}$. If $x_0 \notin \widehat{\Omega}$ then we have $|\nabla\zeta(x)| \leq |\nabla\zeta(x_0)| \leq 1$ and we are done. Otherwise the maximum principle yields

$$\xi(x_0) \geq e^{-\lambda M}\tilde{\xi}(x_0) \geq e^{-\lambda M} \inf_{\partial\widehat{\Omega}} \tilde{\xi}(x) \geq e^{-2\lambda M} \inf_{\partial\widehat{\Omega}} \xi(x)$$

which gives

$$|\nabla\zeta(x_0)| \leq \sqrt{1 + |\nabla\zeta(x_0)|^2} \leq e^{2M\lambda} \sup_{\partial\widehat{\Omega}} \sqrt{1 + |\nabla\zeta(x)|^2} \leq e^{2M\lambda}(1 + \sup_{\partial\widehat{\Omega}} |\nabla\zeta(x)|) \,.$$

We now note that $\partial\widehat{\Omega} \subset \partial\Omega \cup \{x \in \Omega \,:\, |\nabla\zeta(x)| = 1\}$ which yields

$$\sup_{\partial\widehat{\Omega}} |\nabla\zeta(x)| \leq 1 + \sup_{\partial\Omega} |\nabla\zeta(x)| \,.$$

The desired estimate now follows for $C := e^{2\lambda M}$. $\square$

2. Solution of the Dirichlet problem

In the first section we have provided an interior gradient bound in terms of the boundary gradient. We now need a boundary gradient estimate. Such an estimate holds for a large class of elliptic differential equations including our one (see [6, chapter 14]). However, we need to imposee following convexity assumption on the domain.

Definition 1*:* Let a domain $\Omega \subset \mathbb{R}^n$ be given. We say that Ω satisfies an enclosing sphere condition of radius $R \in (0, +\infty)$ if for each point $x \in \partial\Omega$ there exists a ball $B_R(x_0)$ of radius R centered at some point $x_0 \in \mathbb{R}^n$ such that

$$B_R(x_0) \cap \Omega = \emptyset \quad \text{and} \quad x \in \partial B_R(x_0)$$

holds. We say Ω satisfies an enclosing sphere condition of radius $+\infty$ if Ω is convex.

Note that a domain satifying an enclosing sphere condition is necessarily a convex domain.

Theorem 2*: Assumptions:*

1.) For some $C^{2+\alpha}$-domain Ω let the prescribed mean curvature $H \in C^{1+\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1}, \mathbb{R})$ satisfy the structure condition

$$H(x, z, N) = H_1(x, z, N) + H_2(x, z, N)N_{n+1}$$

for differentiable functions H_1 and H_2 such that H_1 is non-decreasing in the z-variable. Additionally, assume the smallness

$$|H| \leq h \quad \textit{and} \quad |H_1| \leq h_1 \quad \textit{in } \Omega \times \mathbb{R} \times S^n_+$$

for constants $h > 0$, $h_1 \geq 0$.

2.) Let the domain Ω be included in a disc of radius $B_{\frac{1}{h}}(0)$ and satisfy an enclosing sphere condition of radius $\frac{n-1}{nh_1}$.

Then for any Dirichlet boundary values $g \in C^{2+\alpha}(\partial\Omega, \mathbb{R})$ the Dirichlet problem (1) has a solution $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$.

Proof:
Consider the family of Dirichlet problems

$$\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}) \quad , \quad \operatorname{div} \frac{\nabla\zeta}{\sqrt{1 + |\nabla\zeta|^2}} = t\,nH(x, \zeta, N) \quad \text{in } \Omega \quad \text{and} \quad \zeta = t\,g \quad \text{on } \partial\Omega \tag{11}$$

with a parameter $t \in [0, 1]$ and let ζ be a solution for some $t \in [0, 1]$. We now define the spherical caps

$$\eta^{\pm}(x) := \pm\Big(||g||_{C^0(\partial\Omega)} + \sqrt{\frac{1}{h^2} - |x|^2}\Big) \quad \text{for } x \in \overline{\Omega}$$

being well defined due to $\Omega \subset B_{\frac{1}{h}}(0)$ and having constant mean curvature $\mp h$. Noting $\eta^- \leq \zeta \leq \eta^+$ on $\partial\Omega$ the comparision principle for quasilinear eliptic equations (see [6, Theorem 10.1]) yields $\eta^- \leq \zeta \leq \eta^+$ in Ω and we obtain the C^0-estimate

$$||\zeta||_{C^0(\overline{\Omega})} \leq ||g||_{C^0(\partial\Omega)} + \frac{1}{h} \, .$$

Now, in case $h_1 = 0$, we can use the convexity of the domain Ω together with the C^0-estimate to obtain a boundary gradient estimate

$$\sup_{x \in \partial\Omega} |\nabla\zeta(x)| \leq C_1$$

for some constant C_1 only depending on $||g||_{C^2(\partial\Omega)}$, h and h_1, using [6, Corollary 14.3]. Using the enclosing sphere condition, a similar boundary gradient estimate can also be derived from [6, Corollary 14.5] for the case $h_1 > 0$. Applying Corollary 1 we obtain the global C^1-estimate

$$||\zeta||_{C^1(\overline{\Omega})} \leq C_2$$

with a constant C_2 independent of t. The Leray-Schauder method [6, Theorem 13.8] yields a solution of the Dirichlet problem (11) for each $t \in [0,1]$. For $t = 1$ we obtain the desired solution of (1). □

Remarks:

1) The assumption $\Omega \subset B_{\frac{1}{h}}(0)$, needed to obtain a C^0-estimate of the solution, can be replaced by any other assumption assuring the existence of a C^0-estimate, for example $h < (\omega_n/|\Omega|)^{1/n}$, see [6, Theorem 10.10].

2) The enclosing sphere condition on the domain is needed to obtain a boundary gradient estimate. In case $H = H(x,z)$, i.e. the prescribed mean curvature does not depend on the normal, we could relax the enclosing sphere condition the a weaker mean-convexity assumption (see [6, Corollary 14.8]). However, it is not clear wether this result can be generalised to the case of prescribed mean curvature also depending on the normal N.

Example 1*:* We consider the following version of the half space model of hyperbolic space

$$\mathbb{H}^{n+1} := \{(x_1, x_2 \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 > 0\}$$

equipped with the Riemannian metric

$$ds^2 = \frac{1}{x_1^2}(dx_1^2 + \cdots + dx_{n+1}^2)\,.$$

We now take a domain $\Omega \subset \{x \in \mathbb{R}^n \mid x_1 > c\}$ for some constant $c > 0$ and a graph $\zeta : \overline{\Omega} \to \mathbb{R}$. Then within $\mathbb{H}^{n+1}$ the graph ζ has a prescribed mean curvature $h : \overline{\Omega} \to \mathbb{R}$ if and only if ζ satisfies the differential equation

$$\operatorname{div} \frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = \frac{n}{x_1}\Big(h(x) + \frac{\zeta_{x_1}}{\sqrt{1+|\nabla\zeta|^2}}\Big) \quad \text{in } \Omega$$

(see [10]). Thus in Euklidian $\mathbb{R}^{n+1}$ this graph has prescribed Euklidian mean curvature

$$H = H(x,N) := \frac{1}{x_1}(h(x) - N \cdot e_1)\,.$$

Note that $H_z = 0$ as H does not depend on z, hence assumption (A) is satisfied. Next we note

$$|H| \leq \frac{1}{c}(|h(x)| + 1) \quad \text{in } \Omega \times \mathbb{R} \times S^n_+\,.$$

Hence we can apply Theorem 2 and construct graphs of prescribed mean curvature in this model hyperbolic space $\mathbb{H}^{n+1}$.

3. Ciritical points and minimizers of a generalized nonparametric area functional

In this section we want to demonstrate that graphs of prescribed mean curvature $H = H(x, z, N)$ naturally appear when considering certain generalised area functionals such as

$$A(\eta) := \int_{\Omega} \Big(a(x,\eta)\sqrt{1+|\nabla\eta|^2} + b(x,\eta) \Big) dx$$

for a domain $\Omega \subset \mathbb{R}^n$ and given functions $a : \overline{\Omega} \times \mathbb{R} \to (0, +\infty)$ and $b : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$.
Examples:

1.) In case $a \equiv 1$ and $b \equiv 0$ we obtain the standard Euklidean area of a graph in $\mathbb{R}^{n+1}$. The Euler equation is the nonparametric minimal surface equation.

2.) For $a \equiv 1$ and $b = b(x_1, \ldots, x_n, z)$ we obtain the functional

$$A(\zeta) = \int_{\Omega} \Big(\sqrt{1+|\nabla\eta|^2} + b(x,\eta) \Big) dx \, .$$

The corresponding Euler equation leads to graphs of prescribed mean curature $H = H(x, \zeta) = \frac{1}{n} b_{x_{n+1}}(x, \zeta)$. Note that here the prescribed mean curvature only depends on the point in space but not on the normal.

3.) Taking some function for $a = a(x_1, \ldots, x_n, z)$ but $b \equiv 0$ one obtains

$$A(\eta) = \int_{\Omega} a(x,\eta)\sqrt{1+|\nabla\eta|^2} dx \, .$$

This functional measures the area of the graph η within the Riemannian space

$$(\mathbb{R}^{n+1}, ds^2) \quad , \quad ds^2 = a(x_1, \ldots, x_{n+1})(dx_1^2 + \cdots + dx_{n+1}^n)$$

which is a space conformally equivalent to Euklidian $\mathbb{R}^{n+1}$. The Euler equation leads to minimal graphs in this Riemannian space. For the special choice $a(x_1) = x_1^{-2}$ we obtain the half space model of hyperbolic space as in Example 1.

4.) Setting

$$U := \{(x,z) \in \mathbb{R}^{n+1} \mid x \in \Omega \, , \; \min(0, \zeta(x)) < z < \max(0, \zeta(x))\}$$

the oriented volume $\mathrm{vol}(U)$ of U within the Riemannian space $(\mathbb{R}^{n+1}, ds^2)$ is computed by

$$\mathrm{vol}(U) = \int_{\Omega} \int_0^{\zeta(x)} \sqrt{a(x_1, \ldots, x_n, z)^{n+1}} dz dx_1 \ldots dx_n \, .$$

We can now define

$$b(x_1, \ldots, x_n, z) := \int_0^z a(x_1, \ldots, x_n, s)^{\frac{n+1}{2}} ds$$

and consider the functional

$$A(\zeta) = \int_{\Omega} \Big(a(x,\eta)\sqrt{1+|\nabla\eta|^2} + nh\, b(x,\eta) \Big) dx$$

with a parameter $h \in \mathbb{R}$. Here one looks for critical points of the area under a volume constraint within the Riemannian space $(\mathbb{R}^{n+1}, ds^2)$ if one considers h to be a Lagrange parameter. The corresponding Euler equation leads to graphs of constant mean curvature h in the Riemannian space.

It is now relatively easy to derive the Euler equation of the functional A. For the case $b \equiv 0$ and $n = 2$ this was already done in [2].

Lemma 1: *Given two functions $a, b \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ with $a > 0$, the Euler equation of the functional A is given by*

$$\operatorname{div} \frac{a\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = a_z\sqrt{1+|\nabla\zeta|^2} + b_z \quad \textit{in } \Omega$$

or equivalenty

$$\operatorname{div} \frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = \frac{\nabla a \cdot N + b_z}{a} \quad \textit{in } \Omega$$

with the upper unit normal $N = N(x)$ of the graph ζ.

Proof:
Setting

$$F(x,z,p) := a(x,z)\sqrt{1+|p|^2} + b(x,z) \quad \text{for } (x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$$

the functional A can written in the form

$$A(\zeta) = \int_\Omega F(x,\zeta,\nabla\zeta)dx \; .$$

We first calculate

$$F_z = a_z\sqrt{1+|p|^2} + b_z \quad , \quad F_{p_i} = \frac{a\,p_i}{\sqrt{1+|p|}} \; .$$

Given a test function $\delta \in C_0^\infty(\Omega)$ and a parameter $t \in \mathbb{R}$ we set $\zeta^t := \zeta + t\delta$ and obtain for the first variation

$$\begin{aligned} 0 &= \frac{d}{dt}A(\zeta^t)\Big|_{t=0} = \int_\Omega (F_z\delta + F_{p_i}\delta_{x_i})dx \\ &= \int_\Omega \Big(F_z\delta - \operatorname{div}(F_p)\,\delta\Big)dx = \int_\Omega \Big(F_z - \operatorname{div}(F_p)\Big)\delta\,dx \; , \end{aligned}$$

In the last step we have integrated by parts and used the zero boundary values of δ. As this equation has to hold for all function $\delta \in C_0^\infty(\Omega)$ we obtain as the Euler equation

$$\begin{aligned} 0 &= F_z - \operatorname{div}(F_p) = a_z\sqrt{1+|\nabla\zeta|^2} + b_z - \operatorname{div}\frac{a\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} \\ &= \frac{a_z(1+|\nabla\zeta|^2)}{\sqrt{1+|\nabla\zeta|^2}} + b_z - a\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} - \sum_{i=1}^n \frac{(a_{x_i} + a_z\zeta_{x_i})\zeta_{x_i}}{\sqrt{1+|\nabla\zeta|^2}} \\ &= \frac{a_z - a_{x_1}\zeta_{x_1} - \cdots - a_{x_n}\zeta_{x_n}}{\sqrt{1+|\nabla\zeta|^2}} + b_z - a\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} \\ &= \nabla a \cdot N + b_z - a\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} \quad \text{in } \Omega \end{aligned}$$

with the normal $N(x)$ of the graph. Using the assumption $a > 0$, after some regrouping we obtain

$$\operatorname{div} \frac{\nabla \zeta}{\sqrt{1+|\nabla \zeta|^2}} = \frac{1}{a}(\nabla a \cdot N + b_z) \quad \text{in } \Omega \,,$$

which is the Euler equation in the desired form. □

If we now define

$$H = H(x,z,N) := \frac{\nabla a(x,z) \cdot N + b_z(x,z)}{n\, a(x,z)} \tag{12}$$

then by Lemma 1 we see that any critical point the functional A is a solution of

$$\operatorname{div} \frac{\nabla \zeta}{\sqrt{1+|\nabla \zeta|^2}} = nH(x,\zeta,N) \quad \text{in } \Omega \,, \tag{13}$$

i.e. ζ is a graph of prescribed mean curvature H. We now want to apply Theorem 2, but we first have to check when this special function H satisfies assumption (A) needed of Theorem 2. We therefore write

$$H = \frac{\nabla a \cdot N + b_z}{n\, a} = \Big(\sum_{i=1}^{n} \frac{a_{x_i} N_i}{n\, a} + \frac{b_z}{n\, a} \Big) + \frac{a_z}{n\, a} N_{n+1} =: H_1(x,z,N) + H_2(x,z,N)\, N_{n+1} \,.$$

Now, since assumption (A) is satisfied if $(H_1)_z \geq 0$, we calculate

$$n(H_1)_z = \sum_{i=1}^{n} \Big(\frac{a_{x_i}}{a}\Big)_z N_i + \frac{b_{zz} a - b_z a_z}{a^2} \geq 0 \,.$$

As this inequality must hold for all $N_i \in (-1,1)$ and $i = 1, \dots, n$, we assume both $b_{zz}a - b_z a_z \geq 0$ as well as

$$0 = \Big(\frac{a_{x_i}}{a}\Big)_z = (\log a)_{z x_i} = 0 \quad \text{for all } i = 1, \dots, n$$

which is equivalent to the product representation $a(x,z) = a_1(x) a_2(z)$ with certain functions $a_1 : \mathbb{R}^n \to (0,+\infty)$ and $a_2 : \mathbb{R} \to (0,+\infty)$. We can now prove

Theorem 3*:*
Assumptions:

a) Let $a, b \in C^{2+\alpha}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfy $a(x,z) = a_1(x) a_2(z)$ with functions $a_1 : \overline{\Omega} \to (0,+\infty)$, $a_2 : \mathbb{R} \to (0,+\infty)$ and

$$b_{zz} a - b_z a_z \geq 0 \quad \text{in } \Omega \times \mathbb{R} \,.$$

b) For some constant $d \geq 0$ assume that

$$\frac{|\nabla a| + |b_z|}{a} \leq d \qquad \text{in } \Omega \times \mathbb{R} \,.$$

c) Let a $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^n$ be given satisfying an enclosing sphere condition of radius $\frac{n-1}{d}$.

Then for any Dirichlet boundary values $g \in C^{2+\alpha}(\partial\Omega, \mathbb{R})$ there exists at least one solution $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ of the Euler equation (13) of the functional A.
In the case $a = a(x)$, i.e. $a_2(z) \equiv 1$, that solution ζ is unique and minimizes the functional A within the class of $C^1(\overline{\Omega}, \mathbb{R})$-functions with boundary values g.

Proof:
We set

$$H(x,z,N) := \frac{1}{na(x,z)}\Big(\nabla a(x,z)\cdot N + b_z(x,z)\Big)\ .$$

We have already shown that this H satisfies assumption (A) needed for Theorem 2. Now note that

$$|H| \le \frac{|\nabla a| + |b_z|}{na} \le \frac{d}{n}\ .$$

Since by assumption Ω satisfies an enclosing sphere condition of redius $\frac{n-1}{d}$, Theorem 2 yields a solution of (13) having prescribed boundary values $g \in C^{2+\alpha}(\partial\Omega,\mathbb{R})$.

We now consider the case $a = a(x)$. Assumption a) of this theorem then gives $b_{zz} \ge 0$. By Lemma 1 the Euler equation reads

$$\operatorname{div}\frac{a\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = b_z(x,\zeta) \quad \text{in } \Omega\ .$$

For any function $\eta \in C^1(\overline{\Omega},\mathbb{R})$ with $\eta = g$ on $\partial\Omega$ we will now show $A(\zeta) \le A(\eta)$ and use arguments similar to those given in [7, chapter 13] for the standard nonparametric area functional. By the divergence theorem we first have

$$\begin{aligned} 0 &= \int\limits_\Omega \operatorname{div}\frac{a(\eta-\zeta)\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}}dx = \int\limits_\Omega \Big((\eta-\zeta)\operatorname{div}\frac{a\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} + \frac{a\nabla\eta\cdot\nabla\zeta - a|\nabla\zeta|^2}{\sqrt{1+|\nabla\zeta|^2}}\Big)dx \\ &= \int\limits_\Omega \Big((\eta-\zeta)b_z + \frac{a\nabla\eta\cdot\nabla\zeta - a|\nabla\zeta|^2}{\sqrt{1+|\nabla\zeta|^2}}\Big)dx\ . \end{aligned}$$

We use this to obtain

$$\begin{aligned} A(\zeta) &= \int\limits_\Omega \Big(a(x)\sqrt{1+|\nabla\zeta|^2} + b(x,\zeta)\Big)dx \\ &= \int\limits_\Omega \Big(a(x)\frac{1+|\nabla\zeta|^2}{\sqrt{1+|\nabla\zeta|^2}} + b(x,\zeta)\Big)dx \\ &= \int\limits_\Omega \Big(a(x)\frac{1+\nabla\zeta\cdot\nabla\eta}{\sqrt{1+|\nabla\zeta|^2}} + (\eta-\zeta)b_z(x,\zeta) + b(x,\zeta)\Big)dx \\ &\le \int\limits_\Omega \Big(a(x)\sqrt{1+|\nabla\eta|^2} + (\eta-\zeta)b_z(x,\zeta) + b(x,\zeta)\Big)dx \\ &\le \int\limits_\Omega \Big(a(x)\sqrt{1+|\nabla\eta|^2} + b(x,\eta)\Big)dx = A(\eta)\ . \end{aligned}$$

In the last step we have used the assumption $b_{zz} \ge 0$. Thus we have shown $A(\zeta) \le A(\eta)$ and equality can only hold if $\nabla\zeta \equiv \nabla\eta$ which by the same boundary values of g is only possible for $\zeta \equiv \eta$. This also shows the uniqueness of the solution ζ. □

Remarks:

1.) In the paper [15] of Tausch a minimizer for functionals including

$$A(\zeta) = \int_\Omega a(x,\zeta)\sqrt{1+|\nabla\zeta|^2}dx$$

in the class $C^2(\Omega)\cap C^{0,1}(\overline{\Omega})$ was constructed using nonparametric variational methods where the product splitting $a(x,z) = a_1(x)a_2(z)$ is needed too. This functional corresponds to our functional for the case $b \equiv 0$. However, the existence result proven there is not applicable in the case $b \neq 0$.

2.) Uniqueness of solutions still holds under the assumption $H_z \geq 0$ on the prescribed mean curvature function H of (12). This is, under the assumptions of Theorem 3, equivalent to $(\log a)_{zz} \geq 0$, i.e. the function $\log a$ is convex as a function of the z-variable. Under this assumption one can also show that the second variation of the functional A is positive (see the author's dissertation thesis [1]).

3.) However, if one does not assume the function $\log a$ to be convex in the z-variable, the solution may not be unique anymore. Also, critical points of the functional A obtained by Theorem 3 may not be minimizers anymore.

4.) For $n = 2$ the Plateau problem for the parametric version of our functional

$$A(X) := \int_B a(X)|X_u \wedge X_v|dudv$$

was treated in [2] (see also [4]). For more general parametric functionals of the form

$$A(X) := \int_B F(X, X_u \wedge X_v)dudv$$

a projectability theorem can be found in [3], which says that under certain assumptions any stable parametric solution X of the Plauteau problem must be a graph over the x,y-plane.

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