IS MORTALITY DEAD? STOCHASTIC FORWARD FORCE OF MORTALITY RATE DETERMINED BY NO ARBITRAGE

KRISTIAN R. MILTERSEN AND SVEIN-ARNE PERSSON

Abstract. Our idea is to introduce the concept of *forward force of mortality* without any limitations on the dependence between the stochastic behavior of the forward force of mortality rate and the stochastic behavior of the forward rate. Heath, Jarrow, and Morton (1992) derive the behavior of future term structures of interest rates by no-arbitrage conditions. We use the same approach to develop the stochastic behavior of future *term structures* of forces of mortality rates. Relying on the insight from the Heath-Jarrow-Morton model, we back out the market price of mortality risk by assuming that we can observe the initial term structure of forward forces of mortality rates. This approach can be used to price force of mortality derivatives without having to make any ad hoc assumptions about the market price of mortality risk.

1. Introduction

This study deals with issues related to brevity and longevity risk. Our approach is fundamentally different from the existing literature. Our main idea is to use price data from the market to back out the *market's* best view of the force of mortality rate today and, more importantly, the *market's* best view of *the stochastic behavior* of the future force of mortality rate, that is, the brevity and longevity risk. That is, we will denote $\mu(t, T; x)$ as the market's best view at date t of the force of mortality rate at date T of a person who is x years old at date T. Since the future force of mortality rate is stochastic this force of mortality rate will, in general, not be equal to the observed *spot* force of mortality rate, $\mu(T; x)$, which is not revealed until date T. With our approach, we do not have to apply sophisticated statistical models to estimate quantities such as expected remaining life time or various death- or survival probabilities. Instead we assume that some basic mortality/survival products exist and are traded in a competitive market with no possibilities for arbitrage opportunities. This assumption may not be as far-fetched as it seems because mortality and survival risk are definitely both priced and analyzed by the insurance/pension industry, although one may say that these market prices are not as transparent as market prices we observe at certain financial exchanges—at least not yet.

Our approach is inspired by and closely related to the seminal model in financial economics by Heath, Jarrow, and Morton (1992), which is used for pricing interest rate derivatives. They model the stochastic behavior of the future term structure of interest rates so that it is consistent with currently observed bond prices. Their approach restrict the stochastic behavior of the future term structure of interest rates by applying no-arbitrage conditions on the corresponding future bond prices. In this way the market price of interest rate risk is endogenously determined in the model. The market price of interest rate risk is, so to speak, determined by the initially observed yield curve. This is made explicit in the embedding of the models by, e.g., Vasiček (1977) and Cox, Ingersoll, and Ross (1985) into the Heath-Jarrow-Morton model. See, e.g., Hull and White (1993) and Miltersen and Persson (1999). Extensions similar to what we have in

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mind for stochastic term structures of forces of mortality rates was done for stochastic term structures of convenience yields in Miltersen and Schwartz (1998) and in Miltersen (2003). The Heath-Jarrow-Morton model is used to price all kinds of interest rate derivatives including minimum rate of return guarantees written on bond and short rate products. The only information needed for this pricing is the initial yield curve and the volatility of the forward rate process.

Cairns, Blake, and Dowd (2004) add uncertainty about the future force of mortality rate to the Heath-Jarrow-Morton model. They introduce the concept of *forward forces of mortality rates* as a similar concept as *forward interest rates*. However, their definition of the forward force of mortality rate and their model for its future stochastic behavior is limited to the case where the uncertainty about the future development of the force of mortality rate is independent of the uncertainty about the future development of the interest rate. We agree that the death of individual policy holders for all practical matters can be considered independent of the development of the financial market and the general macroeconomic development of a society. But we would argue that the force of mortality rate in the future, i.e, the average number of death per unit of time is to a large extent dependent on the development of the economy in general[1] and thereby also the development of the financial market, and the future interest rates in particular. If there is such a dependence between changes in the force of mortality rate over time and the development of the financial market, the market price of mortality risk could very well be different from zero. This fact implies that the pricing of all types of life-insurance products—in particular derivative products related to changes in mortality—rely on a good estimate of this market price of mortality risk. The main point of the Heath-Jarrow-Morton model is to use market data to back out the market price of interest rate risk. Given the crucial independence assumption in Cairns, Blake, and Dowd (2004), it is not possible with their approach to backup the market price of mortality risk. Their approach, therefore, excludes some of the interesting practical applications of such a theory. Put differently, Cairns, Blake, and Dowd (2004) assume *a priori* that mortality risk has a beta of zero in a CAPM way of thinking.

Our idea is to introduce the concept of *forward force of mortality rate* without any limitations on the dependence between the stochastic behavior of the forward force of mortality rate and the stochastic behavior of the forward rate. Similar to what the Heath-Jarrow-Morton approach derives for the behavior of future term structures of interest rates, we derive no-arbitrage conditions on the stochastic behavior of future *term structures* of forces of mortality rates. Relying on the insight from the Heath-Jarrow-Morton model, we back out the market price of mortality risk by assuming that we can observe the initial term structure of forward forces of mortality rates.

To be more concrete, we take as given an observed *term structure* of *pure endowment* contracts from the market[2] to back out today's forward force of mortality rate. By explicitly modeling the volatility of the dynamics of the force of mortality rate, we derive no-arbitrage restrictions on the drift of the dynamics of the force of mortality rate. These no-arbitrage conditions comes from an assumption of that there cannot exist arbitrage opportunities in the market for pure endowment contracts. This derived dynamics of the term structure of the forces of mortality rates may again be utilized to price other products such as more complex life insurance/pension contracts and/or derivatives on mortality. The mortality indexed bond issued by Swiss Re in 2003 is one example of a real world product that could be priced by our

[1]Demographic studies by Jan M. Hoem at the Max Planck Institute for Demographic Research give examples of this dependence. For example he demonstrates (i) a clear dependence between the political events in the last 40 years in Russia and the expected life time of the Russian population and (ii) a convergence of the one year death probability of the Eastern German population toward the one year death probability of the Western German population right after the unification of Germany.

[2]In principle, we would need prices of the pure endowment insurances for all maturity dates and all ages.

approach. As more innovative products based on mortality or survival factors become available in real world markets we believe that our proposed methodology would constitute a valuable valuation tool.

2. Defining the Forward Force of Mortality Rate

Let $\{r_s\}_{s\in[0,\infty)}$ be a stochastic instantaneous interest rate process fulfilling standard regularity conditions. Standard no-arbitrage arguments yield

$$P(t,T) = E_t^{\mathsf{Q}}[e^{-\int_t^T r_s\,ds}],$$

where E_t^{Q} denotes the conditional expectation given all information up to (and including) date t under a pricing measure, Q, and $P(t,T)$ denotes the price at date t of a claim that pays \$1 for sure at date T. We have implicitly assumed that $t \leq T$. We define the instantaneous continuously compounded forward rate, $f(t,T)$, as the date-t observable function fulfilling

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}. \tag{1}$$

So $f(t,T)$ is the forward rate for date T seen from date t. A few lines of algebra give that

$$f(t,T) = \frac{-P_2(t,T)}{P(t,T)},$$

where $P_2(t,T) = \frac{\partial}{\partial T}P(t,T)$. Particularly, $f(s,s) = r_s$.

Let $\{\mu(t;x)\}_{t\in[0,\infty),x\in[0,\infty)}$ be a stochastic force of mortality rate process. That is, $\mu(t;x)$ is the instantaneous force of mortality rate at date t for a person at age x at date t.

Define the σ-algebra $\mathcal{G}_t$ as

$$\mathcal{G}_t = \sigma\{r_s, \mu(s;x) | s \in [0,t], x \in [0,\infty)\}$$

and the corresponding filtration (fulfilling the usual conditions), $\mathbb{G}$, as $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,\infty)}$.

Define $\tau(t;x)$ as

$$\tau(t;x) = \inf\left\{T \in [t,\infty) \,\middle|\, \int_t^T \mu(s;x+s-t)\,ds \geq E_1\right\},$$

where E_1 is exponentially distributed with unit intensity and independent of $\mathbb{G}$. Then the stochastic variable $\tau(t;x)$ has the same distribution as the remaining lifetime of a person of age x at date t, who's (stochastic) force of mortality rate follows $\{\mu(s;x+s-t)\}_{s\in[t,\infty)}$, but where the actual event of death is independent of the mortality distribution.

Define the σ-algebra $\mathcal{H}_t$ as

$$\mathcal{H}_t = \sigma\{1_{\{\tau(s;x)\leq u\}} | s \in [0,t], u \in [s,t], x \in [0,\infty)\}$$

and the corresponding filtration (also fulfilling the usual conditions), $\mathbb{H}$, as $\mathbb{H} = \{\mathcal{H}_t\}_{t\in[0,\infty)}$. For completeness we will also define the σ-algebra $\mathcal{F}_t$ as

$$\mathcal{F}_t = \sigma\{r_s, \mu(s;x), 1_{\{\tau(s;x)\leq u\}} | s \in [0,t], u \in [s,t], x \in [0,\infty)\} = \mathcal{G}_t \vee \mathcal{H}_t$$

and the corresponding filtration (also fulfilling the usual conditions), $\mathbb{F}$, as $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,\infty)}$. When we say that we condition on all information up to (and including) date t, we, of course, mean that we condition with the σ-algebra $\mathcal{F}_t$, i.e., $E_t[\cdot] = E[\cdot|\mathcal{F}_t]$.

The price, at date t, of a pure endowment insurance for a person of age x, promising the holder \$1 at date T, if and only if she or he is still alive at that date, can now be calculated as

$$Pe(t,T;x) = E_t^{\mathsf{Q}}[e^{-\int_t^T r_s\,ds}\,1_{\{\tau(t;x)>T\}}].$$

If we condition the expectation on the right hand side with $\mathcal{G}_T$, we get

$$\begin{aligned} E^{\mathsf{Q}}[e^{-\int_t^T r_s\,ds}\,1_{\{\tau(t;x)>T\}}|\mathcal{G}_T\vee\mathcal{F}_t] &= e^{-\int_t^T r_s\,ds}\,E^{\mathsf{Q}}[1_{\{\tau(t;x)>T\}}|\mathcal{G}_T\vee\mathcal{F}_t] \\ &= e^{-\int_t^T r_s\,ds}\,\mathsf{Q}(\tau(t;x)>T|\mathcal{G}_T\vee\mathcal{F}_t) \\ &= e^{-\int_t^T r_s\,ds}\,e^{-\int_t^T \mu(s;x+s-t)\,ds} \\ &= e^{-\int_t^T (r_s+\mu(s;x+s-t))\,ds}. \end{aligned}$$

That $\mathsf{Q}[\tau(t;x)>T|\mathcal{G}_T\vee\mathcal{F}_t] = e^{-\int_t^T \mu(s;x+s-t)\,ds}$ is per definition of the stopping time $\tau(t;x)$. Hence by iterating expectations, we have

$$\begin{aligned} Pe(t,T;x) &= E_t^{\mathsf{Q}}[e^{-\int_t^T r_s\,ds}\,1_{\{\tau(t;x)>T\}}] \\ &= E^{\mathsf{Q}}[E^{\mathsf{Q}}[e^{-\int_t^T r_s\,ds}\,1_{\{\tau(t;x)>T\}}|\mathcal{G}_T\vee\mathcal{F}_t]|\mathcal{F}_t] \\ &= E^{\mathsf{Q}}[e^{-\int_t^T (r_s+\mu(s;x+s-t))\,ds}|\mathcal{F}_t] \\ &= E_t^{\mathsf{Q}}[e^{-\int_t^T (r_s+\mu(s;x+s-t))\,ds}]. \end{aligned} \tag{2}$$

A similar derivation shows up in Lando (1994), who considers credit risk models.

Inspired by the term structure of interest rate models, equation (1) in particular, we define the forward force of mortality rate, $\mu(t,T;x)$, as the date-t observable function fulfilling

$$Pe(t,T;x) = e^{-\int_t^T (f(t,s)+\mu(t,s;x+s-t))\,ds}. \tag{3}$$

We call $\mu(t,T;x)$ the forward force of mortality rate for date T for a person of age x at date T (i.e. the person was of age $x+t-s$ at date t) seen from date t. Again, a few lines of algebra give that

$$\mu(t,T;x) = \frac{-Pe_2(t,T;x+t-T)}{Pe(t,T;x+t-T)} - f(t,T), \tag{4}$$

where $Pe_2(t,T;x) = \frac{\partial}{\partial T}Pe(t,T;x)$. The $x+t-T$-term in the third parameter of Pe and Pe_2 on the right hand side in equation (4) is simply related to the fact that the age x in the forward mortality rate function, $\mu(t,T;x)$, is related to the realization date T, whereas in the pure endowment price, $Pe(t,T;x)$, the age x is related to the pricing date t. Note that by differentiating, in equation (2) and (3), with respect to T, dividing by $Pe(t,T;x)$ and taking the limit $T\downarrow t$ gives the connection between the forward force of mortality rate and the (spot) force of mortality rate

$$\mu(t,t;x) = \mu(t;x).$$

Note also that if $\{\mu(t;x)\}_{t\in[0,\infty),x\in[0,\infty)}$ is deterministic—it may still be a function of both t and x, but the whole function is known at date zero—then

$$\mu(t,T;x) = \mu(T;x)$$

for all T an x and for any $t\le T$. This is so, because in that case equation (2) simplifies to

$$Pe(t,T;x) = e^{-\int_t^T \mu(s;x+s-t)\,ds}E_t^{\mathsf{Q}}[e^{-\int_t^T r_s\,ds}] = e^{-\int_t^T \mu(s;x+s-t)\,ds}P(t,T).$$

Hence, using equation (1),

$$Pe(t,T;x) = e^{-\int_t^T (f(t,s)+\mu(s;x+s-t))\,ds}. \tag{5}$$

Comparing equation (5) with equation (3) we see that $\mu(t,T;x)$ must be equal to $\mu(T;x)$.

Under an explicit assumption that the stochastic force of mortality rate is independent of the stochastic forward rate under any pricing measure, Q, Cairns, Blake, and Dowd (2004) also define a forward force

of mortality rate, which they term $\bar{\mu}(t,T,x)$, as

$$\bar{\mu}(t,T,x) = \frac{-B_2(t,T,x-T)}{B(t,T,x-T)}, \tag{6}$$

where $B_2(t,T,x) = \frac{\partial}{\partial T} B(t,T,x)$. Cairns, Blake, and Dowd (2004) have defined $B(t,T,x)$ as

$$B(t,T,x) = E_t^{\mathsf{Q}}[e^{-\int_0^T \mu(s;x+s)\,ds}]$$

translated into our notation. That is, the age x in their $B(t,T,x)$ is related to the age at date zero. (This is why we have $x-T$ in the third parameter of B and B_2 on the right hand side in equation (6).) Given their independence assumption, equation (2) reduces to

$$Pe(t,T;x) = E_t^{\mathsf{Q}}[e^{-\int_t^T r_s\,ds}]\, E_t^{\mathsf{Q}}[e^{-\int_t^T \mu(s;x+s-t)\,ds}] = P(t,T)\,\frac{B(t,T,x-t)}{B(t,t,x-t)},$$

so it is clear that the two forward force of mortality rate definitions are identical when there is independence between the stochastic force of mortality rate and the stochastic forward rate. However, equation (3) would *not* hold, in general, with their definition, $\bar{\mu}(t,T,x)$, of the forward force of mortality rate from equation (6). If the independence assumption does not hold, one would need some correlation adjustments.

If we apply Jensen's inequality on the convex function $x \mapsto e^{-x}$ in equation (2), we have

$$Pe(t,T;x) \geq e^{-\int_t^T (E_t^{\mathsf{Q}}[r_s] + E_t^{\mathsf{Q}}[\mu(s;x+s-t)])\,ds},$$

for all t, T, and x with $t \leq T$. Hence, if, in the pricing of pure endowment insurance from equation (3), one would rely on using $f(t,T) = E_t^{\mathsf{Q}}[r_T]$ and $\mu(t,T;x) = E_t^{\mathsf{Q}}[\mu(T;x)]$, one would get a price which would be too low. A different way of interpreting this result is,

$$f(t,T) + \mu(t,T;x) \leq E_t^{\mathsf{Q}}[r_T] + E_t^{\mathsf{Q}}[\mu(T;x)],$$

for all t, T, and x with $t \leq T$. That is, since $x \mapsto e^{-x}$ is a decreasing function, we would have to set $f(t,T) + \mu(t,T;x)$ less than $E_t^{\mathsf{Q}}[r_T] + E_t^{\mathsf{Q}}[\mu(T;x)]$ in order to get $Pe(t,T;x)$ to be greater than $e^{-\int_t^T (E_t^{\mathsf{Q}}[r_s] + E_t^{\mathsf{Q}}[\mu(s;x+s-t)])\,ds}$. We know also from Jensen's inequality that $f(t,T) \leq E_t^{\mathsf{Q}}[r_T]$, for all t and T with $t \leq T$. Hence, in the general case, we are not able to say anything about the mortality term in itself. That is, we cannot, in general, say, whether $\mu(t,T;x)$ is greater than, less than, or equal to $E_t^{\mathsf{Q}}[\mu(T;x)]$. If we assume independence between the stochastic force of mortality rate and the stochastic forward rate, we would have

$$\frac{Pe(t,T;x)}{P(t,T)} = E_t^{\mathsf{Q}}[e^{-\int_t^T \mu(s;x+s-t)\,ds}] \geq e^{-\int_t^T E_t^{\mathsf{Q}}[\mu(s;x+s-t)]\,ds}.$$

Hence, under the independence assumption, we would be able to conclude that

$$\mu(t,T;x) \leq E_t^{\mathsf{Q}}[\mu(T;x)],$$

for all t, T, and x with $t \leq T$. This result would normally carry over to the physical measure, since the independence assumption normally would imply that the market price of mortality risk is zero.

This analysis is very similar to the analysis about expectations hypotheses in interest rate models, cf. Cox, Ingersoll, and Ross (1981).

Alternatively, we can look at term insurance. The price at date t of a term insurance for a person of age x, promising the (heirs of the) holder \$1 at the date of death, if and only if death occur before (or at) date T, can now be derived as

$$Ti(t,T;x) = E_t^{\mathsf{Q}}[1_{\{\tau(t;x)\leq T\}}\, e^{-\int_t^{\tau(t;x)} r_s\,ds}].$$

If we again condition the expectation on the right hand side with $\mathcal{G}_T$, we get

$$E^{\mathsf{Q}}[1_{\{\tau(t;x)\leq T\}}\, e^{-\int_t^{\tau(t;x)} r_s\, ds}|\mathcal{G}_T \vee \mathcal{F}_t] = E^{\mathsf{Q}}[\int_t^T e^{-\int_t^u r_s\, ds}\, e^{-\int_t^u \mu(s;x+s-t)\, ds}\, \mu(u;x+u-t)\, du|\mathcal{G}_T \vee \mathcal{F}_t]$$
$$= \int_t^T e^{-\int_t^u (r_s+\mu(s;x+s-t))\, ds}\, \mu(u;x+u-t)\, du.$$

The first equality sign comes from that the conditional density (under the pricing measure Q) of the stopping time, $\tau(t;x)$ is $f^{\mathsf{Q}}_{\tau(t;x)}(u|\mathcal{G}_T \vee \mathcal{F}_t) = e^{-\int_t^u \mu(s;x+s-t)\, ds}\, \mu(u;x+u-t)$. Again, by iterating expectations, we have

$$\begin{aligned} Ti(t,T;x) &= E^{\mathsf{Q}}_t[1_{\{\tau(t;x)\leq T\}}\, e^{-\int_t^{\tau(t;x)} r_s\, ds}] \\ &= E^{\mathsf{Q}}[E^{\mathsf{Q}}[1_{\{\tau(t;x)\leq T\}}\, e^{-\int_t^{\tau(t;x)} r_s\, ds}|\mathcal{G}_T \vee \mathcal{F}_t]|\mathcal{F}_t] \\ &= E^{\mathsf{Q}}[\int_t^T e^{-\int_t^u (r_s+\mu(s;x+s-t))\, ds}\, \mu(u;x+u-t)\, du|\mathcal{F}_t] \\ &= E^{\mathsf{Q}}_t[\int_t^T e^{-\int_t^u (r_s+\mu(s;x+s-t))\, ds}\, \mu(u;x+u-t)\, du]. \end{aligned} \tag{7}$$

Hence, we can also define a forward force of mortality rae, $\tilde{\mu}(t,T;x)$, based on term insurance as the date-t observable function fulfilling

$$Ti(t,T;x) = \int_t^T e^{-\int_t^u (f(t,s)+\tilde{\mu}(t,s;x+s-t))\, ds}\, \tilde{\mu}(t,u;x+u-t)\, du.$$

We are not sure whether the two definitions of the forward force of mortality rate are identical, in general. Actually, we think they are not identical based on the following (very intuitive) argument: Since $x \mapsto xe^{-x}$ is a concave function for $x \leq 2$ (which we would claim is the relevant values), we would argue that it is most likely that

$$Ti(t,T;x) \leq \int_t^T e^{-\int_t^u (E^{\mathsf{Q}}_t[r_s]+E^{\mathsf{Q}}_t[\mu(s;x+s-t)])\, ds}\, E^{\mathsf{Q}}_t[\mu(u;x+u-t)]\, du, .$$

for all t, T, and x with $t \leq T$. If we again assume independence between the stochastic force of mortality rate and the stochastic forward rate, we would moreover argue that

$$\tilde{\mu}(t,T;x) \geq E^{\mathsf{Q}}_t[\mu(T;x)],$$

for all t, T, and x with $t \leq T$, and most likely we would also have this result under the physical measure. This argument is based on, that the function $x \mapsto xe^{-x}$ is decreasing for $x \geq 1$ (which we would, again, claim is the relevant values). Based on these intuitive arguments, we are quite sure that, in general, our two definitions of the forward force of mortality rate, $\tilde{\mu}(t,T;x)$ and $\mu(t,T;x)$, would be different, since we have argued that one of them is typically greater than the conditional expected future force of mortality rate, whereas the other one is smaller. We admit that these arguments are far from rigorous, we would have to do a comprehensive simulation study in order to get some more confidence in these results. Surely, if $\{\mu(t;x)\}_{t\in[0,\infty),x\in[0,\infty)}$ is deterministic, we would have

$$Ti(t,T;x) = \int_t^T e^{-\int_t^u (f(t,s)+\mu(s;x+s-t))\, ds}\, \mu(u;x+u-t)\, du$$

and therefore

$$\tilde{\mu}(t,T;x) = \mu(T;x),$$

for all T an x and for any $t \leq T$, which would of course also give us that $\tilde{\mu}(t,T;x) = \mu(t,T;x)$, for all t, T, and x with $t \leq T$, so in the case of deterministic mortality rate, our two definitions of the forward force of mortality rate are identical.

3. Stochastic Forward Force of Mortality Rate

As in Heath, Jarrow, and Morton (1992), we specify the dynamics of the stochastic forward rate as

$$df(\cdot,s)_t = \nu_f(t,s)\,dt + \sigma_f(t,s)\,dW_t,$$

where W is a (possible multidimensional) Wiener process under a pricing measure Q. The usual Heath-Jarrow-Morton drift restriction (Heath, Jarrow, and Morton 1992), which gives that zero-coupon bond prices in the savings account numeraire are martingales, is given by

$$\nu_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,s)\,ds. \tag{8}$$

Similarly, we specify the dynamics of the stochastic forward force of mortality rate as

$$d\mu(\cdot,s;x)_t = \nu_\mu(t,s;x)\,dt + \sigma_\mu(t,s;x)\,dW_t.$$

We will use the same (possible multidimensional) Wiener process as used to describe the dynamics of the stochastic forward rate. Hence, possible co-movements of the two processes are captured in the volatility terms. Define the stochastic variable

$$X(t,T;x,t_0) = \int_t^T \big(f(t,s) + \mu(t,s;x+s-t_0)\big)\,ds.$$

Then the date-t price of a pure endowment insurance for a person of age x at a given reference date t_0 is

$$Pe(t,T,x+t-t_0) = e^{-X(t,T;x,t_0)}.$$

We are interested in the dynamics of the stochastic process, $\{Pe(t,T,x+t-t_0)\}_{t\in[0,\infty)}$, since this is the price process of a marketed security. By an extension of Itô's lemma, we have

$$\begin{aligned} dX(\cdot,T;x,t_0)_t = &\left(\int_t^T \big(\nu_f(t,s) + \nu_\mu(t,s;x+s-t_0)\big)\,ds - f(t,t) - \mu(t,t;x+t-t_0)\right)dt \\ &+ \left(\int_t^T \big(\sigma_f(t,s) + \sigma_\mu(t,s;x+s-t_0)\big)\,ds\right)dW_t. \end{aligned}$$

Again by Itô's lemma, we get the dynamics of $\{Pe(t,T,x+t-t_0)\}_{t\in[0,\infty)}$ as

$$\begin{aligned} dPe(\cdot,T;x+\cdot-t_0)_t = Pe(t,T;x+t-t_0)\Bigg(\Bigg(&f(t,t) + \mu(t,t;x+t-t_0) \\ &- \int_t^T \big(\nu_f(t,s) + \nu_\mu(t,s;x+s-t_0)\big)\,ds \\ &+ \frac{1}{2}\left(\int_t^T \big(\sigma_f(t,s) + \sigma_\mu(t,s;x+s-t_0)\big)\,ds\right)^2\Bigg)\,dt \\ &- \left(\int_t^T \big(\sigma_f(t,s) + \sigma_\mu(t,s;x+s-t_0)\big)\,ds\right)dW_t\Bigg). \end{aligned}$$

The date-t price of the pure endowment insurance in the savings account numeraire is

$$Pe^*(t,T,x+t-t_0) = e^{-\int_0^t f(s,s)\,ds}\,Pe(t,T,x+t-t_0).$$

Hence,

$$\begin{aligned} dPe^*(\cdot,T;x+\cdot-t_0)_t = Pe^*(t,T;x+t-t_0)\Bigg(\Bigg(&\mu(t,t;x+t-t_0) \\ &- \int_t^T \big(\nu_f(t,s)+\nu_\mu(t,s;x+s-t_0)\big)\,ds \\ &+ \frac{1}{2}\bigg(\int_t^T \big(\sigma_f(t,s)+\sigma_\mu(t,s;x+s-t_0)\big)\,ds\bigg)^2\Bigg)\,dt \\ &- \bigg(\int_t^T \big(\sigma_f(t,s)+\sigma_\mu(t,s;x+s-t_0)\big)\,ds\bigg)\,dW_t\Bigg). \end{aligned}$$

The pure endowment insurance is not a true non-dividend paying security. Consider a person, A, of age x at date t_0. A financial entity (who is not itself exposed to mortality risk) who invests in the pure endowment insurance (written on the person A) pays a lower price for the police at date t than at date $t+dt$ not only because of the interest rate effect, but also because of the risk that person A may die between date t and date $t+dt$. Said differently, the chance that person A survives to date T is marginally higher (actually, $\mu(t,t;x+t-t_0)\,dt$ higher) at date $t+dt$ than it was at date t. Hence, the pure endowment insurance, which only gives a payoff, if the person A survives up to date T, is more expensive at date $t+dt$ than it was at date t, since at date $t+dt$ it is now clear that the person A did not die between date t and date $t+dt$. This corresponds to that the security $Pe(\cdot,T;x+\cdot-t_0)$ has an implicit negative dividend. Hence, the gains process of the pure endowment insurance in the savings account numeraire is

$$GPe^*(t,T,x+t-t_0) = Pe^*(t,T,x+t-t_0) - \int_0^t Pe^*(s,T,x+s-t_0)\mu(s,s;x+s-t_0)\,ds.$$

Here we have abstracted from the fact that person A may have a negative age for s close to zero. By no arbitrage, $GPe^*(\cdot,T,x+\cdot-t_0)$ should be a martingale. By Itô's lemma

$$\begin{aligned} dGPe^*(\cdot,T;x+\cdot-t_0)_t = -Pe^*(t,T;x+t-t_0)\Bigg(\Bigg(&\int_t^T \big(\nu_f(t,s)+\nu_\mu(t,s;x+s-t_0)\big)\,ds \\ &- \frac{1}{2}\bigg(\int_t^T \big(\sigma_f(t,s)+\sigma_\mu(t,s;x+s-t_0)\big)\,ds\bigg)^2\Bigg)\,dt \\ &+ \bigg(\int_t^T \big(\sigma_f(t,s)+\sigma_\mu(t,s;x+s-t_0)\big)\,ds\bigg)\,dW_t\Bigg). \end{aligned}$$

Hence,

$$\begin{aligned} \int_t^T \big(\nu_f(t,s)+\nu_\mu(t,s;x+s-t_0)\big)\,ds = &\frac{1}{2}\bigg(\int_t^T \sigma_f(t,s)\,ds\bigg)^2 + \frac{1}{2}\bigg(\int_t^T \sigma_\mu(t,s;x+s-t_0)\,ds\bigg)^2 \\ &+ \bigg(\int_t^T \sigma_f(t,s)\,ds\bigg)\bigg(\int_t^T \sigma_\mu(t,s;x+s-t_0)\,ds\bigg). \end{aligned}$$

Taking into account the Heath-Jarrow-Morton drift restriction, cf. equation (8), we get the drift restriction for the stochastic forward force of mortality rate

$$\begin{aligned} \int_t^T \nu_\mu(t,s;x+s-t_0)\,ds = &\frac{1}{2}\bigg(\int_t^T \sigma_\mu(t,s;x+s-t_0)\,ds\bigg)^2 \\ &+ \bigg(\int_t^T \sigma_f(t,s)\,ds\bigg)\bigg(\int_t^T \sigma_\mu(t,s;x+s-t_0)\,ds\bigg) \end{aligned}$$

which we can reduce to

$$\begin{aligned}\nu_\mu(t,T;x+T-t_0) &= \big(\sigma_f(t,T)+\sigma_\mu(t,T;x+T-t_0)\big)\int_t^T \sigma_\mu(t,s;x+s-t_0)\,ds \\ &\quad + \sigma_\mu(t,T;x+T-t_0)\int_t^T \sigma_f(t,s)\,ds.\end{aligned} \tag{9}$$

This drift restriction nests the drift restriction derived by Cairns, Blake, and Dowd (2004). Cairns, Blake, and Dowd (2004) derive their result under the assumption that the stochastic force of mortality rate is independent of the stochastic forward rate under any pricing measure, Q.[3] In that case the correlation terms vanish, i.e.,

$$\sigma_f(t,T)\int_t^T \sigma_\mu(t,s;x+s-t_0)\,ds = 0$$

and

$$\sigma_\mu(t,T;x+T-t_0)\int_t^T \sigma_f(t,s)\,ds = 0.$$

So under the assumptions by Cairns, Blake, and Dowd (2004), equation (9) reduces to

$$\nu_\mu(t,T;x+T-t_0) = \sigma_\mu(t,T;x+T-t_0)\int_t^T \sigma_\mu(t,s;x+s-t_0)\,ds,$$

which is of the same form as the standard Heath-Jarrow-Morton drift restriction, cf. (8).

4. Conclusion

We have introduced the concept of *forward force of mortality rate* without any limitations on the dependence between the stochastic behavior of the forward force of mortality rate and the stochastic behavior of the forward rate. We derived the forward force of mortality rate no arbitrage Heath-Jarrow-Morton drift restriction which is essential for pricing all types of life-insurance products—in particular derivative products related to changes in mortality. So far we have left the volatility of the dynamics of the forward force of mortality rate undetermined. This has to estimated either from a historical time series of the spot force of mortality rate or backed out from liquidly traded mortality derivatives. At the present, we do not believe that such derivatives markets exist, hence we would have to rely on time series data in order to estimate the volatility. In particular, our approach is well suited to price the mortality indexed bonds issued by Swiss Re in 2003.

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[3]Cairns, Blake, and Dowd (2004) base their no-arbitrage argument on the pricing of the claim that pays $B(T,T,x)$ at date T. Surely, this is a non-dividend paying tradable security. Its price process is

$$PB(t,T;x) = E_t^{\mathsf{Q}}[e^{-\int_t^T r_s\,ds}B(T,T,x)]$$

and the corresponding price process in the savings account numeraire is

$$PB^*(t,T;x) = e^{-\int_0^t f(s,s)\,ds}PB(t,T;x)$$

Hence, the process, $\{PB^*(t,T;x)\}_{t\in[0,T]}$, should be a martingale under any pricing measure Q. Under the assumption that the stochastic force of mortality rate is independent of the stochastic forward rate under any pricing measure, Q, $\{PB^*(t,T;x)\}_{t\in[0,T]}$ is a martingale if and only if $\{B(t,T,x)\}_{t\in[0,T]}$ is a martingale. The drift restriction derived by Cairns, Blake, and Dowd (2004) is based on that $\{B(t,T,x)\}_{t\in[0,T]}$ is a martingale under any pricing measure Q. Their drift restriction is not, in general, correct if the independence assumption does not hold.

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Norwegian School of Economics and Business Administration, Department of Finance and Management Science, Helleveien 30, NO–5045 Bergen, Norway

E-mail address: kristian.r.miltersen@nhh.no

Norwegian School of Economics and Business Administration, Department of Finance and Management Science, Helleveien 30, NO–5045 Bergen, Norway

E-mail address: svein-arne.persson@nhh.no