Large Portfolio Approximation in an Elliptical Distributions Framework

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Financial Modelling Workshop Ulm, September 2005
Outline

1. Introduction

2. Our Contribution
   - The basic structure and assumptions
   - Main Result
   - Example
We want to price and analyze **Credit Structures** such as CDOs.

Essentially needed: **distribution of losses** in underlying portfolio
Credit structures with large underlying portfolio (> 100 names)

**Problem:**
portfolio loss distribution
→ mathematically difficult to obtain;
pricing and sensitivity analysis via MC-Simulation
→ very time consuming

**Aim/Solution:**
factor setup
→ reduction of dimensionality;
approximation of portfolio losses
→ distribution to be obtained (semi)-analytically
Like Merton’s model [2] (one-period model \([0, T]\)):

default is triggered by fall of firm value \(S_j\) below “appropriate” threshold at time \(T\).

**Figure:** 5000 realizations of \((S_1, S_2)\) with different dependence structures

More general: we want to use **elliptical distributions.**
The central objects

Company \( j, j = 1, \ldots, n \), with asset value \( S_j \) at time \( T \):

<table>
<thead>
<tr>
<th></th>
<th>Merton</th>
<th>our setup</th>
</tr>
</thead>
<tbody>
<tr>
<td>ratings at ( t = 0 ):</td>
<td>(non-)default</td>
<td>( k_j \in {1, \ldots, r} ); ( k = 1 ) for default;</td>
</tr>
<tr>
<td>given probabilities:</td>
<td>PDs</td>
<td>( p_{kl} ): going from ( k ) at ( t = 0 ) ( \rightarrow ) ( l ) at ( t = T ); ( p_{11} = 1 );</td>
</tr>
<tr>
<td>thresholds:</td>
<td>( DT_j )</td>
<td>( c_{kl}^{(j)} \in \mathbb{R} ): corresponding to ( p_{kl} );</td>
</tr>
<tr>
<td>rating at ( t = T ):</td>
<td>( 1 { S_j &lt; DT_j } )</td>
<td>( L_j := \sum_{l=1}^{r} 1 { c_{kj,l-1}^{(j)} &lt; S_j \leq c_{kj,l}^{(j)} } \cdot l );</td>
</tr>
<tr>
<td>losses at ( t = T ):</td>
<td>( (Not. - S_j)^+ )</td>
<td>( \pi(j, k_j, l_j, \Psi) ), if from ( k_j ) to ( L_j = l_j ), macro-influence ( \Psi ).</td>
</tr>
</tbody>
</table>

→ characteristics \((S_j, k_j, L_j, \pi(j, k_j, L_j, \Psi))\).

(See e.g. also [3] for setup with several rating classes)
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Definition (Portfolio credit loss $C_n$ at $t = T$)

$$C_n := \sum_{j=1}^{n} \pi(j, k_j, L_j, \Psi) = \sum_{j=1}^{n} \sum_{l_j=1}^{r} \pi_{j,l_j} \cdot Z_{j,l_j}$$

where $Z_{j,l} := \mathbf{1}_{\{L_j = l_j\}}$ and $\pi_{j,l} := \pi(j, k_j, l, \Psi)$.

Definition (asset value vector $S^{(n)}$)

$$S^{(n)} := (S_1, \ldots, S_n)^T = \beta^{(n)} F + \varepsilon^{(n)},$$

$$\rightarrow S_j = \beta_j^T F + \varepsilon_j$$

where

$\beta^{(n)} = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^{n \times m}$, $\beta_j \in \mathbb{R}^m$ : factor-loadings ($m << n$)

$\varepsilon^{(n)} = (\varepsilon_1, \ldots, \varepsilon_n)^T$, $\varepsilon_j$ : individual behavior / idiosyncratic risk.

$\varepsilon^{(n)}$ independent of $(F, \Psi)$. 
Portfolio Credit loss and Factor structure

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where

- $\beta^{(n)} = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^{n \times m}$, $\beta_j \in \mathbb{R}^m$ : factor-loadings ($m <<< n$)
- $\varepsilon^{(n)} = (\varepsilon_1, \ldots, \varepsilon_n)^T$, $\varepsilon_j$: individual behavior / idiosyncratic risk.
- $\varepsilon^{(n)}$ independent of $(F, \Psi)$.
Assumptions (1) - An elliptical distribution framework

\[ S^{(n)} = \beta^{(n)} F + \varepsilon^{(n)} \]

Special choice of

\[ F \sim EC_m(0, \Omega_F, \phi_F) \quad \text{and} \quad \varepsilon^{(n)} \sim EC_n(0, \Sigma_n, \phi) : \]

\[
\begin{align*}
F & \overset{d}{=} R_1 \cdot W \\
R_1 & \geq 0, R_1 \sim G_1 \\
W & \sim N_m(0, \Omega_F) \\
\Omega_F & > 0 \\
\mathbb{E}(R_1^2) & = 1 \\
R_1, R, W, Y^{(n)} & \text{ ind.}
\end{align*}
\]

\[
\begin{align*}
\varepsilon^{(n)} & \overset{d}{=} R \cdot Y^{(n)} \\
R & \geq 0, R \sim G_2 \\
Y^{(n)} & \sim N_n(0, \Sigma_n) \\
\Sigma_n & = \text{diag}(\omega_1, \ldots, \omega_n) > 0 \\
\mathbb{E}(R^2) & = 1
\end{align*}
\]

\[ R_1, R, W, Y^{(n)} \text{ ind.} \]

\[ \implies S_j = \beta_j^T F + \varepsilon_j \sim F_j = H_{\beta_j^T \Omega_F \beta_j, \omega_j} \quad \text{with} \]

\[ H_{\sigma_1^2, \sigma_2^2}(x) := \int_{-\infty}^{x} \int_{0}^{\infty} \int_{0}^{\infty} n(y; 0, r^2 \sigma_1^2 + s^2 \sigma_2^2) dG_{R_1}(r) dG_{R_2}(s) dy. \]
Assumptions (2)

**Assumption 1:**

a.) \( \exists \) Borel fct. \( f_j : \mathbb{R}^{2+m+d} \rightarrow \mathbb{R} \) s.t.

\[ \pi(j, k_j, L_j, \Psi) = f_j(Y_j, R, F, \Psi); \]

b.) \( \exists \) strictly increasing \( (b_n)_{n \geq 0} \) with \( b_n \rightarrow \infty \), for \( n \rightarrow \infty \), s.t.

\[ \sum_{n=1}^{\infty} \left( \frac{\log n}{b_n} \right)^2 \mathbb{E} \left( [\pi_{n,L_n} - \mathbb{E}(\pi_{n,L_n}|R, F, \Psi)]^2 \right) < \infty. \]

**Assumption 2:**

a.) \( \pi(j, k_j, l_j, \Psi) \) measurable w.r.t. \( \sigma(R, F, \Psi) \), for all \( 1 \leq l_j \leq r \).

b.) Case 1: \( \Psi \) ind. of \( \sigma(R, F) \lor \sigma(S_j) \subseteq \sigma(R, F, Y_j) \),

or case 2: \( \sigma(R, F, \Psi) = \sigma(R, F) \)
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Approximating result

Under above factor structure and assumptions 1 and 2, $C_n \in L_1$:

**Definition (Conditional portfolio credit losses $(B_n)_{n \geq 0}$)**

\[
B_n := \mathbb{E}(C_n| R, F, \Psi) = \sum_{j=1}^{n} \sum_{l=1}^{r} \pi(j, k_j, l_j, \Psi) \cdot \hat{\Phi}_{j,l}(F, R)
\]
\[
\hat{\Phi}_{j,l}(f, r) := \Phi \left( \frac{c^{(j)}_{k_j,l} - \beta^T_j f}{\sqrt{\omega_j r}} \right) - \Phi \left( \frac{c^{(j)}_{k_j,l-1} - \beta^T_j f}{\sqrt{\omega_j r}} \right), \quad f \in \mathbb{R}^m, \ r \in \mathbb{R}.
\]

**Theorem**

\[
\frac{C_n - B_n}{b_n} \to 0, \quad \text{as } n \to \infty, \text{ almost surely.}
\]
Approximating result

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$$B_n := \mathbb{E}(C_n | R, F, \Psi) = \sum_{j=1}^{n} \sum_{l_j=1}^{r} \pi(j, k_j, l_j, \Psi) \cdot \hat{\Phi}_{j,l}(F, R)$$

with

$$\hat{\Phi}_{j,l}(f, r) := \Phi\left(\frac{c_{k_j,l}^{(j)} - \beta^T_j f}{\sqrt{\omega_j r}}\right) - \Phi\left(\frac{c_{k_j,l-1}^{(j)} - \beta^T_j f}{\sqrt{\omega_j r}}\right), f \in \mathbb{R}^m, r \in \mathbb{R}.$$

**Theorem**

$$\frac{C_n - B_n}{b_n} \rightarrow 0, \text{ as } n \rightarrow \infty, \text{ almost surely}.$$

C. Prestele (University of Ulm) Large Portfolio Approximation Workshop Ulm, Sept. 2005
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2 rating classes: 1 for default, 2 for non-default; let

- \( m = 1, \beta_j \equiv \rho, \var{Var}(F) = 1, \omega_j \equiv 1 - \rho^2 \).
- \( \pi(j, 1, l, \psi) = \begin{cases} N_j, & \text{for } l_j = 1; \\ 0, & \text{for } l_j = 2. \end{cases} \); \( N_j \): size of loan \( j \).

\[ B_n = \Phi \left( \frac{\rho F - c_{2,2}}{\sqrt{1 - \rho^2 R}} \right) \cdot \sum_{j=1}^{n} N_j \]

\[ C_n \frac{1}{b_n} - \frac{1}{b_n} \sum_{j=1}^{n} N_j \cdot \Phi \left( \frac{\rho F - c_{2,2}}{\sqrt{1 - \rho^2 R}} \right) \to 0, \quad \text{as } n \to \infty, \text{ a.s..} \]
If $\frac{1}{b_n} \sum_{j=1}^{n} N_j \xrightarrow{n \to \infty} 1$:

For portfolio loss distribution:

in the limit only distribution of $\Phi \left( \frac{\rho F - c_{2,2}}{\sqrt{1 - \rho^2 R}} \right)$ important!

This (scaled) approximated portfolio loss distribution is directly given via a two-dimensional integral → analytical evaluation:

$$x \mapsto \int_{0}^{\infty} \int_{0}^{\infty} \Phi \left( \frac{\sqrt{1 - \rho^2 r \Phi^{-1}(x) + c_{2,2}}}{\rho r_1} \right) dG_1(r_1) dG(r).$$
Factor model plus Gaussian mixture distributions yield approximation for portfolio credit losses.
Approximation gives rise to (semi-)analytical valuation and analysis of credit structures.

Outlook

- Consequences for the analysis of credit structures?
- Does this model imply a correlation smile similar to the market implied correlation smile?
For Further Reading I

C. Bluhm, L. Overbeck and C. Wagner. 
*An Introduction to Credit Risk Modeling, Financial Mathematics Series.*

R. Merton. 
On the pricing of corporate debt: the risk structure of interest rates. 

André Lucas, Pieter Klaassen, Peter Spreij, and Stefan Straetmans. 
*An Analytic Approach to Credit Risk of Large Corporate Bond and Loan Portfolios.* 
Andersen, Leif and Sidenius, Jakob. 
Extensions to the Gaussian Copula: Random Recovery and Random Factor Loadings. 

Hull, John and White, Alan. 
Valuation of a CDO and an nth to Default CDS Without Monte Carlo Simulation. 