Modelling and pricing in electricity markets

Fred Espen Benth

Work in collaboration with D. Frestad, J. Kallsen, S. Koekebakker and T. Meyer-Brandis

Centre of Mathematics for Applications (CMA)
University of Oslo, Norway

Universität Ulm, April 2007
Overview of the lectures

1. The Nordic Power Market NordPool
2. The spot and electricity forward relation
   ▶ Establishing some mathematical connections and notation
3. Modelling the spot price of electricity
   ▶ Arithmetic models
   ▶ Pricing of forwards and options
4. Market models for the forward dynamics
   ▶ Seasonality of volatility
The NordPool Market

The spot and electricity forward relation
Spot price modelling
HJM approach to forwards
Conclusions
The NordPool market organizes trade in
- Hourly spot electricity, next-day delivery
- Financial forward contracts
  - In reality mostly futures, but we make no distinction here
  - Frequently called swaps
- European options on forwards

Difference from “classical” forwards:
- Delivery over a period rather than at a fixed point in time

Crucial point in modeling
Elspot: the spot market

- A (non-mandatory) hourly market with physical delivery of electricity
- Participants hand in bids before noon *the day ahead*
  - Volume and price for each of the 24 hours next day
  - Maximum of 64 bids within technical volume and price limits
- NordPool creates demand and production curves for the next day before 1.30 pm
The system price is the equilibrium
Reference price for the forward market
Due to congestion (non-perfect transmission lines), area prices are derived
  - Sweden and Finland separate areas
  - Denmark split into two
  - Norway may be split into several areas
The area prices are the actual prices for the consumers/producers in the area in question
Historical system price from the beginning in 1992
The forward market

- Forward with delivery over a period
- Financial market
- Settlement with respect to system price in the delivery period
- Delivery periods
  - Next day, week or month
  - Quarterly (earlier seasons)
  - Yearly
- Overlapping settlement periods (!)
- Contracts also called *swaps*: Fixed for floating price
The forward curve March 25, 2004
The option market

- European call and put options on electricity forwards
  - Quarterly and yearly electricity forwards
- Low activity on the exchange
- OTC market for electricity derivatives huge
  - Average-type (Asian) options, swing options ....
### Option prices May 4, 2006

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Electricity finance: problems encountered

- Spot price is mean-reverting, with seasonality and spikes
- Electricity is non-storable
  - Incomplete market
- Forward contracts deliver over a period
  - Heath-Jarrow-Morton approach?
- Pricing of options?
The NordPool Market
The spot and electricity forward relation
Spot price modelling
HJM approach to forwards
Conclusions

The spot and electricity forward relation
The spot and electricity forward relation

- Let $S(t)$ be the spot price
  - Not necessarily a semimartingale
- Consider a forward contract delivering (financially) electricity over a period $[\tau_1, \tau_2]$
- Payoff from a long forward position entered at time $t \leq \tau_1$

$$\int_{\tau_1}^{\tau_2} S(t) \, dt - (\tau_2 - \tau_1) F(t, \tau_1, \tau_2)$$

- The forward price $F(t, \tau_1, \tau_2)$ denoted in Euro/MWh
From general theory:
- Price of any derivative is given as the present expected value with respect to a risk-neutral measure $Q$
- The spot $S(t)$ not storable
  - Any $Q \sim P$ risk-neutral
- Cost of entering the contract should be zero
- Price of a forward with constant interest rate
  - Assuming financial settlement at maturity $\tau_2$
  - Using adaptedness of $F(t, \tau_1, \tau_2)$

$$F(t, \tau_1, \tau_2) = \mathbb{E}_Q \left[ \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} S(u) \, du \mid \mathcal{F}_t \right]$$
Interchanging expectation and integration leads to

\[ F(t, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f(t, u) \, du \]

Here, \( f(t, u) \) is the price of a forward with fixed-delivery time at \( u \),

\[ f(t, u) = \mathbb{E}_Q [S(u) \mid \mathcal{F}_t] \]

The spot \( S(t) \) must satisfy some integrability conditions in \( t \) and \( \omega \) under \( Q \).
No unique forward prices

- All $Q \sim P$ are risk-neutral probabilities
- Restrict to a subclass of measures $Q$
  - Usual choice: Esscher transform
  - Structure preserving
- Essentially, a measure change introduces an modification in the spot drift
  - Coined the *market price of risk*
- For the arithmetic model
  - Deterministic change of jump characteristics
The Heath-Jarrow-Morton approach

- Motivated from interest rate theory: Model the electricity forwards directly
- Problems:
  - Fixed-delivery $f(t, u)$, or the actually traded ones $F(t, \tau_1, \tau_2)$?
  - Former is not traded, so data needs to be constructed
- Complications with no-arbitrage models if insisting on a model for the whole forward curve dynamics $F(t, \tau_1, \tau_2)$
- Using *market models* to resolve this issue
  - Motivated from LIBOR models
  - Model only the traded contracts
Modelling issues include

Marginal:
- Are logreturns normal, or leptokurtic?
- Volatility term structure?

Multivariate:
- What is the dependency structure across contracts?

Direct modeling of forwards, or implied through options
- Latter raises the question of liquidity
Options

- Given a forward price model, we can price call and put options

- Problems here:
  - Model for fixed-delivery forwards $f(t, u)$ leads to options on the average
  - Approximations available?
  - Numerical procedures?

- Fourier transform yields prices in terms of characteristic functions
Spot price modelling
Spot price modelling

- What do we want from a spot model?
  1. Reflect prices in a statistically sound way
  2. Allow for pricing of derivatives

- The derivatives we have in mind are forwards and options

- Other objectives:
  - Portfolio management
  - Risk measurement
  - Swing options
Arithmetic model

- The spot price as a sum of non-Gaussian OU-processes

\[
S(t) = \Lambda(t) + \sum_{i=1}^{n} Y_i(t)
\]

\[
dY_i(t) = -\lambda_i Y_i(t) \, dt + \sigma_i(t) \, dL_i(t)
\]

- \(\Lambda(t)\) deterministic seasonality function, or seasonal floor
- \(L_i(t)\) are independent increasing time-inhomogeneous pure jump Lévy processes
  - Called an independent increment process
Lévy processes

- $L(t)$ is a Lévy process if
  1. The increments $L(t) - L(s)$ are stationary
  2. The increments $L(t) - L(s)$ are independent

- Inhomogeneous Lévy process: Remove condition 1

- Poisson random measure associated to $L$: For $A \subset \mathbb{R}\{0\}$, Borel

\[
  N(A \times (0, t]) = \sum_{0<s\leq t} 1(\Delta L(s) \in A), \quad \Delta L(s) = L(s) - L(s-) \]

\[
  \sum_{0<s\leq t} 1(\Delta L(s) \in A)
\]
- $N(A \times (0, t])$ counts number of jumps of size $A$ up to time $t$
- Compensator measure is defined as

$$\ell(A \times (0, t]) = \mathbb{E} [N(A \times (0, t])]$$

- The process

$$t \mapsto N(A \times (0, t]) - \ell(A \times (0, t])$$

is a (local) martingale
The Lévy-Kintchine formula (or the characteristic function of $L(t)$)

\[
\ln \mathbb{E} \left[ \exp(i\theta(L(t) - L(s))) \right] = \psi(s, t, \theta)
\]

\[
\psi(s, t, \theta) = i\theta(\gamma(t) - \gamma(s)) - \frac{1}{2}\theta^2(C(t) - C(s)) + \int_s^t \int_{\mathbb{R}} \left\{ e^{i\theta z} - 1 - i\theta z 1(|z| < 1) \right\} \ell(dz, du)
\]

$\gamma$ is the drift, $C$ is an increasing function
C is the covariance of a continuous martingale function

- In case of Lévy: \( C(t) - C(s) = \sigma(t - s) \)
- Continuous martingale is the Brownian motion

\( L(t) \) is called a pure-jump process if \( C = 0 \)
\( L(t) \) is a Lévy process if

\[ \ell(dz, dt) = dt \ell(dz) \]

\( \ell(dz) \) called the Lévy measure
Back to the arithmetic spot model....

- Jump (compensator) measure of $L_i(t)$ is of the form

$$\ell_i(dt, dz) = \rho_i(t) dt \ell_i(dz)$$

- $\ell_i(dz)$ supported on $\mathbb{R}_+$, $L_i$ increasing, pure jump process
- $\rho_i(t)$ models the seasonal variation of jump intensity.
- $\sigma_i(t)$ models the seasonal variation of jump sizes
- $\lambda_i$ different levels of mean-reversion

- Reversion to the floor $\Lambda$
The model guarantees positive prices because the $L_i(t)$’s are increasing.

- Upward jumps followed by downward drops
  - Sharpness is controlled by the corresponding $\lambda_i$
  - Gives upward spikes

- Apparent downward spikes when fast mean-reversion out-performs upward jumps

- NordPool spot prices
  - Sharp upward jumps followed by strong mean-reversion in the winter
  - “Normal” variations over the year

- Main advantage: The model allows for analytical pricing of electricity forward contracts
Case study: NordPool spot price

- Fitting of the arithmetic model
- Study by T. Meyer-Brandis
  - Based on data from January 1997 until December 2000
  - Weekends discarded
Proceeding in four steps (using $n = 2$):

1. Identification of the OU-process $Y_2(t)$ modelling the seasonal spikes
2. Fit a deterministic seasonal floor $\Lambda(t)$ of cosines to the data series
3. Find the mean-reversion of the remaining series using the autocorrelation function, assuming stationarity
4. Use the stationary distributions to calibrate the jumps
1. Identification of $Y_2(t)$ modelling the seasonal spikes

Simulated spike:

Zooming in the 3 biggest spikes of 1998, 1999, 2000:
For an estimated threshold $T = 23.5$, pick all positive jumps bigger than $T$ and construct the corresponding path $\hat{Y}_2(t)$.

Fit an OU-process $Y_2$

- Estimated mean reversion $\hat{\lambda}_2 = 1.12$
- Corresponds to $2/3$ decay of a spike after one day

Suppose the following parametrization of $Y_2(t)$

$$
Y_2 \begin{bmatrix}
\lambda_2 \\
\sigma_2(t) \\
\ell_2(dz) \\
\rho_2(t)
\end{bmatrix}
= 
\begin{bmatrix}
1.12 \\
1 \\
\exp(180) \\
0.07 \cdot 
\left( 
\frac{2}{\left| \sin\left(\frac{\pi(t-6)}{261}\right)\right| + 1} - 1 
\right)
\end{bmatrix}
$$
2. Fitting a deterministic seasonal floor $\Lambda(t)$

- Consider a stationary OU-processes

$$dY_1(t) = -\lambda_1 Y_1(t) \, dt + dL_1(t)$$

- Note that asymptotically

$$\mathbb{E}[\Lambda(t) + Y_1(t)] = \Lambda(t) + \frac{1}{\lambda_1} \int_0^\infty z \ell_1(dz) = \text{seasonal}(t)$$

- Fit a series of cosines using least squares to $\text{spot}(t) - \hat{Y}_2(t)$

- Adjust seasonal function so that data series $\text{spot}(t) - \Lambda(t) - \hat{Y}_2(t)$ remains positive
3. Find the mean-reversions of the stationary OU-processes

- The r’th lag of the ACF of \( Y_1(t) \) is given by
  \[ e^{-\lambda_1 r}, \]

- Fitting the ACF of \( Y_1(t) \) to the empirical gives the mean reversion parameter \( \lambda_1 \)
\( Y_1(t) \) with \( \lambda_1 = 0.0846 \) gives good fit
4. Finding the stationary distribution

- Assume $Y_1 \sim \text{Gamma}(\nu, \alpha)$
- ML-estimation gives $\nu = 8.055, \alpha = 0.132$
To the given stationary distribution of $Y_1$, there exists a Lévy process $L_1(t)$.

$L_1(t)$ is commonly referred to as the *background driving Lévy process*.

The Lévy measure of $L_1$ is

$$\ell_1(dz) = \nu\alpha \exp(-\alpha z) \, dz$$

Basis for simulating the path (my Rosinski’s method)
Simulation of estimated process
Empirical moments of NordPool spot price versus simulated moments (averaged over 3000 simulation paths):

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<th>Mean</th>
<th>Std. Dev.</th>
<th>Skewness</th>
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Electricity forward pricing

- Pricing of electricity forwards for arithmetic model
- Introduce risk-neutral measures $Q$ using the Esscher transform
  - Structure preserving transform
  - Change of jump characteristics under $Q$ by simple scaling
  - A drift is implicitly introduced
- Jump measure under $Q$

$$
\ell^Q_i(dz, dt) = e^{\theta_i(t)z} \ell_i(dz, dt)
$$

- Parametrization via $\theta_i$, deterministic function
Radon-Nikodym derivative for measure change:

\[
\frac{dQ}{dP}\big|_{\mathcal{F}_t} = \prod_{i=1}^{n} Z_i(t)
\]

\(Z_i\) martingales defined as

\[
Z_i(t) = \exp\left(\int_0^t \theta_i(s) \, dL_i(s) - \psi_i(0, t, -i\theta_i(\cdot))\right)
\]

\(\psi(0, t, -i\theta)\) is the log-moment generating function
Derivation of the forward price (assuming \( n = 1 \)):

- Express \( F \) in terms of \( f \)
- State \( Y \)-dynamics under \( Q \)
- Using martingale property to derive the price dynamics for \( f \)
- Integrate over delivery period

\[
F(t, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f(t, u) \, du
\]

\[
f(t, u) = \Lambda(t) + \mathbb{E}_Q[Y(u) | \mathcal{F}_t]
\]
Explicit solution of $Y$ under $Q$

$$Y(u) = Y(t)e^{-\lambda(u-t)} + \int_t^u \sigma(s)e^{-\lambda(u-s)} dL(s)$$

$$= Y(t)e^{-\lambda(u-t)} + \int_t^u \sigma(s)e^{-\lambda(u-s)} d\gamma(u)$$

$$+ \int_t^u \int_{\mathbb{R}^+} \sigma(s)e^{-\lambda(u-s)}z\left\{e^{\theta(s)z} - 1 \mid |z| < 1\right\} \ell(dz, ds)$$

$$+ \int_t^u z\sigma(s)e^{-\lambda(u-s)} \tilde{N}^Q(dz, ds)$$

$\tilde{N}^Q$ compensated Poisson random measure of $L$ under $Q$
Integration wrt $\tilde{N}^Q$ is a martingale under $Q$

Conditional expectation becomes zero

\[
f(t, u) = Y(t)e^{-\lambda(u-t)} + \int_t^u \sigma(s)e^{-\lambda(u-s)} d\gamma(u)
+ \int_t^u \int_{\mathbb{R}^+} \sigma(s)e^{-\lambda(u-s)}z\{e^{\theta(s)z} - 1|z|<1\} \ell(dz, ds)
\]

Integrating over the delivery period $[\tau_1, \tau_2]$ yields the forward price
In conclusion:

\[
F(t, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Lambda(u) du + \Theta(t, \tau_1, \tau_2, \theta) + \sum_{i=1}^{n} \frac{e^{-\lambda_i(\tau_1-t)} - e^{-\lambda_i(\tau_2-t)}}{\lambda_i(\tau_2 - \tau_1)} Y_i(t),
\]

where

\[
(\tau_2 - \tau_1) \Theta(t, \tau_1, \tau_2, \theta) = \sum_{i=1}^{n} \int_{t}^{\tau_2} \int_{\max(v, \tau_1)}^{\tau_2} \sigma(v) e^{-\lambda_i(u-v)} \, du \, d\gamma_i(v) \\
+ \sum_{i=1}^{n} \int_{t}^{\tau_2} \int_{\mathbb{R}^+} \int_{\max(v, \tau_1)}^{\tau_2} \sigma_i(v) e^{-\lambda_i(u-v)} z\{e^{\theta(v)z} - 1\} \, du \, \ell_i(dz, dv)
\]
Not possible to express $F$ in terms of $S$

For delivery periods in the far future, we have

$$ F(t, \tau_1, \tau_2) \approx \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Lambda(u) du + \Theta(t, \tau_1, \tau_2, \theta) $$

Here, $\tau_2 - \tau_1$ is kept fixed, and $\tau_1$ gets large.

In the long end, the forward prices should not vary much stochastically.
Forward dynamics

Appealing to the Itô Formula for jump processes

\[
dF(t, \tau_1, \tau_2) = \sum_{i=1}^{n} \Sigma_i(t, \tau_1, \tau_2) \int_{\mathbb{R}_+} z \tilde{N}_i^\theta(dz, dt)
\]

\[
\Sigma_i(t, \tau_1, \tau_2) = \frac{\sigma_i(t)}{\lambda_i(\tau_2 - \tau_1)} \left( e^{-\lambda_i(\tau_1-t)} - e^{-\lambda_i(\tau_2-t)} \right)
\]

Average Samuelson effect in the scaling of the jumps

Average of \(\exp(-\lambda(u-t))\) over the delivery period

The closer \(t\) is to delivery \(u\), the bigger volatility
Pricing of options on forwards

- Let $g$ be the payoff of an option
  - E.g., a put option $g(x) = \max(K - x, 0)$
  - Call options require a damping factor in what follows

- Option price is

$$p(t, T; \tau_1, \tau_2) = e^{-r(T-t)}\mathbb{E}_Q[\max(K - F(T, \tau_1, \tau_2), 0) \mid \mathcal{F}_t]$$

- Calculate this using Fourier transformation
  - Pricing expression suitable for FFT
Using the inverse Fourier transform:

\[ g(x) = \frac{1}{2\pi} \int \hat{g}(y) \exp(ixy) \, dy \]

By the independent increment property (using \( n = 1 \))

\[
\mathbb{E}_Q \left[ g(F(T, \tau_1, \tau_2)) \mid \mathcal{F}_t \right] = \frac{1}{2\pi} \int \hat{g}(y) \mathbb{E}_Q \left[ e^{iyF(T, \tau_1, \tau_2)} \mid \mathcal{F}_t \right] \, dy
\]

\[
= \frac{1}{2\pi} \int \hat{g}(y) e^{iyF(t, \tau_1, \tau_2)} \mathbb{E}_Q \left[ e^{iy \int_t^T \int_0^\infty \Sigma(s, \tau_1, \tau_2) z \tilde{N}_\theta \, dz, ds} \mid \mathcal{F}_t \right] \, dy
\]

\[
= \frac{1}{2\pi} \int \hat{g}(y) e^{iyF(t, \tau_1, \tau_2)} \mathbb{E}_Q \left[ e^{iy \int_t^T \int_0^\infty \Sigma(s, \tau_1, \tau_2) z \tilde{N}_\theta} \, dz, ds} \right] \, dy
\]
Introducing a cumulant $\tilde{\psi}$

$$\mathbb{E}_Q [g(F(T,\tau_1,\tau_2)) | \mathcal{F}_t] = \frac{1}{2\pi} \int \hat{g}(y)e^{\tilde{\psi}(t,T,y\Sigma(\cdot,\tau_1,\tau_2))}e^{iyF(0,\tau_1,\tau_2)} dy$$

$$\tilde{\psi}(t, T, \theta) = \int_t^T \int_0^\infty \left\{ e^{i\theta(s)z} - 1 \right\} \ell_i^Q (dz, ds)$$

Fourier expression for option price ($\ast$ the convolution product)

$$p(t, T; \tau_1, \tau_2) = e^{-r(T-t)} (g \ast \Phi_{t,T})(F(t, \tau_1, \tau_2))$$

where

$$\Phi_{t,T}(y) = \exp \left( \sum_{i=1}^n \tilde{\psi}_i(t, T, y\Sigma(\cdot, \tau_1, \tau_2)) \right)$$
HJM approach to forwards
Heath-Jarrow-Morton (HJM) approach

- Use ideas from interest rate theory to model electricity markets
- Heath-Jarrow-Morton 1992:
  - Model the complete term structure dynamics of interest rates directly under the risk-neutral probability
  - Analogue in electricity: Model the term structure dynamics of forward/futures prices
  - Problem: Electricity forwards has a delivery period!
- Goal of modelling: Models which can be used for derivatives pricing and risk analysis
Look at two alternative HJM-approaches for electricity markets

1. The “fixed-delivery” approach: Modelling $f(t,u)$
2. The direct approach: Modelling $F(t,\tau_1,\tau_2)$

Goals

- Create reasonable models for the risk
- Reveal the volatility for option pricing purposes
HJM-pricing: The “fixed-delivery” approach

- Assume a geometric Brownian motion dynamics for the fixed-delivery forwards (under $Q$)

\[ df(t, u) = \sigma(t, u)f(t, u) \, dW(t) \]

- Forward with delivery over a period $[\tau_1, \tau_2]$

\[ F(t, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f(t, u) \, du \]

- **Q1:** What is the implied dynamics of the electricity forward?
- **Q2:** How to fit model to data?
Q1: Dynamics of $F(t, \tau_1, \tau_2)$:

- Using stochastic Fubini and integration-by-parts

\[ F(t, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f(s, u) \, du \]

\[ = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f(0, u) \, du + \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \int_{0}^{t} \sigma(s, u)f(s, u) \, dW(s) \, du \]

\[ = F(0, \tau_1, \tau_2) + \frac{1}{\tau_2 - \tau_1} \int_{0}^{t} \int_{\tau_1}^{\tau_2} \sigma(s, u)f(s, u) \, du \, dW(s) \]

\[ = F(0, \tau_1, \tau_2) + \int_{0}^{t} \sigma(s, \tau_2)F(s, \tau_1, \tau_2) \, dW(s) \]

\[ - \int_{0}^{t} \int_{\tau_1}^{\tau_2} \frac{u - \tau_1}{\tau_2 - \tau_1} \partial_u \sigma(s, u)F(s, \tau_1, u) \, du \, dW(s) \]
The dynamics becomes

$$dF(t, \tau_1, \tau_2) = \left\{ \sigma(t, \tau_2)F(t, \tau_1, \tau_2) - \int_{\tau_1}^{\tau_2} \frac{u - \tau_1}{\tau_2 - \tau_1} \partial_u \sigma(t, u)F(t, \tau_1, u) DU \right\} dW(t)$$

Electricity forward does not have the lognormal property

- Note the infinite dimensional structure
One approach is to *assume* that this is the case!

- Bjerksund et. al 2000.

Approximation of the dynamics

\[
dF(t, \tau_1, \tau_2) = \left\{ \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \sigma(t, u) \, du \right\} F(t, \tau_1, \tau_2) \, dW(t)
\]

- Industry standard (VizRisk)
Q2: How to fit the “fixed-delivery model” to data?

- Only forwards with delivery period are traded
- Standard approach is to smoothen the electricity forward curve
- Idea: Today’s forward curve is factorized into a seasonal and a correction term
  \[ f(u) := f(0, u) = \Lambda(u) + \epsilon(u) \]

- Seasonal function can be a given prior
  - From a spot model or a fundamental model
- Correction term function is a spline
Fit using two criteria:

1. Constrained on the bid-ask spread, or actual prices

\[
\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f(u) \, du = F(0, \tau_1, \tau_2)
\]

2. Maximum smoothness, in the sense

\[
\max \int_0^\tau \epsilon''(u) \, du
\]

In addition, natural to suppose \( \epsilon'(\tau) = 0 \)

- Forward prices in the long end are risk-adjusted seasonal prices
- \( \epsilon \) 4th order polynomial spline
Empirical study and applications to volatility estimation in B., Koekebakker and Ollmar 2005
Drawbacks with the fixed-delivery HJM-approach:

1. Model of non-existing forwards
2. Estimation uses data which must be transformed (smoothed)
3. Implied electricity forward dynamics is very involved, even for a GBM-model
HJM-pricing: the direct approach

- Model the electricity forward dynamics directly
  - B. and Koekebakker 2005
- No-arbitrage condition: Overlapping forwards must satisfy
  \[ F(t, \tau_1, \tau_N) = \sum_{i=1}^{N-1} \frac{\tau_{i+1} - \tau_i}{\tau_N - \tau_1} F(t, \tau_i, \tau_{i+1}) \]
- If market trades in forwards with all possible delivery periods
  \[ F(t, \tau_1, \tau_N) = \frac{1}{\tau_N - \tau_1} \int_{\tau_1}^{\tau_N} F(t, u, u) \, du \]
GBM-model does not in general satisfy this condition

- The exception is constant volatility

\[
\frac{1}{\tau_N - \tau_1} \int_{\tau_1}^{\tau_N} F(t, u, u) \, du = \frac{1}{\tau_N - \tau_1} \int_{\tau_1}^{\tau_N} F(0, u, u) \, du + \frac{1}{\tau_N - \tau_1} \int_t^0 \sigma \frac{1}{\tau_N - \tau_1} \int_{\tau_1}^{\tau_N} F(0, u, u) \, du \, dW(s)
\]

- Hence, if initial forward curve satisfies the no-arbitrage condition

\[
F(t, \tau_1, \tau_N) = \frac{1}{\tau_N - \tau_1} \int_{\tau_1}^{\tau_N} F(t, u, u) \, du
\]
What if volatility is delivery-period dependent?

Differentiate wrt $\tau_2$ on both sides

\[
F(t, \tau_1, \tau_2) \left( \frac{1}{\tau_2 - \tau_1} + \frac{1}{2} \int_0^t \partial_{\tau_2} \Sigma^2(s, \tau_1, \tau_2) \, ds \right)
- \int_0^t \partial_{\tau_2} \Sigma(s, \tau_1, \tau_2) \, dW(s) \right) = \frac{1}{\tau_2 - \tau_1} F(t, \tau_1, \tau_2)
\]

RHS is positive, while LHS may be arbitrary negative
In conclusion: Hard to state models which are
- realistic,
- easy to estimate,
- and satisfy the no-arbitrage condition

A practical approach: Model only the existing forwards in the market
1. Single out the “smallest” forwards (the building blocks)
2. Model these
3. Forwards with larger delivery period are modelled by the no-arbitrage relation

Purpose is to reveal the volatility term structure
Market models

- Suppose \([\tau_1^i, \tau_2^i]\) is a sequence of delivery periods for the building block forwards
- Suppose each forward is modelled as a GBM

\[
dF_i(t) = \Sigma^i(t)F_i(t) \, dW_i(t)
\]

- The volatilities \(\Sigma^i\) will depend on the start and end of delivery period
- \(W^i\) are Brownian motions
  - With a correlation structure
- Options priced using Black-76, with time-dependent volatility
Case study from Nordpool

- Suppose $W^i = W$, e.g., one common risk factor for all contracts
- Fitted to data from NordPool
  - Extracting only non-overlapping forwards
  - Using more than 10,000 price data
- Assumed constant market price of risk
  - Model for $F^i$ specified under $Q$
  - Add a constant drift for model under $P$
- Volatility with seasonality and maturity effect

\[
\Sigma(t, \tau_1, \tau_2) = \frac{\sigma}{a(\tau_2 - \tau_1)} \left\{ e^{-a(\tau_1-t)} - e^{-a(\tau_2-t)} \right\} + s(t)
\]

- $s(t)$ truncated Fourier series
The NordPool Market
The spot and electricity forward relation
Spot price modelling
HJM approach to forwards
Conclusions

Introduction
Market models
Extensions

A) Implied time dependent volatility for each contract

B) Maturity effect

C) Implied seasonal spot price vol
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Extensions

Questions:

1. Are logreturns normally distributed?
2. What about correlations structure for the $W^i$'s

Presentation of some initial findings

Work in progress by Dennis Frestad
- Distributions of electricity forwards log-returns are non-normal
  - Tails are heavy
  - Symmetric
The normal inverse Gaussian distribution

- The observed heavy tails are not captured by the normal distribution.
- Motivated from finance, use the normal inverse Gaussian distribution (NIG)
  - Barndorff-Nielsen 1998
- Four-parameter family of distributions
  - $\alpha$: tail heaviness
  - $\delta$: scale (or volatility)
  - $\beta$: skewness
  - $\mu$: location
Density of the NIG

\[ f(x; \alpha, \beta, \delta, \mu) = c \exp(\beta(x - \mu)) \frac{K_1 \left( \alpha \sqrt{\delta^2 + (x - \mu)^2} \right)}{\sqrt{\delta^2 + (x - \mu)^2}} \]

where \( K_1 \) is the modified Bessel function of the third kind with index one

\[ K_1(x) = \frac{1}{2} \int_{0}^{\infty} \exp \left( -\frac{1}{2} x (z + z^{-1}) \right) \, dz \]

Explicit (log-)moment generating function

\[ \phi(u) := \ln \mathbb{E}[e^{uL}] = u\mu + \delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2} \right) \]
The NordPool Market
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Empirical returns, first week

Empirical returns, first block

Empirical returns, first season

Empirical returns, last season
Correlation among forwards with one week delivery
- Estimated from smoothed forward curves
  - week contracts are extracted over regular times
- Correlation not stationary as a function of distance-between-delivery
To specify a market model for all forwards including correlation:

- Need non-stationary correlation structures

Correlation function of

- Time to delivery
- Time of the year when delivery takes place
- Length of delivery

Work in progress to understand the stylized facts of the correlation structure
Conclusions

- **Arithmetic spot model**
  - Allows for analytical forward pricing for contracts with periodic delivery
  - Option pricing by Fourier transforms
  - Allows for calibration

- **HJM/Market models for electricity forwards**
  - Seasonal volatility
  - Option pricing using Black-76 with time-dependent volatility
  - Non-normal returns and the correlation structure
Coordinates

- fredb@math.uio.no
- www.math.uio.no/~fredb
- www.cma.uio.no
References


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