2nd sheet of exercises
due: 15 November 2007

Exercise 5. Let us consider the Fourier regression model where we observe the data

$$Y_{j,n} \;=\; \sum_{k=-m}^{m} a_k \exp(2\pi i k j/n) \;+\; \varepsilon_{j,n}\,, \quad \forall j \in \{0,\ldots,n-1\}$$

with the design points $x_{j,n} = j/n$, where the ε_j are i.i.d., $E\varepsilon_{j,n} = 0$ and $E|\varepsilon_{j,n}|^2 = 1$. Show that

$$\hat{a}_k \;:=\; \frac{1}{n}\sum_{j=0}^{n-1} \exp(-2\pi i k j/n) Y_{j,n}\,,$$

is an unbiased estimator of a_k (i.e. $E\hat{a}_k = a_k$); and calculate its variance.

Exercise 6. Consider the kernel regression estimator (2.6) with a non-negative, compactly supported and bounded kernel function K; further, select K so that $K(0) \neq 0$ and K is continuous at $x = 0$. Choose the bandwidth so that $h \to 0$ and $nh^d \to \infty$ as $n \to \infty$ is satisfied. Also, assume that

$$\#\big(\{x_{1,n},\ldots,x_{n,n}\} \cap B_\varepsilon(x)\big) \;\geq\; \text{const.} \cdot n\varepsilon^d\,, \, \forall n \in \mathbb{N}\,, \, \forall \varepsilon > n^{-1/d}\,,$$

where $B_\varepsilon(x)$ denotes the open ball around x with the radius $\varepsilon > 0$. Verify the conditions of Lemma 2.4 in the current setting.

Exercise 7. We consider the nonparametric problem where the right endpoint of a density f_X, i.e.

$$\theta(f_X) \;:=\; \inf\Big\{x \,:\, \int_{y \geq x} f_X(y)dy = 0\Big\}\,,$$

shall be estimated based on the i.i.d. data $X_1,\ldots,X_n$ having the density f_X. The class consisting of all densities f which satisfy $\theta(f) \in [\theta_0,\theta_1]$ and $f(x) \geq c$ for LB-almost all $x \in [\theta(f)-\varepsilon, \theta(f))]$ is denoted by $\mathcal{F}_{\theta_0,\theta_1,c,\varepsilon}$. We have $\theta_0 < \theta_1$, $c > 0$, $\varepsilon > 0$.

Show that the estimator $\hat{\theta} := \max\{X_1,\ldots,X_n,\theta_0\}$ achieves the following rate of convergence:

$$\sup_{f_X \in \mathcal{F}_{\theta_0,\theta_1,c,\varepsilon}} E|\hat{\theta} - \theta(f_X)|^2 \;=\; O(n^{-2})\,.$$

Exercise 8. Consider the nonparametric regression problem with fixed design. Assume that there exists some $\varepsilon > 0$ so that

$$\#\Big(B_\varepsilon(x) \cap \bigcup_{n \in \mathbb{N}} \{x_{1,n},\ldots,x_{n,n}\}\Big) \;<\; \infty\,.$$

Show that there is no consistent estimator $\hat{g}(x)$ in the weak sense, i.e. $\hat{g}(x) \to g(x)$ in probability as $n \to \infty$ even if g is continuous at x.