5th sheet of exercises
due: 10 January 2008

Exercise 17. We observe the i.i.d. data $X_1, \ldots, X_n$ which are normally distributed with the unknown expectation $\mu \in \mathbb{R}$ and the variance 1. Use Proposition 4.1 to show that there is no estimator $\hat{\mu} = \hat{\mu}(X_1, \ldots, X_n)$ of μ so that

$$\liminf_{n\to\infty} n \cdot \sup_{\mu\in\mathbb{R}} E|\hat{\mu} - \mu|^2 = 0 .$$

Exercise 18. We observe the i.i.d. data $X_1, \ldots, X_n$ which are uniformly distributed on some interval $[0, \theta]$. Show by Proposition 4.1 that there exists no estimator $\hat{\theta} = \hat{\theta}(X_1, \ldots, X_n)$ so that

$$\liminf_{n\to\infty} n^2 \cdot \sup_{\theta\in[\theta_0,\theta_1]} E|\hat{\theta} - \theta|^2 = 0 ,$$

for some fixed $0 < \theta_0 < \theta_1 < \infty$. Conclude that the estimator defined in Exercise 7 achieves optimal convergence rates in the above problem.

Exercise 19. Given the i.i.d. observations $X_1, \ldots, X_n$, we do not want to estimate their density f_X itself but some characteristic of f_X. In the mathematical model, we aim at estimating $\Phi(f_X)$ where Φ is a function mapping some density class $\mathcal{F}$ to some metric space $(\mathcal{L}, l)$.

(a) Show that $\int |f(x) - g(x)| dx \geq H^2(f, g)$ for all densities f, g where H denotes the Hellinger distance.

(b) We assume there exists a uniformly consistent estimator $\hat{\phi}$ of $\Phi(f_X)$, i.e.

$$\sup_{f_X\in\mathcal{F}} E\, l(\hat{\phi}, \Phi(f_X)) \stackrel{n\to\infty}{\longrightarrow} 0 ,$$

for some density class $\mathcal{F}$. Show by Proposition 4.1 that Φ is uniformly continuous with respect to the $L_1(\mathbb{R})$-norm on $\mathcal{F}$ and the l-metric on $\mathcal{L}$.

Exercise 20. As a non-uniform version of the assumption in Exercise 19, assume that there exists a consistent estimator $\hat{\phi}$ of $\Phi(f_X)$, i.e.

$$E\, l(\hat{\phi}, \Phi(f_X)) \stackrel{n\to\infty}{\longrightarrow} 0 , \quad \forall f_X \in \mathcal{F} .$$

Show that there is no simple analogous result in the individual case, i.e. Φ is not necessarily continuous on the whole of $\mathcal{F}$ by constructing a counterexample.