CONVEXITY INEQUALITIES FOR POSITIVE OPERATORS

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Abstract. We prove pointwise convexity inequalities of the form $F(Tf) \le T[F(f)]$ where F is a convex function defined on a convex subset of some Banach space X and T is the X-valued extension of a positive operator on some function space. Special examples include the pointwise Hölder inequality $T(fg) \le (Tf^p)^{\frac{1}{p}} (Tf^q)^{\frac{1}{q}}$ for a positive sublinear operator T. As applications we consider vector-valued conditional expectation and a "real" proof of the Riesz-Thorin theorem for positive (sublinear) operators yielding the optimal constant.

1. Introduction

The basic version of the classical Hölder inequality reads

$$\int fg \, d\mu \le \left(\int f^p \, d\mu \right)^{\frac{1}{p}} \left(\int g^{p'} \, d\mu \right)^{\frac{1}{p'}} \tag{1}$$

where f, g are positive functions on a measure space (Ω, Σ, μ) and $1 < p, p' < \infty$ is a pair of conjugated exponents, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. One possible proof proceeds in the following way. By elementary calculus one verifies

$$ab = \inf_{t>0} \frac{t^p}{p} a^p + \frac{t^{-p'}}{p'} b^{p'}$$

for all real numbers $a, b \ge 0$. This yields

$$fg \le \frac{t^p}{p} f^p + \frac{t^{-p'}}{p'} g^{p'} \tag{2}$$

for all $t > 0$, where this inequality is to be understood in the space $\mathfrak{M}(\Omega, \mu)$ of μ-equivalence classes of measurable scalar functions on (Ω, Σ). (See Section 3 for notation.) Integrating gives

$$\int fg \, d\mu \le \frac{t^p}{p} \int f^p \, d\mu + \frac{t^{-p'}}{p'} \int g^{p'} \, d\mu$$

for all $t > 0$, and taking the infimum with respect to t again we arrive at (1).

This elegant and short proof can be found in [5, p.43], but it does not seem to be so widely known. Unfortunately, one may say, since it reveals a certain scheme, namely a variational approach to integral inequalities. Indeed, also the Minkowski inequality can be proved in the same way, utilizing the identity

$$(a^{\frac{1}{p}} + b^{\frac{1}{p}})^p = \inf_{0<t<1} t^{-p} a + (1-t)^{-p} b.$$

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The main idea of this paper is that this method of proof generalizes to positive operators (on spaces of measurable functions) instead of just integrals. The pattern is an inequality of the form

$$F(Tf, Tg) \leq T[F(f, g)]$$

for some concave, positively homogeneous, and upper semicontinuous function $F : \mathbb{R}_+^2 \longrightarrow [0, \infty)$, cp. [8, Lemma 1]. This can be seen as a 2-dimensional Jensen-type inequality. Indeed, for T being integration, this is the approach of Bourbaki [2, I.1 & I.2].

As is well-known in convex analysis, by means of epigraphs (see Section 5) one can reduce such inequalities to invariance properties of convex sets (in higher dimensional spaces) of the form: If $f(\omega) \in C$ for almost all $\omega \in \Omega$ then $[Tf](\omega') \in C$ for almost all $\omega' \in \Omega'$. Our main Theorems 5.3 and 5.4 give Banach space-valued versions of these facts. Before, we have to talk about the canonical extension of positive operators to Banach space-valued function spaces (Section 3) and about characterizing closed convex subsets of separable Banach spaces by *countably* many linear inequalities (Section 4). Let us stress the fact that if the Banach space is finite-dimensional the operator in question need not be linear.

As applications, in Section 6 we first realize that the construction of the vector-valued conditional expectation operator as well as the Jensen inequality for it are just instances of our general results. Second, using the Hölder inequality for operators we give an elementary and "real" proof of the Riesz-Thorin Theorem for positive (sublinear and positively homogeneous) operators, yielding the best constant.

2. The Hölder and Minkowski Inequalities for Positive Sublinear Operators

Let (Ω, Σ, μ) and (Ω', Σ', ν) be measure spaces, which are assumed to be σ-finite. By $\mathfrak{M}(\Omega, \Sigma, \mu)$ we denote the space of μ-equivalence classes of real-valued Σ-measurable functions on Ω. When Σ is understood, we just write $\mathfrak{M}(\Omega, \mu)$ for it. We will consider a subcone $C \subset \mathfrak{M}(\Omega, \mu)_+$ and an operator $T : C \longrightarrow \mathfrak{M}(\Omega', \nu)_+$ which is monotone, sublinear and positively homogeneous. This means, we require the following properties:

a) If $f \leq g$, then $Tf \leq Tg$,

b) $T(f + g) \leq Tf + Tg$,

c) $T(\alpha f) = \alpha Tf$,

for all $f, g \in C$, $\alpha \geq 0$. For such operators, Hölder's inequality holds.

Proposition 2.1 (Hölder inequality). *Let (Ω, Σ, μ) and (Ω', Σ', ν) be measure spaces, $C \subset \mathfrak{M}(\Omega, \mu)_+$ be a subcone and $T : C \longrightarrow \mathfrak{M}(\Omega', \nu)_+$ be monotone, sublinear and positively homogeneous. If $f, g \in \mathfrak{M}(\Omega, \mu)_+$ are such that $f^p, g^{p'}$ and fg are in C then*

$$T(fg) \leq (Tf^p)^{\frac{1}{p}} \, (Tg^{p'})^{\frac{1}{p'}}$$

in $\mathfrak{M}(\Omega', \nu)$.

Proof. Repeat the proof given in the Introduction with integration against μ being replaced by the application of T. If one hesitates in taking the infimum over the uncountable family $\{t \in \mathbb{R} \mid t > 0\}$, just note that one can restrict to positive rationals here. □

Remark 2.2. At least for the linear operators T, this result is well known, cf. [6, Lemma 7.4] (where the proof is a bit less elegant), [5, p.246] and [8, Remark 1]. See [7] for an interesting historical account.

As we have said in the Introduction, we can cast our initial inequality (2) into the form

$$a^{\frac{1}{p}}\, b^{\frac{1}{p'}} = \inf_{t>0} \left(\frac{t^p}{p}\, a + \frac{t^{-p'}}{p'}\, b \right)$$

and observe that the function

$$F : \mathbb{R}^2_+ \longrightarrow [0,\infty), \quad F(a,b) := a^{\frac{1}{p}}\, b^{\frac{1}{p'}}$$

is concave and positively homogeneous. The Hölder inequality is then equivalent to

$$T(F(f,g)) \le F(Tf, Tg)$$

for suitable $f, g \in \mathfrak{M}(\Omega,\mu)$. Thus we arrive at the following result.

Proposition 2.3 (**Jensen inequality**). *Let $F : \mathbb{R}^d_+ \longrightarrow [0,\infty)$ be concave, positively homogeneous and upper semicontinuous. Assume that (Ω,Σ,μ) and (Ω',Σ',ν) be measure spaces, $C \subset \mathfrak{M}(\Omega,\mu)_+$ is a subcone and $T : C \longrightarrow \mathfrak{M}(\Omega',\nu)_+$ is monotone, sublinear and positively homogeneous. If $f_1,\ldots,f_d \in C$ are such that $F(f_1,\ldots,f_d)$ is in C then*

$$T\left(F(f_1,\ldots,f_d)\right) \le F\left(Tf_1,\ldots,Tf_d\right)$$

in $\mathfrak{M}(\Omega',\nu)$.

Proof. We will show below (see Remark 5.5) that F has a representation

$$F(x_1,\ldots,x_d) = \inf_{a\in A} \sum_{j=1}^{d} a_j x_j$$

for a certain countable set $A \subset \mathbb{R}^d_+$. Then the argument is the same as in the proof of Proposition 2.1. Compare [2, Prop. I.1]. □

As an immediate application we obtain also Minkowski's inequality.

Corollary 2.4 (**Minkowski Inequality**). *Let (Ω,Σ,μ) and (Ω',Σ',ν) be measure spaces, $C \subset \mathfrak{M}(\Omega,\mu)_+$ be a subcone and $T : C \longrightarrow \mathfrak{M}(\Omega',\nu)_+$ monotone, sublinear and positively homogeneous, and let $1 \le p < \infty$. If $f, g \in \mathfrak{M}(\Omega,\mu)_+$ are such that f^p, g^p and $(f+g)^p$ are in C then*

$$(T(f+g)^p)^{\frac{1}{p}} \le (Tf^p)^{\frac{1}{p}} + (Tg^p)^{\frac{1}{p}}$$

in $\mathfrak{M}(\Omega',\nu)$.

Proof. Apply Proposition 2.3 to the function

$$F : \mathbb{R}^2_+ \longrightarrow [0,\infty), \quad F(a,b) = (a^{\frac{1}{p}} + b^{\frac{1}{p}})^p = \inf_{0<t<1} t^{-p}a + (1-t)^{-p} b.$$

□

Clearly, the usual Hölder and Minkowski inequalities are corollaries by taking Ω' a one-point space. In the following sections we generalize Proposition 2.3 to situations, where $\mathbb{R}^d$ is replaced by any Banach space X. Of course there is a straightforward extension of Proposition 2.3 to sequence spaces. However, if a general Banach space is considered, one has to assume the operator T to be linear in order to define a canonical extension to the vector-valued setting. This is the topic of the next secttion.

3. Vector-valued Extensions of Positive Linear Operators

The contents of this section are well-known and only included for the convenience of the reader. Suppose you are given a (σ-finite) measure space (Ω, Σ, μ) and a Banach space X with norm $|\cdot|_X$. We denote by

$$\mathrm{St}(\Omega, \Sigma, \mu; X) := \{\sum_{j=1}^{n} \mathbf{1}_{A_j} \otimes x_j \mid A_j \in \Sigma,\ \mu(A_j) < \infty,\ x_j \in X\ (1 \le j \le n)\}$$

the space of *step functions*. Here we have used the notational convention

$$f \otimes x := (\omega \mapsto f(\omega)x) : \Omega \longrightarrow X.$$

By $\mathfrak{M}(\Omega, \Sigma, \mu; X)$ we denote the vector space of all μ-equivalence classes of all mappings $f : \Omega \longrightarrow X$ which can be approximated pointwise almost-everywhere by a sequence of step functions. A well-known result of measure theory says that a mapping $f : \Omega \longrightarrow X$ defines an element of $\mathfrak{M}(\Omega, \mu, \Sigma; X)$ if and only if f is essentially separably-valued and essentially measurable (where X bears its Borel structure), see [4, III.6.10]. If the σ- algebra Σ is understood, we simply write $\mathrm{St}(\Omega, \mu; X)$ and $\mathfrak{M}(\Omega, \mu; X)$. If also $X = \mathbb{R}$, we again shorten this to $\mathrm{St}(\Omega, \mu)$ and $\mathfrak{M}(\Omega, \mu)$.

There is a natural (F-space) topology on $\mathfrak{M}(\Omega, \mu; X)$, namely the topology (locally) in measure. We abbreviate convergence in measure by $f_\alpha \to_\mu f$. It is given by

$$f_\alpha \to_\mu f \quad \Longleftrightarrow \quad \mu(A \cap \{|f_\alpha - f|_X > t\}) \to 0 \quad \forall\ A \in \Sigma,\ \mu(A) < \infty,\ t > 0,$$

and can be characterized by convergence pointwise μ-a.e. of subsequences. The space $\mathrm{St}(\Omega, \mu; X)$ of all X-valued step functions is dense in $\mathfrak{M}(\Omega, \mu; X)$.

A *Banach function space* on (Ω, Σ, μ) is a subspace $E \subset \mathfrak{M}(\Omega, \mu)$ together with a norm $\|\cdot\|_E$ such that the following holds:

a) E is an ideal, i.e., if $|f| \le |g|$ and $g \in E$, then $f \in E$.

b) The norm is a lattice norm, i.e. $f, g \in E$, $|f| \le |g|$ implies $\|f\|_E \le \|g\|_E$.

c) $(E, \|\cdot\|_E)$ is complete and the inclusion mapping $E \longrightarrow \mathfrak{M}(\Omega, \mu)$ is continuous.

Now, let X be a (real) Banach space with norm $|\cdot|_X$). We consider the space

$$\widehat{E(X)} := \{f \in \mathfrak{M}(\Omega, \mu; X) \mid |f|_X \in E\}$$

and the norm $\|f\|_{E(X)} := \||f|_X\|_E$ thereon. By the ideal property a), this definition is meaningful and $\widehat{E(X)}$ is in fact a space. Moreover, we have $E \otimes X \subset \widehat{E(X)}$, where $E \otimes X = \mathrm{span}\{f \otimes x \mid f \in E,\ x \in X\}$.

Lemma 3.1. *The space $\widehat{E(X)}$ is complete under the norm $\|f\|_{E(X)}$ and the inclusion mapping $\widehat{E(X)} \longrightarrow \mathfrak{M}(\Omega, \mu; X)$ is continuous.*

Proof. Let $(f_n)_n$ be a Cauchy sequence in $\widehat{E(X)}$, i.e., $\||f_n - f_m|_X\|_E \to 0$. Then, by property c), $|f_n - f_m|_X \to 0$ in measure. Since $\mathfrak{M}(\Omega, \mu; X)$ is complete, there is $f \in \mathfrak{M}(\Omega, \mu; X)$ such that $|f_n - f|_X \to 0$ in measure. Now, one obviously has $||f_n|_X - |f_m|_X| \le |f_n - f_m|_X$ and property b) shows that $(|f_n|_X)_n$ is a Cauchy sequence in E. Since E is assumed to be complete we obtain $|f|_X \in E$ by property c). Hence $f \in \widehat{E(X)}$. Because of $||f_n - f|_X - |f_m - f|_X| \le |f_n - f_m|_X$ and property b), the sequence $(|f_n - f|_X)_n$ is E-Cauchy. From this follows readily $\|f_n - f\|_{E(X)} \to 0$. $\square$

Remark 3.2. Our definition of Banach function space coincides with the definition of *ideal lattice* in [5, p. 40]. There it is shown that continuity of the embedding $E \longrightarrow \mathfrak{M}(\Omega, \mu)$ is automatic from the other hypotheses. Note that sometimes in the literature, the term 'Banach function space' has a slightly different (but close) meaning.

We will denote by

$$E(X) := \overline{E \otimes X}^{\|\cdot\|_{E(X)}} \subset \widehat{E(X)}$$

the closure of $E \otimes X$ with respect to the norm $\|\cdot\|_{E(X)}$. By Lemma 3.1, the space $E(X)$ is a Banach space.

Examples 3.3. Let $1 \leq p < \infty$ and set $E := \mathbf{L^p}(\Omega, \mu)$. Then $E(X) = \widehat{E(X)} = \mathbf{L^p}(\Omega, \mu; X)$ is the usual Bochner space. In case $p = \infty$, $\widehat{E(X)} = \mathbf{L^\infty}(\Omega, \mu; X)$ is the Bochner space whereas the space $E(X)$ in this case is strictly smaller in general.

We now start with a linear and continuous operator $T : E \longrightarrow \mathfrak{M}(\Omega', \nu)$ and consider its well-defined extension

$$T \otimes I : E \otimes X \longrightarrow \mathfrak{M}(\Omega', \nu) \otimes X \subset \mathfrak{M}(\Omega', \nu; X).$$

By abuse of notation, we will frequently write T instead of $T \otimes I$. It is an intrigueing question whether $T = T \otimes I$ is in fact continuous with respect to the norm $\|\cdot\|_{E(X)}$. In general, this will depend on the particular choice of T and the Banach space X. However, there are two cases where one has complete information: if $X = H$ is a Hilbert space then there is no restriction on the operator T, and if T is positive there is no restriction on the Banach space X. We are going to prove the latter.

Lemma 3.4. *Let T be positive. Then $|Tf|_X \leq T\,|f|_X$ for all $f \in E \otimes X$.*

Proof. The mapping f takes values in a separable (in fact: finite-dimensional) space F. Hence there is a countable subset $A \subset B_{X'}$ with $|x|_X = \sup_{a \in A} \langle a, x \rangle$ for all $x \in F$. Therefore, one has $\langle a, f \rangle \leq |f|_X$ and $\langle a, Tf \rangle = T\,\langle a, f \rangle \leq T\,|f|_X$ for all $a \in A$. Taking the supremum again yields $|Tf|_X \leq T\,|f|_X$. □

By Lemma 3.4, $f_n \to 0$ in $E(X)$ implies $Tf_n \to 0$ in $\mathfrak{M}(\Omega', \nu; X)$. Hence one can extend $T = T \otimes I$ to the space $E(X)$ to obtain a continuous linear operator

$$T = T \otimes I : E(X) \longrightarrow \mathfrak{M}(\Omega', \nu; X).$$

Let $(Y, |\cdot|_Y)$ be any other Banach space. Each bounded linear mapping $S : X \longrightarrow Y$ induces a continuous linear mapping

$$I \otimes S : \mathfrak{M}(\Omega, \mu; X) \longrightarrow \mathfrak{M}(\Omega, \mu; Y). \tag{3}$$

which we will also abbreviate by S. By restriction, one obtains a bounded operator $S\big| : \widehat{E(X)} \longrightarrow \widehat{E(Y)}$. Since clearly $S(E \otimes X) \subset E \otimes Y$, a bounded linear operator $S : E(X) \longrightarrow E(Y)$ is induced.

The extensions of T and S commute, i.e.

$$(T \otimes I_Y) \circ (I_{\mathfrak{M}(\Omega,\mu)} \otimes S) = (I_{\mathfrak{M}(\Omega',\nu)} \otimes S) \circ (T \otimes I_X). \tag{4}$$

In particular, T commutes with linear functionals $x' \in X'$.

Proposition 3.5. *Let E be a Banach function space on the measure space (Ω, μ) and $T : E \longrightarrow \mathfrak{M}(\Omega', \nu)$ positive. Let $(X, |\cdot|_X)$ a Banach space and denote by T the canonical extension of $T \otimes I_X$ from $E \otimes X$ to $E(X)$. Then*

$$|Tf|_X \leq T\,|f|_X$$

for all $f \in E(X)$.

Proof. This follows easily from Lemma 3.4 by continuity. However, one could also repeat its proof. ☐

Let us remark, that the use of Proposition 3.5 could considerably shorten the proof of the "spectral bound equals growth bound"-theorem for positive semigroups on $\mathbf{L}^{\boldsymbol{p}}$-spaces, given in [1, p.351-355].

4. Closed Convex Sets in Banach Spaces

Note that Proposition 3.5 is of the following type: $C := X$ is a closed convex set in X and $F = \|\cdot\|_X : V \longrightarrow \mathbb{R}$ is a convex mapping. If $f \in E(X)$ takes its values in C (which is trivially true here) then also Tf does so and one has $F(Tf) \leq T(F(f))$. As already remarked above, this property reduces to an invariance result for closed convex subsets of $X \times \mathbb{R}$ (see Theorem 5.4). For this we need the well-known fact that closed convex sets in Banach spaces can be described by linear inequalities. However, since we want to take infima pointwise almost everywhere, we need to ensure a countability condition.

Proposition 4.1. *Let X be a* separable *Banach space and $C \subset X$ a closed convex subset. Then there is a* countable *subset $A \subset X' \times \mathbb{R}$ such that*

$$C = \bigcap_{(x',\,b)\,\in\,A} \{x \in X \mid \langle x', x\rangle \leq b\}$$

Proof. By means of the Hahn-Banach theorem one can find some set $\tilde{A} \subset X' \times \mathbb{R}$ such that $C = \bigcap_{(x',\,b)\,\in\,\tilde{A}} \{x \in X \mid \langle x', x\rangle \leq b\}$. Using the following lemma yields the assertion. ☐

Lemma 4.2. *Let X be a separable Banach space and $\tilde{A} \subset X'$. Then there is a countable set $A \subset \tilde{A}$ which is weak*-sequentially dense in $\tilde{A}$.*

Proof. One can assume without restriction, that $\tilde{A}$ is bounded, i.e. there is $r > 0$ such that $\tilde{A} \subset \{x' \in X' \mid \|x'\| \leq r\}$. By the Banach-Alaoglu theorem, the latter set is compact. But it is also metrizable, since X is separable. Hence $\tilde{A}$ is a subset of a second countable metric space whence it is by itself second countable and metric. A fortiori, it has a countable (sequentially) dense subset. ☐

Corollary 4.3. *Let X be a* separable *Banach space and $C \subset X$ a closed convex subset.*

a) One has $0 \in C$ if and only if there is a countable subset $A \subset X' \times [0, \infty)$ such that

$$C = \bigcap_{(x',\,b)\,\in\,A} \{x \in X \mid \langle x', x\rangle \leq b\}$$

b) The set C is a cone if and only if there is a countable subset $A \subset X'$ such that

$$C = \bigcap_{x'\,\in\,A} \{x \in X \mid \langle x', x\rangle \leq 0\}$$

Proof. This is easy by Proposition 4.1. ☐

5. The Main Results

Let $E \subset \mathfrak{M}(\Omega, \mu)$ be a Banach function space, $T : E \longrightarrow \mathfrak{M}(\Omega', \nu)$ a linear, positive, continuous operator, X a real Banach space, and $T := T \otimes I_X$ the canonical extension of T to an operator $T : E(X) \longrightarrow \mathfrak{M}(\Omega', \nu; X))$ (cp. Section 3).

Definition 5.1. Let $E \subset \mathfrak{M}(\Omega, \mu)$ be a sublattice. A positive operator $T : E \longrightarrow \mathfrak{M}(\Omega', \nu)$ is called *markovian* (*submarkovian*), if one has

$$f \in E,\ f \leq b \quad \Longrightarrow \quad Tf \leq b$$

for *all* (*all positive*) real numbers $b \in \mathbb{R}$.

Here is a more common description of these notions.

Lemma 5.2. *Let $E \subset \mathfrak{M}(\Omega, \mu)$ be a sublattice and $T : E \longrightarrow \mathfrak{M}(\Omega', \nu)$ be a positive operator.*

a) The operator T is submarkovian, if and only if the implication

$$0 \leq f \leq 1,\ f \in E \quad \Longrightarrow \quad Tf \leq 1$$

holds.

b) If $\mathbf{1} \in E$, then T is markovian if and only if $T\mathbf{1} = \mathbf{1}$.

Proof. Ad a). One implication is trivial. For the nontrivial part, take $b \geq 0$. Since T is positive, we can assume $b > 0$, hence $b = 1$ without restriction. Let $f \in E, f \leq 1$. Then $f = f^+ - f^-$ and both $f^+, f^- \in E$, since E is a lattice. Clearly $0 \leq f^+ \leq 1$ and $f^- \geq 0$. By hypothesis it follows that $0 \leq T[f^+] \leq 1$ and $T[f^-] \geq 0$, whence $Tf = T[f^+] - T[f^-] \leq T[f^+] \leq 1$.

b). Suppose $\mathbf{1} \in E$ and T is markovian. Since $\mathbf{1} \leq 1$ we obtain $T\mathbf{1} \leq 1$. On the other hand, $-\mathbf{1} \leq -1$, whence $-T\mathbf{1} = T(-\mathbf{1} \leq -1$. This gives $T\mathbf{1} \geq 1$, and therefore $T\mathbf{1} = \mathbf{1}$. Assuming $T\mathbf{1} = \mathbf{1}$ and $f \leq b = b\mathbf{1}$, we use positivity and linearity to infer $Tf \leq T(b\mathbf{1}) = bT\mathbf{1} = b\mathbf{1} \leq b$. $\square$

Now we can prove the main result.

Theorem 5.3. *Let $E \subset \mathfrak{M}(\Omega, \mu)$ be a Banach function space, $T : E \longrightarrow \mathfrak{M}(\Omega', \nu)$ a positive, linear, continuous operator. Let X be a Banach space and $C \subset X$ a closed convex subset of X. Then the implication*

$$f \in E(X),\ f(\omega) \in C \ (\mu\text{-almost all } \omega) \quad \Longrightarrow \quad [Tf](\omega') \in C \ (\nu\text{-almost all } \omega')$$

holds in each of the following cases:

a) C is a cone (and no further restrictions on T).

b) $0 \in C$ and T is submarkovian.

c) T is markovian (and no further restrictions on C).

Proof. Assume a) and take $f \in E(X)$ such that $f \in C$ μ-a.e.. Since each f is essentially separably-valued, one can assume without restriction that X is separable. Apply Corollary 4.3 and find a countable set $A \subset X'$ with $x \in C \quad \Leftrightarrow \quad \langle x', x\rangle \leq 0$ for all $x' \in A$. Hence we have $\langle x', f\rangle \leq 0$ and therefore, by (4) and the positivity of T, also $\langle x', Tf\rangle = T\,\langle x', f\rangle \leq 0$ for all $x' \in A$. Since A is countable, this implies $Tf \in C$ ν-a.e..

The proof in the other cases b) and c) is essentially the same. The properties required for T in each case ensure that the linear relations $\langle x', f\rangle \leq b$ which describe C are preserved by T. $\square$

By turning to the so-called *epigraph*

$$\operatorname{epi}(F) := \{(x, t) \mid x \in C,\ F(x) \leq t\} \subset X \oplus \mathbb{R} \tag{5}$$

of a convex mapping F, we obtain the following.

Theorem 5.4. *Let the assumptions be as in Theorem 5.3. Let $F : C \longrightarrow \mathbb{R}$ be a mapping which is convex and lower semicontinuous. Then the implication*

$$f \in E(X),\ f(\omega) \in C \ (\mu\text{-almost all } \omega),\ F(f) \in E \quad \Longrightarrow \quad F(Tf) \leq T[F(f)]$$

holds in each of the following cases:

a) C is a cone, F is positively homogeneous (and no further restrictions on the operator T).

b) $0 \in C$, $F(0) \leq 0$ and T is submarkovian.

c) T is markovian (and no further restrictions on C or F).

Proof. Consider the epigraph $\operatorname{epi}(F)$ of F, see (5). Since F is lower semicontinuous and convex, $\operatorname{epi}(F)$ is a closed convex subset of $X \oplus \mathbb{R}$. In case b), $0 \in \operatorname{epi}(F)$ and in case a), $\operatorname{epi}(F)$ is even a cone. Hence we can apply Theorem 5.3 to the closed convex set $\operatorname{epi}(F)$. Namely, if we start with $f \in E(X)$, $f \in C$ (μ-a.e.), $F(f) \in E$ we see that $g := (f, F(f)) \in E(X \oplus \mathbb{R})$ and $g \in \operatorname{epi}(F)$ (μ-a.e.). By Theorem 5.3 we obtain $(Tf, T[F(f)]) = Tg \in \operatorname{epi}(F)$ (ν-a.e.) which means $F(Tf) \le T[F(f)]$. □

Remark 5.5. Although concealed, the proof of Theorem 5.4 is essentially the same as the proof of Hölder's ineequality 2.1. Namely, let C and F be as in the hypotheses of Theorem 5.4. Then one has

$$\operatorname{epi}(F) = \bigcap_{(x',r,b)\in A} \{(x,t) \mid \langle x', x\rangle + rt \le b\}$$

for some subset $A \subset X' \times \mathbb{R} \times \mathbb{R}$ which can be chosen countable when X is separable. One can conclude from this that

$$F(x) = \sup_{(x',b)\in B} \langle x', x\rangle - b \quad (x \in C).$$

for a certain subset $B \subset X' \times \mathbb{R}$ wich can be chosen to be countable if X is separable. In case a) all $b = 0$, and in case b) all $b \ge 0$. Now the proof of Proposition 2.1 can be transferred to this situation.

6. Applications and Remarks

6.1. **Vector-Valued Conditional Expectation.** Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $\mathcal{F} \subset \Sigma$ a sub-σ-algebra. The associated *conditional expectation operator*

$$T := \mathbb{E}(\cdot \,|\, \mathcal{F}) : \mathbf{L^1}(\Omega, \Sigma, \mathbb{P}) \longrightarrow \mathbf{L^1}(\Omega, \mathcal{F}, \mathbb{P})$$

is a positive, linear, and markovian operator. If X is any Banach space, as we have seen in Section 3, this operator has a natural extension to an X-valued analogue

$$T := \mathbb{E}(\cdot \,|\, \mathcal{F}) : \mathbf{L^1}(\Omega, \Sigma, \mathbb{P};\, X) \longrightarrow \mathbf{L^1}(\Omega, \mathcal{F}, \mathbb{P};\, X).$$

For each $f \in \mathbf{L^1}(\Omega, \Sigma, \mathbb{P};\, X)$, the function Tf is is characterized by the properties

$$Tf \in \mathbf{L^1}(\Omega, \mathcal{F}, \mathbb{P};\, X), \quad \text{and} \quad \int_A f\, d\mathbb{P} = \int_A Tf\, d\mathbb{P}, \quad (A \in \mathcal{F}).$$

Our approach shows that the *ad hoc* constructions prevalent in the literature can be replaced by a general scheme. Moreover, applying Theorem 5.4 to this special case also yields the *Jensen Inequality* for (scalar- and vector-valued) conditional expectations. Compare [3, Chapter V.1].

6.2. **The Riesz-Thorin Theorem.** Recall the setting of the Riesz-Thorin Interpolation Theorem (for positive operators). Namely, assume one is given measure spaces (Ω, μ) and (Ω', ν) and a positive linear operator $T : \operatorname{St}(\Omega, \mu; \mathbb{R}) \longrightarrow \mathfrak{M}(\Omega', \nu; \mathbb{R})$ which satisfies the inequalities

$$\|Tf\|_{q_0} \le M_0 \|f\|_{p_0}, \qquad \|Tf\|_{q_1} \le M_1 \|f\|_{p_1}$$

for all $f \in \operatorname{St}(\Omega, \mu)$. Here $1 \le p_k, q_k \le \infty$ for $k = 0, 1$ are arbitrary exponents and M_0, M_1 are positive constants.

Result: *In the situation above, let $\theta \in (0,1)$ and define $1 \le p, q \le \infty$ by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then $\|Tf\|_q \le M_0^{1-\theta} M_1^{\theta} \|f\|_p$ for all $f \in \operatorname{St}(\Omega, \mu)$.

Proof. By the usual Liapunov inequality one can assume $p_0 \neq p_1$, and obviously the assumption $p_0 \leq p_1$ is not a restriction. Define $1 < r < \infty$ and $1 \leq s, l \leq \infty$ by

$$\frac{1}{r} := \theta, \ \frac{1}{s} = \frac{(1-\theta)p}{p_0}, \ \frac{1}{l} = \frac{(1-\theta)q}{q_0}.$$

Then $\frac{1}{r'} = 1-\theta, \frac{1}{s'} = \frac{\theta p}{p_1}$ and $\frac{1}{l'} = \frac{\theta q}{q_1}$. (We will use the pair (l, l') only when $q < \infty$ in which case the definition makes sense.)

To prove the assertion, by $|Tf| \leq T|f|$ one can restrict to positive f. Then we can apply the Hölder inequality to obtain

$$Tf = T(f^{\frac{1}{s'}} f^{\frac{1}{s}}) \leq (Tf^{\frac{r'}{s'}})^{\frac{1}{r'}} (Tf^{\frac{r}{s}})^{\frac{1}{r}} = (Tf^{\frac{p}{p_1}})^{\theta} (Tf^{\frac{p}{p_0}})^{1-\theta}. \tag{6}$$

In the case $q_0 = q = q_1 = \infty$ and $p_1 < \infty$ we can continue

$$\leq \left(M_1 \left(\int (f^{\frac{p}{p_1}})^{p_1}\right)^{\frac{1}{p_1}}\right)^{\theta} \left(M_0 \left(\int (f^{\frac{p}{p_0}})^{p_0}\right)^{\frac{1}{p_0}}\right)^{1-\theta} = M_1^{\theta} M_0^{1-\theta} \left(\int f^p\right)^{\frac{1}{p}}$$

which is what we wanted. If $p_1 = \infty$, obvious modifications to this argument also yield the desired estimate.

Now, let us assume that $q_0, q_1, p_1 < \infty$. Then we conclude from (6) by means of the Hölder inequality for integrals that

$$\begin{aligned}
\int [Tf]^q &\leq \int (Tf^{\frac{p}{p_1}})^{q\theta} (Tf^{\frac{p}{p_0}})^{q(1-\theta)} \leq \left(\int (Tf^{\frac{p}{p_1}})^{q\theta l'}\right)^{\frac{1}{l'}} \left(\int (Tf^{\frac{p}{p_0}})^{q(1-\theta)l}\right)^{\frac{1}{l}} \\
&= \left(\int (Tf^{\frac{p}{p_1}})^{q_1}\right)^{\frac{1}{l'}} \left(\int (Tf^{\frac{p}{p_0}})^{q_0}\right)^{\frac{1}{l}} \\
&\leq M_1^{\frac{q_1}{l'}} M_0^{\frac{q_0}{l}} \left(\int f^p\right)^{\frac{q_1}{p_1 l'}} \left(\int f^p\right)^{\frac{q_0}{p_l}} = M_1^{q\theta} M_0^{q(1-\theta)} \left(\int f^p\right)^{\frac{q}{p}}.
\end{aligned}$$

Again, if one of the exponents p_1, q_0, q_1 equals infinity, obvious changes of the above computations have to be made. □

Remark 6.1. We note that in order to prove the result we do not actually need T to be linear nor that T is defined on step functions only. More precisely, T might be defined on a vector sublattice E of $\mathfrak{M}(\Omega, \mu)$, with the following properties

$$0 \leq f \leq g \rightarrow Tf \leq Tg, \quad |T(f+g)| \leq T|f| + T|g|, \quad T(\alpha f) = \alpha Tf$$

for all $f, g \in E, \alpha > 0$. Such an operator may arise as a maximal operator $Tf := \sup_n T_n |f|$ for a set of linear positive operators T_n.

Of course, the linear version of the Riesz-Thorin theorem given here is by no means new. In fact, it follows from the usual (complex) Riesz-Thorin theorem by complexification. The inequality $|Tf| \leq T|f|$ (which is Lemma 3.4 of course) ensures that the constants M_0, M_1 do not change during this process. However it seems interesting that one can give a totally "real" proof when the operator T is positive. Krengel in [6, p.65] notes the special case of $p_0 = q_0 = \infty, p_1 = q_1 = 1$. The same can be found in Schaefer [9, Prop. V.8.2] ("The Little Riesz Theorem") with a much more involved proof using the Stone representation space. For general indices $p_0 = q_0, q_1 = p_1$, the proof is implicit in [5, p.246]. However, the "sublinear version" (cf. Remark 6.1) may be new.

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