GAUSSIAN ESTIMATES FOR SECOND ORDER OPERATORS WITH UNBOUNDED COEFFICIENTS

STEFAN KARRMANN

Abstract. We study second order differential operators A with lower order coefficients in some $\mathbf{L}_q + \mathbf{L}_\infty$. We prove the generation of positive, quasi-contractive C_0-semigroups on $\mathbf{L}_p$ for all $p \in (1, \infty)$. If the second order coefficients are in some $\mathbf{L}_q + \mathbf{L}_\infty$ we get upper pseudo Gaussian bounds of the heat kernel.

Maximal regularity, spectral independence on $\mathbf{L}_p$, and analyticity of the generated semigroup on $\mathbf{L}_1$ are studied for these operators.

Introduction

There is now an extensive literature on Gaussian bounds for heat kernels associated with strongly elliptic operators (see Davies [8], Robinson [19] and references therein). The best upper bounds have been derived by a technique introduced by Davies in [7]. We use this technique and ideas developed in Stampacchia [20], Daners [6], Arendt and ter Elst [2] to analyse the Cauchy Problem

$$\left.\begin{aligned} d_t u(t) + Au(t) &= 0 && \text{on } \Omega, \text{ for all } t \in (0,\infty) \\ u(0) &= u_0 && \text{on } \Omega \\ u(t) &= 0 && \text{on } \partial\Omega, \text{ for all } t \in (0,\infty) \end{aligned}\right\} \tag{CP}$$

where Ω is an open subset of $\mathbf{R}^n$ and the operator A is associated with the closure of the second order form $\mathcal{E}$ given by

$$\mathcal{E}(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{i,j}(x)\partial_i u(x)\partial_j v(x)\,dx + \sum_{j=1}^{n} \int_\Omega b_{1,j}(x)\partial_i u(x) v(x)\,dx + \\ + \sum_{j=1}^{n} \int_\Omega b_{2,j}(x) u(x)\partial_j v(x)\,dx + \int_\Omega c(x)u(x)v(x)\,dx. \tag{1}$$

This form is well defined for all $u, v \in \mathbf{W}^1_{\infty,c}$ if the coefficients $a_{i,j}, b_{l,j}$ and c are in $\mathbf{L}_{1,\mathrm{loc}}$ and real valued.

Suppose $c_- \in \mathbf{Ł}_q := \mathbf{L}_q + \mathbf{L}_\infty$ and $b_{l,j} \in \mathbf{Ł}_{2q}$ for some $q \geq 1$. Then we show that the operator $-A$ generates a semigroup T on $\mathbf{L}_2$. We prove that in this case T can be extended to consistent, quasi-contractive, positive C_0-semigroups on $\mathbf{L}_p$ for all $p \in (1, \infty)$. If $b_{l,j} \in \mathbf{Ł}_p$ and $c_- \in \mathbf{Ł}_q$ for some $p > 2$, $q > 1$ then $T(t)$ is in $\mathcal{L}(\mathbf{L}_1, \mathbf{L}_\infty)$ and is quasi

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ultra-contractive, in particular $\|T(t)\|_{\mathcal{L}(\mathbf{L}_1,\mathbf{L}_\infty)} \leq Ct^{-n/2}e^{\omega t}$. Of course, in this situation T is associated with a kernel k such that

$$T(t)f(x) = \int_\Omega k(t,x,y)f(y)\,dy \qquad \text{for all } f \in \mathbf{L}_1. \tag{2}$$

Assume, in addition, that $a_{i,j} \in \text{Ł}_p$ for some $p > 2$. Then we prove a pseudo Gaussian bound of order $m \geq 2$, explicitly

$$|k(t,x,y)| \leq Ct^{-n/2}\exp(\omega_1 t - \omega_2(|x-y|^m/t)^{\frac{1}{m-1}}) \tag{3}$$

for almost all $x, y \in \Omega, t \in (0,\infty)$ and some $C, \omega_i > 0$. If $a_{i,j} \in \mathbf{L}_\infty$ we show $m = 2$ and obtain a classical Gaussian kernel as an upper bound. This extends the result of Aronson [4], Arendt and ter Elst [2] and Daners [6]. Using estimate (2) we obtain some results about spectral independence, maximal regularity and analyticity.

The plan of this article is as follows. In section 1 we state the exact assumptions and the main results. Then we do some estimates for the coefficients in section 2. In section 3 we derive the key estimate (6) and use it to show that the form $\mathcal{E}$ generates a C_0-semigroup on $\mathbf{L}_2$. That $\mathcal{E}$ generates quasi-dissipative C_0-semigroups on $\mathbf{L}_p$ $(1 < p < \infty)$ is proved in section 4. There we give an $\mathbf{L}_1$-$\mathbf{L}_\infty$ estimate, too. In section 5 we derive the upper Gaussian estimates and apply these results in section 6.

In this article we fix the open set $\Omega \subset \mathbf{R}^n$ and use the following notation. By $\mathbf{N}$ and $\mathbf{R}$ we denote the set of all natural numbers (without zero) and the set of all real numbers, respectively. We use $\mathbf{R}^+ = [0,\infty)$ for short. We write $\mathbf{L}_p$ to denote the real Banach space of all functions from Ω into $\mathbf{R}$ with norm $\|f\|_p = \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p}$, if $1 \leq p < \infty$ and $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\Omega} |f(x)|$. We use $\text{Ł}_p = \mathbf{L}_p + \mathbf{L}_\infty$ for short. If $p = 2$ then $\mathbf{L}_2$ is a Hilbert space with scalar product $\langle u, v\rangle = \int_\Omega u(x)v(x)\,dx$. By $\mathbf{L}_{1,\text{loc}}$ we denote the set of all real, measurable functions, that are integrable on every compact subset of Ω. The space $\mathbf{W}^1_{\infty,c}$ is the subset of $\mathbf{L}_\infty$, whose functions have compact support and (weak) derivatives in $\mathbf{L}_\infty$, while $\mathcal{D}$ is the set of all infinitely differentiable functions on Ω with compact support. By $\mathbf{H}^1$ we denote the Hilbert space of all functions in $\mathbf{L}_2$, whose first (weak) derivatives are in $\mathbf{L}_2$, too, with scalar product $\langle u, v\rangle_{\mathbf{H}^1} = \langle u, v\rangle + \sum_{k=1}^n \langle \partial_k u, \partial_k v\rangle$. Finally, $\mathbf{H}^1_0$ denotes the closure of $\mathcal{D}$ with respect to $\mathbf{H}^1$.

For functions with two arguments like $f(t,x)$ we write $f(t)$ for the mapping $x \mapsto f(t,x)$ and $f(x)$ for the mapping $t \mapsto f(t,x)$ if there is no ambiguity. By $f_\pm$ we denote $\max\{0, \pm f\}$ for short, while $\operatorname{sgn} f = \mathbf{1}_{\{x; f(x)\neq 0\}} f/|f|$ where $\mathbf{1}_A$ is the characteristic function of the set A. Given a form $\mathcal{E}$ on a Hilbert space H, we write $\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha\langle u, v\rangle_H$ where $\alpha \in \mathbf{R}$.

A family $(A_p)_{p\in\Pi}$ with $\Pi \subset [1,\infty]$ and $A_p : \mathbf{L}_p \to \mathbf{L}_p$ for all $p \in \Pi$ is called *consistent*, if for all $p, q \in \Pi, f \in \mathbf{L}_p \cap \mathbf{L}_q$ holds $A_p f = A_q f$. A continuous form $\mathcal{E}$ on some Hilbert space H is called *coercive*, if there exists a constant $c > 0$ such that $\mathcal{E}(u,u) \geq c\|u\|_H^2$ for all $u \in D(\mathcal{E})$.

1. Conditions and Main Results

We distinguish three different conditions; each one enables us to state interesting theorems.

Condition 1.1. The coefficients of the form $\mathcal{E}$ satisfy

(a) the growth condition: We have $a_{i,j}, c \in \mathbf{L}_{1,\mathrm{loc}}$, $b_{l,j} \in \mathbf{L}_{2q}$ for $i,j = 1,\dots,n$, $l = 1,2$ and $c_- \in \mathbf{L}_q$ where

$$q \in \begin{cases} \{1\} & \text{if} \quad n = 1 \\ (1,\infty) & \text{if} \quad n = 2 \\ [n/2,\infty) & \text{if} \quad n > 2 \end{cases}.$$

(b) the strong ellipticity condition: There exists $\nu \in (0,\infty)$ such that

$$\nu\,|\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \tag{4}$$

for all $\xi \in \mathbf{R}^n$ and almost all $x \in \Omega$.

(c) the antisymmetric boundedness condition: For $i,j = 1,\dots,n$ we have $a_{i,j} - a_{j,i} \in \mathbf{L}_\infty$.

These requirements are quite weak. The second order coefficients are as arbitrary as possible and the lower order ones may still be unbounded. The boundedness of $a_{i,j} - a_{j,i}$ is only needed in Lemma 3.5 to prove the continuity of $\mathcal{E}$. The closure of $\mathcal{E}$ is associated with an generator A and its generated C_0-semigroup T. These weak assumptions are strong enough to prove

Theorem 1.2. *If Condition 1.1 is satisfied then there is a consistent family $(T_p)_{p\in(1,\infty)}$ of quasi-contractive, positive C_0-semigroups T_p on $\boldsymbol{L}_p$ with generator $-A_p$ and $A_2 = A$.*

We cannot expect to get always a quasi-contractive semigroup on $\mathbf{L}_1$. There is a counter example by Ouhabaz [18, Remark 4.3.a].

Condition 1.3. In addition to Condition 1.1 we assume that the coefficients of the form $\mathcal{E}$ satisfy $b_{l,j} \in \mathbf{L}_{p_1}$ for all $j = 1,\dots,n, l = 1,2$ and $c_- \in \mathbf{L}_{p_0}$ where $p_1 > 2q$ and $p_0 > q$; i.e., any choice of

$$p_1 \in \begin{cases} (2,\infty] & \text{if} \quad n = 1 \\ (n,\infty] & \text{if} \quad n \ge 2 \end{cases} \qquad\qquad p_0 \in \begin{cases} (1,\infty] & \text{if} \quad n = 1 \\ (n/2,\infty] & \text{if} \quad n \ge 2 \end{cases}$$

is possible, if the coefficients are appropriate.

This condition is strong enough to prove the following kernel bound.

Theorem 1.4. *If Condition 1.3 is satisfied then T is ultra-contractive. Moreover*

$$\|T(t)\|_{\mathcal{L}(\boldsymbol{L}_1,\boldsymbol{L}_\infty)} \le Ct^{-n/2}e^{(\beta)t} \qquad \textit{for all} \quad t \in (0,\infty)$$

and some $C, \beta > 0$ and $\delta(2) > 0$ (see Theorem 3.3 and Remark 4.7).

Condition 1.5. The coefficients of the form $\mathcal{E}$ satisfy condition 1.3 and $a_{i,j} \in \mathbf{L}_{p_2}$ for all $i,j = 1,\dots,n$ and some $p_2 > 2q$.

Theorem 1.6. *If Condition 1.5 is satisfied then the associated semigroup T is given by $(T(t)f)(x) = \int_\Omega k(t,x,y)f(y)\,dy$ and the kernel $k : (0,\infty)\times\Omega\times\Omega \to \boldsymbol{R}$ satisfies $0 \le k(t,x,y)$ and a pseudo Gaussian estimate of order m; i.e.,*

$$|k(t,x,y)| \le Ct^{-n/2}\exp(\omega_1 t - \omega_2(|x-y|^m/t)^{\frac{1}{m-1}}) \tag{5}$$

for all $x,y \in \Omega, t \in (0,\infty)$, some $C, d, \omega_i > 0$ and $m = 2 + \frac{4q}{p_2 - 2q}$. If $m = 2$ then we have the classical Gaussian estimate (of order 2).

Theorem 1.7. *If Condition 1.5 is satisfied then $\sigma(A_p) = \sigma(A_2)$ for all $p \in (1,\infty)$. If the semigroup extends to a C_0-semigroup on $\boldsymbol{L}_1$ with generator A_1 (see Remark 6.4) then $\sigma(A_1) = \sigma(A_2)$, too.*

Theorem 1.8. *If Condition 1.5 is satisfied and $a_{i,j} \in \boldsymbol{L}_\infty$ for all $i,j = 1,\dots,n$ then A has maximal $\boldsymbol{L}_p$-$\boldsymbol{L}_q$ regularity for all $1 < p,q < \infty$; i.e., for every $f \in \boldsymbol{L}_p(\boldsymbol{R}^+, \boldsymbol{L}_q(\Omega))$ exists an unique solution $u \in \boldsymbol{W}_p^1(\boldsymbol{R}^+, \boldsymbol{L}_q(\Omega)) \cap \boldsymbol{L}_p(\boldsymbol{R}^+, D(A_q))$ of $u'(t) = Au(t) + f(t)$ for $t \in \boldsymbol{R}^+$ and $u(0) = 0$.*

Theorem 1.9. *If Condition 1.5 is satisfied and the associated semigroup $T : \Sigma_\theta \to \mathcal{L}(\boldsymbol{L}_2)$ is analytic with $\Sigma_\theta = \left\{re^{\mathrm{i}\varphi}; r > 0, |\varphi| < \theta\right\}$ where $\theta \in (0,\pi/2]$, then for all $0 < \theta_1 < \theta$ there is an analytic mapping $T_1 : \Sigma_{\theta_1} \to \mathcal{L}(\boldsymbol{L}_1)$, such that $T(z)$ and $T_1(z)$ are consistent for all $z \in \Sigma_{\theta_1}$.*

If we have a classical Gaussian estimate of the kernel then we obtain an analytic semigroup on $\boldsymbol{L}_1$.

2. Coefficients

As we are discussing quite general forms $\mathcal{E}$, we introduce $\mathbf{Ł}_p$ a larger space than $\mathbf{L}_p$ as the space where the coefficients are taken from.

Definition 2.1. For $1 \le p \le \infty$ let $\mathbf{Ł}_p = \mathbf{L}_p + \mathbf{L}_\infty$. Moreover, for $0 \le \epsilon < \infty$ and $f \in \mathbf{Ł}_p$ let

$$[f]_{\epsilon,p} = \inf\left\{k \in [0,\infty]; \exists f_i \in \mathbf{Ł}_p : f = f_1 + f_2, \|f_1\|_p \le \epsilon, \|f_2\|_\infty \le k\right\}$$

and $\|f\|_{\mathbf{Ł}_p} = \inf\left\{\epsilon + [f]_{\epsilon,p}; 0 \le \epsilon < \infty\right\}$.

Remark 2.2. Clearly $(\mathbf{Ł}_p, \|\cdot\|_{\mathbf{Ł}_p})$ is a Banach space and $[f]_{\epsilon,p}$ is non-increasing in ϵ. For all $f,g \in \mathbf{Ł}_p$ with $0 \le f \le g$ we have $[f]_{\epsilon,p} \le [g]_{\epsilon,p}$.

Some other properties of the spaces $\mathbf{Ł}_p$ are collected in

Lemma 2.3. *Let $1 \le p \le \infty, f_i \in \mathbf{Ł}_p, g \in \boldsymbol{L}_\infty$ and $0 \le \epsilon < \infty$. Then*

(i) $[g]_{\epsilon,p} \le \|g\|_\infty$,
(ii) *for all* $\theta \in [0,1]$ *we have* $[f_1 + f_2]_{\epsilon,p} \le [f_1]_{\theta\epsilon,p} + [f_2]_{(1-\theta)\epsilon,p}$,
(iii) *for all* $\rho \in \boldsymbol{R}$ *we have* $[\rho f_1]_{\epsilon|\rho|,p} = |\rho|\,[f_1]_{\epsilon,p}$,
(iv) *If* $p < q < \infty$, $\epsilon \ne 0$ *and* $f \in \boldsymbol{L}_q$ *then* $f \in \mathbf{Ł}_p$ *and* $[f]_{\epsilon,p} \le \|f\|_q^{\frac{q}{q-p}}\, \epsilon^{-\frac{p}{q-p}}$.

Proof. (i) and (ii) are clear by the definition of $[f]_\epsilon$. We show (iii).

$$\begin{aligned}
[\rho f_1]_{\epsilon|\rho|,p} &= \inf\left\{k \in [0,\infty]; \exists g_i \in \mathbf{Ł}_p : f = g_1 + g_2, \|\rho g_1\|_p \le \epsilon\,|\rho|\,, \|\rho g_2\|_\infty \le k\right\} \\
&= |\rho| \inf\left\{k \in [0,\infty]; \exists g_i \in \mathbf{Ł}_p : f = g_1 + g_2, \|g_1\|_p \le \epsilon, \|g_2\|_\infty \le k\right\} \\
&= |\rho|\,[f_1]_{\epsilon,p}
\end{aligned}$$

To proof (iv) let $k = \|f\|_q^{\frac{q}{q-p}} \epsilon^{-\frac{p}{q-p}}$ and set $f_k = \min\{k, \max\{-k, f\}\}$, $g_k = f - f_k$. Then we have $\|f_k\|_\infty \le k$ and

$$\|g_k\|_p \le \left\|f \mathbf{1}_{\{|f|>k\}}\right\|_p \le \|f\|_q \left\|\mathbf{1}_{\{|f|>k\}}\right\|_1^{1/p-1/q} \le \|f\|_q \left\|\frac{|f|}{k}\right\|_q^{q(1/p-1/q)} \le \|f\|_q^{q/p} k^{-(q/p-1)} = \epsilon.$$

Thus, $[f]_{\epsilon,p} \le k$. □

To estimate terms in which the coefficients appear, we need the well known

Lemma 2.4 (Sobolev embedding). *Let $n \in \boldsymbol{N}$ and*

$$q \in \begin{cases} \{\infty\} & \text{if} \quad n = 1 \\ [2,\infty) & \text{if} \quad n = 2 \\ [2, \frac{2n}{n-2}] & \text{if} \quad n > 2 \end{cases}.$$

Then exists a constant $C \in \boldsymbol{R}^+$ such that

$$\|u\|_q \le C(\|\nabla u\|_2 + \|u\|_2)$$

for all functions $u \in \mathcal{D}$. We call the infimum of such constants C the Sobolev embedding constant for n and q.

See for example Adams [1, Theorem 5.4].

Lemma 2.5. *Let* $q \in \begin{cases} \{2\} & \text{if} \quad n = 1 \\ (2,\infty] & \text{if} \quad n = 2 \\ [n,\infty] & \text{if} \quad n > 2 \end{cases}$. *For all $a_i \in \boldsymbol{Ł}_q$, $v, u \in \boldsymbol{W}^1_{\infty,c}$, $\epsilon \in (0,\infty)$ we have*

$$\sum_{i=1}^n \|a_i u \partial_i v\|_1 \le \epsilon\left(\|\nabla u\|_2 + \|u\|_2\right)\|\nabla v\|_2 + \left(\sum_{i=1}^n [a_i]^2_{\frac{\epsilon}{\sqrt{n}C_1},q}\right)^{1/2} \|u\|_2 \|\nabla v\|_2$$
$$\le \epsilon \|\nabla u\|_2 \|\nabla v\|_2 + \epsilon \|\nabla v\|_2^2 + \frac{1}{2\epsilon}\left(\epsilon^2 + \sum_{i=1}^n [a_i]_{\frac{\epsilon}{\sqrt{n}C_1},q}\right)\|u\|_2^2$$

with C_1 the Sobolev embedding constant for n and q.

Proof. For all $\eta > 0$ there are $b_i, c_i \in \mathbf{Ł}_q$ such that $a_i = b_i + c_i$, $\|b_i\|_q \le \frac{\epsilon}{\sqrt{n}C_1}$ and $\|c_i\|_\infty \le [a_i]_{\frac{\epsilon}{\sqrt{n}C_1},q} + \eta$. Let $1/2 = 1/p + 1/q$. Then, by Lemma 2.4,

$$\|a_i u \partial_i v\|_1 \le \|b_i u \partial_i v\|_1 + \|c_i u \partial_i v\|_1 \le \|b_i u\|_2 \|\partial_i v\|_2 + \|c_i\|_\infty \|u \partial_i v\|_1$$
$$\le \|b_i\|_q \|u\|_p \|\partial_i v\|_2 + \left([a_i]_{\frac{\epsilon}{\sqrt{n}C_1},q} + \eta\right)\|u \partial_i v\|_1$$
$$\le \frac{\epsilon}{\sqrt{n}}\left(\|\nabla u\|_2 + \|u\|_2\right)\|\partial_i v\|_2 + \left([a_i]_{\frac{\epsilon}{\sqrt{n}C_1},q} + \eta\right)\|u \partial_i v\|_1.$$

Letting $\eta \searrow 0$ we obtain

$$\|a_i u \partial_i v\|_1 \le \frac{\epsilon}{\sqrt{n}}\left(\|\nabla u\|_2 + \|u\|_2\right)\|\partial_i v\|_2 + [a_i]_{\frac{\epsilon}{\sqrt{n}C_1},q} \|u \partial_i v\|_1.$$

Summing up and using that $\sum_{i=1}^n \|\partial_i f\|_2 \le \sqrt{n}\,\|\nabla f\|_2$ we obtain

$$\begin{aligned}\sum_{i=1}^n \|a_i u \partial_i v\|_1 &\le \frac{\epsilon}{\sqrt{n}}\left(\|\nabla u\|_2 + \|u\|_2\right)\sum_{i=1}^n \|\partial_i v\|_2 + \sum_{i=1}^n [a_i]_{\frac{\epsilon}{\sqrt{n}C_1},q}\, \|u\,|\partial_i v|\|_1 \\ &\le \epsilon\left(\|\nabla u\|_2 + \|u\|_2\right)\|\nabla v\|_2 + \left(\sum_{i=1}^n [a_i]^2_{\frac{\epsilon}{\sqrt{n}C_1},q}\right)^{1/2} \|u\|_2\,\|\nabla v\|_2 \\ &\le \epsilon\,\|\nabla u\|_2\,\|\nabla v\|_2 + \epsilon\,\|\nabla v\|_2^2 + (2\epsilon)^{-1}\left(\epsilon^2 + \sum_{i=1}^n [a_i]^2_{\frac{\epsilon}{\sqrt{n}C_1},q}\right)\|u\|_2^2\end{aligned}$$

by $xy \le \epsilon x^2 + (4\epsilon)^{-1}y^2$ for all $\epsilon > 0$. ☐

Lemma 2.6. *Let* $q \in \begin{cases}\{1\} & \textit{if} \quad n = 1\\ (1,\infty] & \textit{if} \quad n = 2\\ [n/2,\infty] & \textit{if} \quad n > 2\end{cases}$. *For all* $f \in \boldsymbol{Ł}_q, v, u \in \boldsymbol{W}^1_{\infty,c}, \epsilon \in \boldsymbol{R}^+$ *we have*

$$\begin{aligned}\|fuv\|_1 \le \epsilon\left(\|\nabla u\|_2\,\|\nabla v\|_2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2\,\|v\|_2\right) + \\ + \frac{\epsilon}{4}\left(\|u\|_2^2 + \|v\|_2^2\right) + [f]_{\frac{\epsilon}{C_2^2},q}\,\|uv\|_1\end{aligned}$$

with C_2 the Sobolev embedding constant for n and q.

Proof. For all $\eta > 0$ there are $g, h \in Ł_q$ such that $f = g + h$, $\|g\|_q \le \epsilon/C_2^2$ and $\|h\|_\infty \le [f]_{\epsilon/C_2^2,q} + \eta$. Let $1 = 2/p + 1/q$. Then, by Lemma 2.4,

$$\begin{aligned}\|fuv\|_1 &\le \|guv\|_1 + \|huv\|_1 \le \|gu\|_{\frac{p}{p-1}}\,\|v\|_p + \|h\|_\infty\,\|uv\|_1 \\ &\le \|g\|_q\,\|u\|_p\,\|v\|_p + \left([f]_{\frac{\epsilon}{C_2^2},q} + \eta\right)\|uv\|_1 \\ &\le \epsilon\left(\|\nabla u\|_2 + \|u\|_2\right)\left(\|\nabla v\|_2 + \|v\|_2\right) + \left([f]_{\frac{\epsilon}{C_2^2},q} + \eta\right)\|uv\|_1 .\end{aligned}$$

Letting $\eta \searrow 0$ we obtain

$$\begin{aligned}\|fuv\|_1 &\le \epsilon\left(\|\nabla u\|_2 + \|u\|_2\right)\left(\|\nabla v\|_2 + \|v\|_2\right) + [f]_{\frac{\epsilon}{C_2^2},q}\,\|uv\|_1 \\ &\le \epsilon\left(\|\nabla u\|_2\,\|\nabla v\|_2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2\,\|v\|_2\right) + \\ &\quad + \frac{\epsilon}{4}\left(\|u\|_2^2 + \|v\|_2^2\right) + [f]_{\frac{\epsilon}{C_2^2},q}\,\|uv\|_1\end{aligned}$$

by $xy \le x^2 + y^2/4$. ☐

3. Approximation

In this section we prove the key to this article, Theorem 3.3. We use this to prove, that $\mathcal{E}$ is a closable, coercive, continuous form. Since we need estimates about $\mathbf{L}_p$ in the next section, we use a substitution and the $\mathbf{L}_2$-norm. This enables us to approximate the $\mathbf{L}_p$-norm using a technique of Daners [6].

Definition 3.1. For any $x \in \Omega$, $u : \Omega \to \mathbf{R}$, $k \in [1, \infty)$, $p \in [2, \infty)$ let

$$\begin{aligned} f_{k,p}(x) &= \begin{cases} |x|^{p/2} & \text{if } |x| \leq k \\ k^{p/2} + \frac{p}{2} k^{\frac{p}{2}-1} (|x| - k) & \text{if } |x| > k \end{cases}, \\ g_{k,p}(x) &= f_{k,p}(x) \partial f_{k,p}(x), \\ w_{k,p,u}(x) &= f_{k,p}(u(x)), \\ v_{k,p,u}(x) &= g_{k,p}(u(x)). \end{aligned}$$

We put together some properties of these functions.

Lemma 3.2. *Let $u \in \boldsymbol{W}^1_{\mathrm{loc}}$, $k \in [1, \infty)$, $p \in [2, \infty)$, then*

$$\begin{gathered} f_{k,p}(x) = \begin{cases} |x|^{p/2} & \textit{if } |x| \leq k \\ \frac{p}{2} k^{\frac{p}{2}-1} \left(|x| - \frac{p-2}{p} k \right) & \textit{if } |x| > k \end{cases}, \\ f'_{k,p}(x) = \begin{cases} \frac{p}{2} |x|^{\frac{p}{2}-2} x & \textit{if } |x| \leq k \\ \frac{p}{2} k^{\frac{p}{2}-1} |x|^{-1} x & \textit{if } |x| > k \end{cases}, \\ f''_{k,p}(x) = \begin{cases} \frac{p}{2} \frac{p-2}{2} |x|^{\frac{p}{2}-2} & \textit{if } |x| \leq k \\ 0 & \textit{if } |x| > k \end{cases}, \\ f'_{k,p}, g'_{k,p} \in \boldsymbol{L}_\infty(\boldsymbol{R}), \\ w^2_{k,p,u} \leq u v_{k,p,u} \leq \frac{p}{2} w^2_{k,p,u}, \\ \partial_i u v_{k,p,u} = w_{k,p,u} \partial_i w_{k,p,u} \quad \textit{a.e.}, \\ |u \partial_i v_{k,p,u}| \leq (p-1) w_{k,p,u} |\partial_i w_{k,p,u}| \quad \textit{a.e.} \end{gathered}$$

Proof. We obtain the derivatives of $f_{k,p}$ by simple differentiation and see that $f'_{k,p}, g'_{k,p} \in \mathbf{L}_\infty(\mathbf{R})$. Now we fix k, p, u and write $w = w_{k,p,u}, v = v_{k,p,u}$ for short. Then we have $w^2 = |u|^p$ and $uv = \frac{p}{2} |u|^2$ if $|u(x)| \leq k$, while $\frac{w^2}{uv} = 1 - \frac{p-2}{p} \frac{k}{|u|} \in [2/p, 1]$ if $|u(x)| \geq k$. This leads to

$$w^2 \leq uv \leq \frac{p}{2} w^2.$$

The next identity is shown using $\partial_i uv = \partial_i u w f'_{k,p}(u) = w \partial_i w$ a.e. by Gilbarg and Trudinger [11, Theorem 7.8]. Again by [11], we have for $|u(x)| \leq k$

$$u \partial_i v = u \partial_i u \frac{p}{2} |u|^{p-2} \left(\frac{p}{2} + \frac{p-2}{2} \right) = (p-1) w \partial_i w \quad \text{a.e.}$$

and

$$u \partial_i v = u \partial_i u \left(\frac{p}{2} \right)^2 k^{p-2} = \frac{p}{2} k^{\frac{p}{2}-1} |u| \partial_i w \quad \text{a.e.}$$

if $|u(x)| \geq k$. As $k^{\frac{p}{2}-1} |u| = k^{\frac{p}{2}} + k^{\frac{p}{2}-1}(|u| - k) \leq w$ and $\frac{p}{2} \leq p - 1$ the last inequality of the Lemma is proved. □

In section 5 we need to control δ. Hence, we provide its value in the following theorem.

Theorem 3.3. *Let $k \geq 1, \eta, \gamma \in [0, 1), 2 \leq p < \infty$ and $u \in \boldsymbol{W}^1_{\infty,c}$. Assume Condition 1.1 is satisfied. Then*

$$\mathcal{E}(u, v_{k,p}) + \delta(p, \eta, \gamma) \|w_{k,p}\|^2_{\boldsymbol{L}_2} \geq \gamma \mathcal{E}'(u, v_{k,p}) + \nu\eta(1 - \gamma) \|\nabla w_{k,p}\|^2_{\boldsymbol{L}_2} \tag{6}$$

with

$$\delta(p, \eta, \gamma) = \frac{3(1 - \gamma)(1 - \eta)\nu}{8} + \frac{2(p - 1)^2}{(1 - \gamma)(1 - \eta)\nu} \sum_{\substack{j=1 \\ l=1,2}}^{n} [b_{l,j}]^2_{\frac{\nu(1-\eta)(1-\gamma)}{4\sqrt{n}C_1(p-1)}, 2q} + \frac{p}{2} [c_-]_{\frac{\nu(1-\eta)(1-\gamma)}{3C_2^2 p}, q} \tag{7}$$

$$\mathcal{E}'(u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{i,j}(x) \partial_i u(x) \partial_j v(x) \, dx$$

and C_1 (respectively C_2) the Sobolev constant for n and q (respectively $2q$). Let $\delta(p) = \delta(p, 1/2, 0)$ for short.

Proof. Let $E_k = \{x \in \Omega; |u(x)| \leq k\}$, $F_k = \Omega \setminus E_k$. Fix the constants k and p, the functions u, $w = w_{k,p,u}$ and $v = v_{k,p,u}$ in the rest of the proof. Note that $w, v \in \mathbf{W}^1_{\infty,c}$, cf. Gilbarg and Trudinger [11, Theorem 7.8]. This is the reason why we don't use $\mathcal{D}$ to define the form $\mathcal{E}$, although the closure would be the same.

On E_k we have

$$\partial_j w \partial_l w = \frac{p}{2(p - 1)} \partial_j u \partial_l v \quad \text{a.e.}$$

Applying this and Lemma 3.2 to (4) we obtain almost everywhere

$$\begin{aligned}
(1 - \gamma)\nu |\nabla w|^2 &\leq (1 - \gamma) \sum_{i,j=1}^{n} a_{i,j} \partial_i w \partial_j w \leq \frac{(1 - \gamma)p}{2(p - 1)} \sum_{i,j=1}^{n} a_{i,j} \partial_i u \partial_j v \\
&\leq -\gamma a'(\cdot, u, v) + a(\cdot, u, v) + \sum_{j=1}^{n} (|b_{1,j} \partial_j u v| + |b_{2,j} v \partial_j v|) + c_- |uv| \\
&\leq -\gamma a'(\cdot, u, v) + a(\cdot, u, v) + \sum_{j=1}^{n} (|b_{1,j} w \partial_j w| + (p - 1) |b_{2,j} w \partial_j w|) + \frac{p}{2} c_- w^2
\end{aligned}$$

where $a'(x, u, v) = \sum_{i,j=1}^{n} a_{i,j}(x) \partial_i u(x) \partial_j v(x)$ and $a(\cdot, u, v)$ is the integrand of (1).

On F_k we have $\partial_j w \partial_l w = \partial_j u \partial_l v$ almost everywhere. Applying this and Lemma 3.2 to (4) we obtain almost everywhere

$$\begin{aligned}
(1 - \gamma)\nu |\nabla w|^2 &\leq (1 - \gamma) \sum_{i,j=1}^{n} a_{i,j} \partial_i w \partial_j w \leq (1 - \gamma) \sum_{i,j=1}^{n} a_{i,j} \partial_i u \partial_j v \\
&\leq -\gamma a'(\cdot, u, v) + a(\cdot, u, v) + \sum_{j=1}^{n} (|b_{1,j} \partial_j u v| + |b_{2,j} v \partial_j v|) + c_- |uv| \\
&\leq -\gamma a'(\cdot, u, v) + a(\cdot, u, v) + \sum_{j=1}^{n} (|b_{1,j} w \partial_j w| + \frac{p}{2} |b_{2,j} w \partial_j w|) + \frac{p}{2} c_- w^2
\end{aligned}$$

As $2 \leq p$ and $E_k \cup F_k = \Omega$ the inequality for E_k holds on Ω. Integration over Ω gives:

$$\begin{aligned}(1-\gamma)\nu \left\|\nabla w\right\|_{\mathbf{L}_2}^2 &\leq -\gamma\mathcal{E}'(u,v) + \mathcal{E}(u,v) + (p-1)\sum_{\substack{j=1\\l=1,2}}^{n} \left\|b_{l,j} w \partial_j w\right\|_{\mathbf{L}_1} + \frac{p}{2}\left\|c_- w^2\right\|_{\mathbf{L}_1} \\ &\leq -\gamma\mathcal{E}'(u,v) + \mathcal{E}(u,v) + \frac{p}{2}\left(3\epsilon_2 \left\|\nabla w\right\|_{\mathbf{L}_2}^2 + \frac{3\epsilon_2}{2}\left\|w\right\|_{\mathbf{L}_2}^2 + [c_-]_{\frac{\epsilon_2}{C_2^2}}\left\|w\right\|_2^2\right) + \\ &\qquad + (p-1)2\epsilon_1 \left\|\nabla w\right\|_{\mathbf{L}_2}^2 + \frac{p-1}{2\epsilon_1}\left\|w\right\|_{\mathbf{L}_2}^2 \left(\epsilon_1^2 + \sum_{\substack{j=1\\l=1,2}}^{n} [b_{l,j}]^2_{\frac{\epsilon_1}{\sqrt{n}C_1}}\right),\end{aligned}$$

where we used Lemma 2.5, 2.6 and 2.3. By choosing $\epsilon_1 = \frac{\nu(1-\eta)(1-\gamma)}{4(p-1)}$, $\epsilon_2 = \frac{\nu(1-\eta)(1-\gamma)}{3p}$ the Theorem is proved. $\square$

We deduce from this the following corollary.

Corollary 3.4. *The form $\mathcal{E}_{\delta(2,\eta,\gamma)}$ is positive for all $\eta, \gamma \in [0,1)$ if condition 1.1 is satisfied.*

Proof. Choose $p = 2$ in (6) and use $u = v_{k,2,u} = w_{k,2,u} \operatorname{sgn} u$ to see the positivity. $\square$

Lemma 3.5. *If Condition 1.1 is satisfied then the form $\mathcal{E}_{\delta(2,\eta,\gamma)}$ is continuous with respect to its graph norm $\left\|u\right\|^2_{\mathcal{E}_{\delta(2,\eta,\gamma)}} = (1+\delta(2,\eta,\gamma))\left\|u\right\|_2^2 + \mathcal{E}(u,u)$ for all $\eta \in [0,1)$ and $\gamma \in (0,1)$.*

Proof. By Lemma 3.3 and equation (4) we know that $\mathcal{E}(u,u) + \delta(2,\eta,\gamma)\left\|u\right\|_2^2 \geq \gamma\mathcal{E}'(u,u) \geq \gamma\nu\left\|\nabla u\right\|_2^2 \geq 0$. Fix $\eta, \gamma \in [0,1)$ and let $\delta := \delta(2,\eta,\gamma)$ in the following.

Let $\check{\mathcal{E}}(u,v) = \frac{1}{2}(\mathcal{E}(u,v) + \mathcal{E}(v,u))$ and $\check{\mathcal{E}}'(u,v) = \frac{1}{2}(\mathcal{E}'(u,v) + \mathcal{E}'(v,u))$ be the anti-symmetric parts of $\mathcal{E}$ respective $\mathcal{E}'$. Choose $M > 0$ such that $|a_{i,j} - a_{j,i}| \leq M\nu$ for all $i, j = 1, \ldots, n$. Then the Hölder inequality and (4) lead to

$$\begin{aligned}4\left|\check{\mathcal{E}}'(u,v)\right|^2 &\leq \left(\sum_{i,j=1}^{n}\int_\Omega M\nu\left|\partial_i u \partial_i v\right|\right)^2 \leq M^2\left(\int_\Omega \nu\left|\nabla u\right|\left|\nabla v\right|\right)^2 \\ &\leq M^2\left\|\nu\nabla u\right\|_1\left\|\nu\nabla v\right\|_1 \leq M^2\mathcal{E}'(u,u)\mathcal{E}'(v,v) \\ &\leq M^2\gamma^{-2}\mathcal{E}_{\delta+1}(u,u)\mathcal{E}_{\delta+1}(v,v)\end{aligned}$$

for any $u, v \in \mathbf{W}^1_{\infty,c}$. Now we can estimate by Lemma 2.5

$$\begin{aligned}&2\left|\check{\mathcal{E}}(u,v) - \check{\mathcal{E}}'(u,v)\right| \\ &= \left|\sum_{j=1}^{n}(b_{1,j} - b_{2,l})(v\partial_j u - u\partial_j v)\right| \leq \sum_{j=1}^{n}(\left\|(b_{1,j} - b_{2,j})v\partial_j u\right\|_1 + \left\|(b_{1,j} - b_{2,j})u\partial_j v\right\|_1) \\ &\leq \epsilon(\left\|\nabla u\right\|_2 + \left\|u\right\|_2)\left\|\nabla v\right\|_2 + \left(\sum_{j=1}^{n}[b_{1,j} - b_{2,j}]_{\frac{\epsilon}{\sqrt{n}C_1},2q}\right)^{1/2}\left\|u\right\|_2\left\|\nabla v\right\|_2 + \\ &\quad + \epsilon(\left\|\nabla v\right\|_2 + \left\|v\right\|_2)\left\|\nabla u\right\|_2 + \left(\sum_{j=1}^{n}[b_{1,j} - b_{2,j}]_{\frac{\epsilon}{\sqrt{n}C_1},2q}\right)^{1/2}\left\|v\right\|_2\left\|\nabla u\right\|_2\end{aligned}$$

As $\gamma\nu \|\nabla u\|_2^2 \leq \gamma\mathcal{E}'(u,u) \leq \mathcal{E}_{\delta+1}(u,u)$ and $\gamma \|u\|_2^2 \leq \gamma\mathcal{E}_1'(u,u) \leq \mathcal{E}_{\delta+1}(u,u)$ by (4) and Lemma 3.3, we can continue the estimate by

$$2\left|\check{\mathcal{E}}(u,v) - \check{\mathcal{E}}'(u,v)\right| \leq 2\mathcal{E}_{\delta+1}(u,u)^{1/2}\mathcal{E}_{\delta+1}(v,v)^{1/2}\gamma^{-1} \left(\frac{\epsilon}{\nu} + \frac{\epsilon}{\sqrt{\nu}} + \frac{1}{\sqrt{\nu}}\left(\sum_{j=1}^{n}[b_{1,j} - b_{2,j}]_{\frac{\epsilon}{\sqrt{n}C_1},2q}\right)^{1/2}\right)$$

Using these inequalities we obtain

$$\begin{aligned}\left|\check{\mathcal{E}}(u,v)\right| &\leq \left|\check{\mathcal{E}}'(u,v)\right| + \left|\check{\mathcal{E}}(u,v) - \check{\mathcal{E}}'(u,v)\right| \\ &\leq \mathcal{E}_{\delta+1}(u,u)^{1/2}\mathcal{E}_{\delta+1}(v,v)^{1/2}\gamma^{-1}\left(\frac{\epsilon}{\nu} + \frac{\epsilon}{\sqrt{\nu}} + \frac{1}{\sqrt{\nu}}\left(\sum_{j=1}^{n}[b_{1,j} - b_{2,j}]_{\frac{\epsilon}{\sqrt{n}C_1},2q}\right)^{1/2} + M/2\right).\end{aligned}$$

Together with the Cauchy-Schwarz inequality for the symmetric part of $\mathcal{E}_\delta$ this implies

$$|\mathcal{E}_\delta(u,v)| \leq K\mathcal{E}_{\delta+1}(u,u)^{1/2}\mathcal{E}_{\delta+1}(v,v)^{1/2}$$

with $K = 1+\delta+\gamma^{-1}\left(\frac{\epsilon}{\nu} + \frac{\epsilon}{\sqrt{\nu}} + \frac{1}{\sqrt{\nu}}\left(\sum_{j=1}^{n}[b_{1,j} - b_{2,j}]_{\frac{\epsilon}{\sqrt{n}C_1},2q}\right)^{1/2} + \frac{M}{2}\right)$ for all $\epsilon > 0$. This ends the proof of the Lemma. □

Lemma 3.6. *If Condition 1.1 is satisfied, then the form $\mathcal{E}_{\delta(2,\eta,\gamma)}$ is closable for all $\eta \in [0,1)$ and $\gamma \in (0,1)$. The closure is a coercive form. The forms $\mathcal{E}_{\delta(2,\eta,\gamma)}$ and $\mathcal{E}'$ define equivalent norms.*

Proof. Let $\hat{\mathcal{E}}'(u,v) = \frac{1}{2}(\mathcal{E}'(u,v) - \mathcal{E}'(v,u))$ be the symmetric part of $\mathcal{E}'$. Fix $\eta, \gamma \in [0,1)$ and let $\hat{\delta} = \delta(2,\eta,\gamma)$ in the following. Using Lemma 2.5 and 2.6 we see that

$$\begin{aligned}|\mathcal{E}(u,u)| &\leq \left|\hat{\mathcal{E}}'(u,u)\right| + \sum_{j=1}^{n}\int_\Omega |b_{1,j} + b_{2,j}|\,|u\partial_j u| + \int_\Omega |c|\,u^2 \\ &\leq \hat{\mathcal{E}}'(u,u) + 2\epsilon\|\nabla u\|_2^2 + \frac{1}{2\epsilon}\left(\epsilon^2 + \sum_{j=1}^{n}[b_{1,j} + b_{2,j}]_{\frac{\epsilon}{\sqrt{n}C_1},2q}\right)\|u\|_2^2 + \\ &\quad + 3\epsilon\|\nabla u\|_2^2 + (\epsilon + \frac{1}{2\epsilon} + [c]_{\frac{\epsilon}{C_2^2},q})\|u\|_2^2 \\ &\leq \mathcal{E}_1'(u,u)\left(1 + \frac{5\epsilon}{\nu} + \frac{3\epsilon}{2} + \frac{1}{2\epsilon} + \frac{1}{2\epsilon}\sum_{j=1}^{n}[b_{1,j} + b_{2,j}]_{\frac{\epsilon}{\sqrt{n}C_1},2q} + [c]_{\frac{\epsilon}{C_2^2},q}\right),\end{aligned}$$

by (4) and the inequality $\mathcal{E}(u,u)+\delta\|u\|_2^2 \geq \gamma\hat{\mathcal{E}}'(u,u) \geq \gamma\nu\|u\|_2^2 \geq 0$ derived from Theorem 3.3. Using these inequalities we obtain

$$c^{-1}\mathcal{E}_1'(u,u) \leq \mathcal{E}_{\hat{\delta}+\gamma}(u,u) \leq \mathcal{E}_{\hat{\delta}+1}(u,u) \leq c\mathcal{E}_1'(u,u)$$

with $c = \gamma^{-1} + \hat{\delta} + 2 + \frac{5\epsilon}{\nu} + \frac{3\epsilon}{2} + \frac{1}{2\epsilon} + \frac{1}{2\epsilon}\sum_{j=1}^{n}[b_{1,j} + b_{2,j}]_{\frac{\epsilon}{\sqrt{n}C_1},2q} + [c]_{\frac{\epsilon}{C_2^2},q}$. Thus the associated norms are equivalent. By Ma and Röckner [15, I.3 Proposition 3.5] the form $\mathcal{E}_{\hat{\delta}+\gamma}$ is closable since $\mathcal{E}'$ is closable, see [15, II.2d Case 1].

Since $\mathcal{E}_\delta$ is continuous (Lemma 3.5), we have $|\mathcal{E}_\delta(u,v)|^2 \le K^2\mathcal{E}_{\delta+1}(u,u)\mathcal{E}_{\delta+1}(v,v)$ for some $K>0$. Thus

$$|\mathcal{E}_{\delta+1}(u,v)|^2 \le (K+1)^2\hat{\mathcal{E}}_{\hat\delta+1}(u,u)\hat{\mathcal{E}}_{\hat\delta+1}(v,v) \le (K+1)^2c^2\mathcal{E}_1'(u,u)\mathcal{E}_1'(v,v).$$

Together with Ma and Röckner [15, I.3 Proposition 3.5] this shows that the closure of $\mathcal{E}_\delta$ is a closed, coercive form. ☐

Definition 3.7. If Condition 1.1 is satisfied, then $\mathcal{E}_{\delta(2,\eta,\gamma)}$ has a unique, positive, closed, coercive, continuous extension to the real Hilbert space

$$V = \overline{\mathbf{W}^1_{\infty,c}} \quad \text{with respect to} \quad \langle u,v\rangle_V = \langle u,v\rangle + \frac{1}{2}\left(\mathcal{E}'(u,v)+\mathcal{E}'(v,u)\right)$$

for all $\eta\in[0,1), \gamma\in(0,1)$. This extension is the well-known Friedrichs extension and denoted again by $\mathcal{E}_{\delta(2,\nu,\gamma)}$ and $\mathcal{E}$, respectively.

It is well known that in this situation $\mathcal{E}$ is associated with an unique operator A with $D(A)=\{u\in V; \exists w\in \mathbf{L}_2\forall v\in V : \langle w,v\rangle = \mathcal{E}(u,v)\}$ and $Au=w$ for all $u\in D(A)$ where $\langle w,v\rangle=\mathcal{E}(u,v)$ for all $v\in V$. The operator $-A$ is the generator of a C_0-semigroup T on $\mathbf{L}_2$.

Lemma 3.8. *Suppose $f\in\mathcal{C}(\boldsymbol{R})$ is piecewise in $\mathcal{C}^1$ with $f'\in \boldsymbol{L}_\infty(\boldsymbol{R})$ and $f(0)=0$. The substitution $u\mapsto f\circ u$ is continuous on V and on $\boldsymbol{L}_2$.*

Proof. First, we have to show that the range of the substitution is in V. Let $u\in V$ and choose $u_n\in\mathbf{W}^1_{\infty,c}$ such that $u_n\to u$ in V. Then, by equation (4),

$$\begin{aligned}\sup_{n\in\mathbf{N}}\mathcal{E}'(f\circ u_n, f\circ u_n) &= \sup_{n\in\mathbf{N}}\int_\Omega\sum_{i,j=1}^n a_{i,j}\partial_i u_n(x)\partial_j u_n(x)(f'\circ u_n)^2\\ &\le \sup_{n\in\mathbf{N}}\|f'\|_\infty^2\,\mathcal{E}'(u_n,u_n)<\infty\end{aligned}$$

since $\mathcal{E}'(u_n,u_n)\to\mathcal{E}'(u,u)$ for $n\to\infty$ and $\partial_i f\circ u_n = f'\circ u_n\partial_i u_n$, see Gilbarg and Trudinger [11, Theorem 7.8]. By the Banach-Alaoglu Theorem $f\circ u_{n_k}\to v$ weakly in V for some subsequence u_{n_k} of u_n and some $v\in V$. Since $f\circ u_n\to f\circ u$ in $\mathbf{L}_2$ we obtain $f\circ u = v\in V$.

Second, let $u_n\in V$ such that $u_n\to u\in V$.Then, again by equation (4) and Gilbarg and Trudinger [11, Theorem 7.8], we have

$$\begin{aligned}\mathcal{E}'(f\circ u_n, f\circ u_n) &= \int_\Omega\sum_{i,j=1}^n a_{i,j}\partial_i u_n(x)\partial_j u_n(x)(f'\circ u_n)^2\\ &\le \|f'\|_\infty^2\,\mathcal{E}'(u_n,u_n)\to\|f'\|_\infty^2\,\mathcal{E}'(u,u)\end{aligned}$$

for $n\to\infty$. Thus, there exists a subsequence u_{n_k} and a function $g\in\mathbf{L}_1$ such that

$$0\le (f'\circ u_{n_k})^2\sum_{i,j=1}^n a_{i,j}\partial_i u_{n_k}(x)\partial_j u_{n_k}(x)\le g$$

almost everywhere. By the dominated convergence Theorem $\mathcal{E}'(f\circ u_{n_k}, f\circ u_{n_k})$ converges to $\mathcal{E}'(f\circ u, f\circ u)$ for $n\to\infty$. This implies continuity, since there is such a dominated subsequence for all subsequences of u_n.

Third, $f \circ u \in \mathbf{L}_2$ whenever $u \in \mathbf{L}_2$ as $|f \circ u| \leq \|f'\|_\infty |u|$ almost everywhere. Finally, $|f \circ u - f \circ v| \leq \|f'\|_\infty |u - v|$ almost everywhere for all $u, v \in \mathbf{L}_2$. Thus, we have $\|f \circ u - f \circ v\|_2 \leq \|f'\|_\infty \|u - v\|_2$. ☐

Corollary 3.9. *If Condition 1.1 is satisfied, then the semigroup T associated with $\mathcal{E}$ is positive. The mapping $u \mapsto |u|$ is continuous on V.*

Proof. Fix $\eta, \gamma \in [0, 1)$ and $\delta = \delta(2, \eta, \gamma)$ in the proof. Lemma 3.8 implies that V is a lattice and $u \mapsto |u|$ is continuous on V, since $x \mapsto |x|$ is smooth on $\mathbf{R} \setminus \{0\}$ with essentially bounded derivative. Thus, we can apply the Beurling-Deny criteria for non-symmetric forms, see Ouhabaz [18, Theorem 2.4]. Since $\partial_i u_+ = \partial_i u \mathbf{1}_{\{x \in \Omega; u(x) > 0\}}$ we have $\mathcal{E}_\delta(u_+, u_-) = 0$ for any $u \in V$, we obtain the positivity of T immediately. ☐

4. Upper Bounds

From the previous section we know that there are unique solutions of (CP). In this section we derive upper bounds of these solutions. To do this, we need a simple generalisation of Theorem 3.3.

Remark 4.1. Since $f_{k,p}$ and $g_{k,p}$ are piecewise smooth with bounded derivative, see Lemma 3.2, $w_{k,p,u}$, $v_{k,p,u} \in V$ for all $u \in V$, by Lemma 3.8.

Obviously Theorem 3.3 holds for all $u \in V$. Since all terms of (6) are defined and the first part of the proof is a pointwise calculation, while the second part only uses that $w_{k,p,u} \in \mathbf{H}^1 \supset V$, the proof carries over to this more general situation.

Lemma 4.2. *If Condition 1.1 is satisfied and u is an orbit with initial value $u(0) \in D(A)$ then*

$$d_t \|w_{k,p,u}(t)\|_2^2 \leq -\nu \|\nabla w_{k,p,u}(t)\|_2^2 + 2\delta(p, \eta, \gamma) \|w_{k,p,u}(t)\|_2^2$$

for all $t > 0, \eta, \gamma \in [0, 1)$.

Proof. Since $u(0) \in D(A)$ we have $u \in \mathcal{C}(\mathbf{R}^+, D(A)) \cap \mathcal{C}^1(\mathbf{R}^+, \mathbf{L}_2)$. Thus, the mapping $t \mapsto f_{k,p} \circ u(t) : \mathbf{R} \to \mathbf{L}_2$ has the derivative $f'_{k,p} \circ u(t) \partial_t u(t)$ almost everywhere, by Gilbarg and Trudinger [11, Theorem 7.8]. Now, we see that

$$\begin{aligned} d_t \|w_{k,p,u}(t)\|_2^2 &= 2 \left\langle f'_{k,p} \circ u(t) \partial_t u(t), w_{k,p,u}(t) \right\rangle = 2 \left\langle \partial_t u(t), v_{k,p,u}(t) \right\rangle = -2 \left\langle A u(t), v_{k,p,u}(t) \right\rangle \\ &= -2\mathcal{E}(u(t), v_{k,p,u}(t)) \leq -\nu \|\nabla w_{k,p,u}(t)\|_2^2 + 2\delta(p, \eta, \gamma) \|w_{k,p,u}(t)\|_2^2 \end{aligned}$$

as of Remark 4.1. ☐

Proof of Theorem 1.2. By Corollary 3.9 we have the positivity of the semigroup. Thus, we have only to show the quasi-contractivity on $\mathbf{L}_p$.

If u is an orbit with initial value $u(0) \in D(A)$ then by Lemma 4.2 we have

$$d_t \|w_{k,p,u}(t)\|_2^2 \leq -\nu \|\nabla w_{k,p,u}(t)\|_2^2 + 2\delta(p) \|w_{k,p,u}(t)\|_2^2 \leq 2\delta(p) \|w_{k,p,u}(t)\|_2^2 .$$

From this differential inequality it follows that

$$\|w_{k,p,u}(t)\|_2^2 \leq \|w_{k,p,u}(0)\|_2^2 e^{2\delta(p)t}.$$

Since $D(A)$ is dense in $\mathbf{L}_2$, we can choose any initial value in $\mathbf{L}_2(\Omega) \cap \mathbf{L}_p(\Omega)$ by approximation, see Lemma 3.8. Since $w_{k,p,u}(t,x) \nearrow |u(t,x)|^{p/2}$ as $k \to \infty$ we have

$$\left\| |u(t)|^{p/2} \right\|_2^2 \le \left\| |u(0)|^{p/2} \right\|_2^2 e^{2\delta(p)t} \quad \text{and} \quad \|u(t)\|_p \le \|u(0)\|_p e^{2\delta(p)t/p}$$

The proposition is finished by a limiting argument. □

The subsequent $\mathbf{L}_p$-estimates of semigroup kernels are based upon an inequality first derived by Nash, see [17] to discuss parabolic partial differential equations on $\mathbf{L}_2(\mathbf{R}^n)$.

Proposition 4.3 (Nash-inequality). *There exists a constant $M_1 > 0$ such that*

$$\|u\|_2^{2+4/n} \le M_1 \|\nabla u\|_2^2 \|u\|_1^{4/n}$$

for all $u \in \boldsymbol{H}_0^1 \cap \boldsymbol{L}_1$. We call $M_1 = \frac{4\pi n}{n+2}\left(\frac{\Gamma(1+n/2)}{1+n/2}\right)^{2/n}$ the Nash constant.

Proof. Extend the function u by 0 to $\mathbf{R}^n$ and use Nash's result [17, page 936], or see Davies [9, Theorem 2.4.6]. □

Lemma 4.4. *If Condition 1.1 is satisfied and u is an orbit with initial value $u(0) \in D(A)$ then we have*

$$\left(\|w_{k,2p,u}(t)\|_2^{1/p} e^{-\frac{\delta(2p)}{p}t}\right)^{-4p/n} \ge \frac{2\nu}{M_1 n} \int_0^t \left(\|w_{k,p,u}(s)\|_2^{2/p} e^{-\frac{\delta(2p)}{p}s}\right)^{-4p/n} ds$$

for all $t \in J := \{t \in \boldsymbol{R}^+; \inf_{0 \le s \le t} \|u(s)\|_2 > 0\}$, $p \ge 2, k > 1$ and M_1 the Nash constant.

Proof. By Lemma 4.2 and Proposition 4.3 we have

$$\begin{aligned} d_t \|w_{k,2p,u}(t)\|_2^2 &\le -\nu \|\nabla w_{k,2p,u}(t)\|_2^2 + 2\delta(2p) \|w_{k,2p,u}(t)\|_2^2 \\ &\le -\frac{\nu}{M_1} \|w_{k,2p,u}(t)\|_2^{2+4/n} \|w_{k,2p,u}(t)\|_1^{-4/n} + 2\delta(2p) \|w_{k,2p,u}(t)\|_2^2 \\ &\le -\frac{\nu}{M_1} \|w_{k,2p,u}(t)\|_2^{2+4/n} \|w_{k,p,u}(t)\|_2^{-8/n} + 2\delta(2p) \|w_{k,2p,u}(t)\|_2^2 . \end{aligned}$$

The last inequality holds by the definition of $w_{k,p,u}$.

For positive functions f with $f' \le g + cf$ on $\mathbf{R}^+$ we have

$$f^{-M}(t)e^{cMt} \ge -M \int_0^t f^{-M-1}(s)g(s)e^{cMs}\, ds$$

for all $M > 0$. Choosing $M = 2/n$ we obtain

$$\|w_{k,2p,u}(t)\|_2^{-4/n} e^{4\frac{\delta(2p)}{n}t} \ge \frac{2\nu}{M_1 n} \int_0^t \|w_{k,p,u}(s)\|_2^{-8/n} e^{4\frac{\delta(2p)}{n}s}\, ds$$

which implies the claim. □

Lemma 4.5. *If Condition 1.3 is satisfied and $r \ge \max\left\{2 + \frac{4q}{p_1 - 2q}, 1 + \frac{q}{p_0 - q}\right\}$ then there exists $\delta_0 > 0$ such that $\delta(p) \le f(p)\delta_0$ for all $p \ge 2$ with $f(p) = (p-1)^r$. Further we have $\sum_{j=1}^\infty 2^{-j} \log f(2^{j+1}) \le 3r \log 2$.*

Proof. Choose $\hat{b}_{l,j}, \hat{c} \in \mathbf{L}_\infty$, $\check{b}_{l,j} \in \mathbf{L}_{p_1}$, $\check{c} \in \mathbf{L}_{p_0}$ such that $c_- = \check{c} + \hat{c}$, and $b_{l,j} = \check{b}_{l,j} + \hat{b}_{l,j}$. We obtain by a direct estimate using Theorem 3.3 and Lemma 2.3 (iv)

$$\delta_0 = \frac{3}{16}\nu + \frac{4}{\nu}\sum_{\substack{j=1\\ l=1,2}}^{n}\left(\left\|\hat{b}_{l,j}\right\|_\infty + \left\|\check{b}_{l,j}\right\|_{p_1}^{\frac{p_1}{p_1-2q}}\left(\frac{8\sqrt{n}C_1}{\nu}\right)^{\frac{2q}{p_1-2q}}\right)^2 + \\ + \frac{6}{\nu} + \|\hat{c}\|_\infty + \|\check{c}\|_{p_0}^{\frac{p_0}{p_0-q}}\left(\frac{6C_2^2}{\nu}\right)^{\frac{q}{p_0-q}}$$

which satisfies the inequality.

The last inequality of this Lemma is quite easy, since

$$\sum_{j=1}^\infty 2^{-j}\log f(2^{j+1}) = \sum_{j=1}^\infty 2^{-j}\log(2^{j+1})^r \le r\sum_{j=1}^\infty 2^{-j}(j+1)\log 2 = r\log 2\sum_{j=0}^\infty 2^{-j}(j+1) \\ = 2r\log 2$$

using the derivative of the geometric series at 1/2. □

Lemma 4.6. *Let $k > 1, l \in \boldsymbol{N}, p = 2^l$ and u be an orbit. Then there exists $C, \gamma > 0$ such that*

$$\|w_{k,p,u}(t)\|_2^{2/p} \le 2^{-n/4}\sqrt{C}t^{-\frac{n}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}e^{\beta t}\|u(0)\|_2 \tag{8}$$

holds for all $t \in (0,\infty)$ if condition 1.3 is satisfied.

Remark 4.7. Taking δ_0, f, r like in Lemma 4.5 and choosing $\epsilon \in (0,1)$, the constants in Lemma 4.6 are given by

$$\beta = \epsilon\delta_0\delta(2) \quad\text{and}\quad C = \left(\frac{M_1 n 12^r}{\nu\epsilon(3^r-\epsilon)}\right)^{n/2}$$

Proof. Let u be an orbit with initial value $u(0) \in D(A)$, $\epsilon \in (0,1)$, the function $h_2(t) = e^{\delta(2)t}$ and

$$h_{2p}(t) = h_p(t)e^{\frac{\delta(2p)\epsilon}{pf(2p)}t}\left(\frac{M_1 n}{2\nu 3^r\epsilon(3^r-\epsilon)}\right)^{\frac{n}{4p}}(f(2p)-\epsilon)^{\frac{n}{4p}},$$

then $h_p(t) \le 2^{-n/4}\sqrt{C}e^{\beta t}$ holds, by Lemma 4.5, and h_p is non-decreasing. To proof the claim we do an induction over $l = \log_2 p$. We see that $\|w_{k,2}(s)\|_2 = \|u(t)\|_2 \le h_2(t)\|u(0)\|_2$. By Lemma 4.4 and the uniqueness of the zero solution we obtain

$$\|w_{k,2p,u}(t)\|_2^{1/p} \le e^{\frac{\delta(2p)}{p}t}\left(\frac{M_1 n}{2\nu}\right)^{\frac{n}{4p}}\left(\int_0^t\left(\|w_{k,p,u}(s)\|_2^{2/p}e^{-\frac{\delta(2p)}{p}s}\right)^{-\frac{4p}{d}}ds\right)^{-\frac{n}{4p}} \\ \le e^{\frac{\delta(2p)}{p}t}\left(\frac{M_1 n}{2\nu}\right)^{\frac{n}{4p}}\left(\int_0^t\left(s^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}h_p(s)e^{-\frac{\delta(2p)}{p}s}\right)^{-\frac{4p}{d}}ds\right)^{-\frac{n}{4p}}\|u(0)\|_2$$

$$\leq e^{\frac{\delta(2p)}{p}t}\left(\frac{M_1 n}{2\nu}\right)^{\frac{n}{4p}} h_p(t)\left(\int_{t(1-\epsilon/f(2p))}^{t} s^{p-2}\,ds\right)^{-\frac{n}{4p}} e^{-\frac{\delta(2p)}{p}t(1-\epsilon/f(2p))}\left\|u(0)\right\|_2$$

$$\leq e^{\frac{\delta(2p)\epsilon}{pf(2p)}t}\left(\frac{M_1 n}{2\nu}\right)^{\frac{n}{4p}} h_p(t)\left(\frac{1}{p-1}t^{p-1}\left(1-\left(1-\frac{\epsilon}{f(2p)}\right)^{p-1}\right)\right)^{-\frac{n}{4p}}\left\|u(0)\right\|_2$$

$$\leq h_{2p}(t)t^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{2p})}\left\|u(0)\right\|_2$$

since $1-(1-\epsilon/f(2p))^{p-1} \geq \frac{(f(4)-\epsilon)(p-1)}{f(p)-\epsilon}(\epsilon/f(4))$ which can be seen by the following elementary calculus. Firstly, $g(x) = \frac{e^x-1}{x}$ is increasing on $\mathbf{R}$. Secondly, the function $\gamma(p) = -(p-1)\log(1-\epsilon/f(2p)) = \epsilon\frac{p-1}{f(2p)}\sum_{j=0}^{\infty}\frac{\epsilon^j}{(j+1)(f(2p))^j}$ is decreasing, since $f(p) = (p-1)^r$ and $r > 2$. Thirdly, $\tilde{\gamma}(x) = -(x-\epsilon)\log(1-\epsilon/x)$ is increasing for $x > \epsilon$, as its derivative is $\tilde{\gamma}'(x) = -\log(1-\epsilon/x)-\epsilon/x \geq 0$. Finally, $\tilde{\gamma}(f(2p))g(-\gamma(p)) = \frac{f(2p)-\epsilon}{p-1}(1-(1-\epsilon/f(2p))^{p-1})$ is non-decreasing.

Lemma 3.8 with regard to $f_{k,p}$ shows that we can approximate any initial value in $\mathbf{L}_2$. □

Corollary 4.8. *If Condition 1.3 is satisfied then there exists $C, \beta > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(\boldsymbol{L}_2,\boldsymbol{L}_\infty)} \leq 2^{-n/4}\sqrt{C}t^{-\frac{n}{4}}e^{\beta t} \quad \text{for all} \quad t > 0.$$

The constants are given in Remark 4.7.

Proof. Let $h(t) = 2^{-n/4}\sqrt{C}e^{\beta t}$. By definition we have

$$\begin{aligned}\|T(t)\|_{\mathcal{L}(\mathbf{L}_2,\mathbf{L}_\infty)} &= \sup_{\|u_0\|_2=1}\|T(t)u_0\|_\infty \leq \sup_{\|u_0\|_2=1}\limsup_{j\to\infty}\|T(t)u_0\|_{2^j}\\ &= \sup_{\|u_0\|_2=1}\limsup_{j\to\infty}\lim_{k\to\infty}\left\|f_{k,2^j}\circ T(t)u_0\right\|_2^{2/2^j}\\ &\leq \sup_{\|u_0\|_2=1}\limsup_{j\to\infty}\lim_{k\to\infty}t^{-\frac{n}{2}(\frac{1}{2}-2^{-j})}h(t)\left\|u_0\right\|_2\\ &= t^{-\frac{n}{4}}h(t).\end{aligned}$$

□

Proof of Theorem 1.4. By duality we obtain

$$\|T(t)\|_{\mathcal{L}(\mathbf{L}_1,\mathbf{L}_\infty)} \leq \|T(t/2)\|_{\mathcal{L}(\mathbf{L}_2,\mathbf{L}_\infty)}\|T(t/2)\|_{\mathcal{L}(\mathbf{L}_1,\mathbf{L}_2)} \leq Ct^{-n/2}e^{(\delta(2)+\gamma)t}.$$

□

5. Gaussian Estimates

In this section we use a technique developed by Davies [7] to obtain pseudo Gaussian upper bounds of the kernel of T. We assume Condition 1.5 in this section.

Definition 5.1. Let $\Psi = \{\psi \in \mathcal{D}(\mathbf{R}^n); |\nabla\psi| \leq 1\}$. For all real numbers ρ and $\psi \in \Psi$ let

$$\begin{aligned}(B_{\rho,\psi}f)(x) &= e^{\rho\psi(x)}f(x)\\ T_{\rho,\psi}(t) &= B_{\rho,\psi}T(t)B_{-\rho,\psi}.\end{aligned}$$

Lemma 5.2. *For any real ρ and $\psi \in \Psi$ the semigroup $T_{\rho,\psi}$ is associated with the form $\mathcal{E}_{\rho,\psi}$ given by*

$$\mathcal{E}_{\rho,\psi}(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{i,j}\partial_i u \partial_j v + \sum_{j=1}^{n} \int_\Omega (b_{1,j,\rho,\psi}\partial_j uv + b_{2,j,\rho,\psi} v \partial_j v) + \int_\Omega c_{\rho\psi} uv$$

$$\text{where} \quad b_{1,j,\rho,\psi} = b_{1,j} + \rho \sum_{k=1}^{n} a_{j,k}\partial_k \psi, \qquad b_{2,j,\rho,\psi} = b_{2,j} - \rho \sum_{k=1}^{n} a_{k,j}\partial_k \psi,$$

$$c_{\rho,\psi} = c - \rho^2 \sum_{i,j=1}^{n} a_{i,j}\partial_j \psi \partial_i \psi - \rho \sum_{j=1}^{n} (b_{1,j} - b_{2,j})\partial_j \psi$$

for all $u, v \in V$.

Proof. For every $u \in D(A)$, $v \in \mathbf{W}^1_{\infty,c}$ we have

$$\begin{aligned}\partial_t \langle T_{\rho,\psi}(t)u, v\rangle &= \partial_t \left\langle T(t)e^{-\rho\psi}u, e^{\rho\psi}v\right\rangle = \left\langle AT(t)e^{-\rho\psi}u, e^{\rho\psi}v\right\rangle = \mathcal{E}(T(t)e^{-\rho\psi}u, e^{\rho\psi}v) \\ &= \mathcal{E}(e^{-\rho\psi}T_{\rho,\psi}(t)u, e^{\rho\psi}v) = \mathcal{E}_{\rho,\psi}(T_{\rho,\psi}(t)u, v).\end{aligned}$$

By approximation this holds actually for all $u \in V$, from which the Lemma follows by a direct calculation. ☐

Theorem 5.3. *If $\rho \in \boldsymbol{R}, \psi \in \Psi, t \in (0,\infty)2$ then there exists $C, d, \delta_i > 0$ such that*

$$\|T_{\rho,\psi}(t)\|_{\mathcal{L}(\boldsymbol{L}_1,\boldsymbol{L}_\infty)} \leq Ct^{-n/2}\exp((\delta_2\rho^m + \delta_1)t) \tag{9}$$

where

$$m = 2 + \frac{4q}{p_2 - 2q} \in [2,\infty). \tag{10}$$

Remark 5.4. The exact values of the constants are not needed in the following, but are given here for completeness. Choosing $\epsilon \in (0,1)$ and analysing the proof provides

$$C = \left(\frac{M_1 n 12^r}{\nu(3^r - \epsilon)\epsilon}\right)^{n/2}, \quad \delta_i = (1+\epsilon)\hat{\delta}_i,$$

$$f(p) = (p-1)^r, \quad r = \max\left\{2 + \frac{4q}{\min\{p_2,p_1\} - 2q}, 1 + \frac{q}{p_0 - q}\right\}$$

$$\hat{\delta}_1 = \hat{\delta}_2 + \frac{12}{\nu}\sum_{\substack{j=1\\l=1,2}}^{n}\left(\left\|\hat{b}_{l,j}\right\|_\infty + \left\|\check{b}_{l,j}\right\|_{p_1}^{\frac{p_1}{p_1-2q}}\left(\frac{16\sqrt{n}C_1}{\nu}\right)^{\frac{2q}{p_1-2q}}\right)^2 +$$

$$+ \|\hat{c}\|_\infty + \|\check{c}\|_{p_0}^{\frac{p_0}{p_0-q}}\left(\frac{18C_2^2}{\nu}\right)^{\frac{q}{p_0-q}} + \frac{3}{16}\nu$$

$$\hat{\delta}_2 = \frac{12}{\nu}\sum_{j=1}^{n}\left(\left(\sum_{i=1}^{n}\|\hat{a}_{i,j}\|_\infty\right)^2 + \left(\sum_{i=1}^{n}\|\check{a}_{i,j}\|_{p_2}^{\frac{p_2}{p_2-2q}}\left(\frac{32n\sqrt{n}C_1}{\nu}\right)^{\frac{2q}{p_2-2q}}\right)^2 + \right.$$
$$\left. + \left(\sum_{i=1}^{n}\|\hat{a}_{i,j}\|_\infty\right)^2 + \left(\sum_{i=1}^{n}\|\check{a}_{i,j}\|_{p_2}^{\frac{p_2}{p_2-2q}}\left(\frac{32n\sqrt{n}C_1}{\nu}\right)^{\frac{2q}{p_2-2q}}\right)^2\right) +$$
$$+ \sum_{i,j=1}^{n}\left(\|\hat{a}_{i,j}\|_\infty + \|\check{a}_{i,j}\|_{p_2}^{\frac{p_2}{p_2-q}}\left(\frac{18n^2C_2^2}{\nu}\right)^{\frac{q}{p_2-q}}\right) +$$
$$+ \sum_{j=1}^{n}\left(\left\|\hat{b}_{1,j} - \hat{b}_{2,j}\right\|_\infty + \left\|\check{b}_{1,l} - \check{b}_{2,l}\right\|_{p_1}^{\frac{2p_1}{p_1-q}}\left(\frac{18nC_2^2}{\nu}\right)^{\frac{q}{2p_1-q}}\right)$$

for any choice of $\hat{a}_{i,j}, \hat{b}_{l,j}, \hat{c} \in \mathbf{L}_\infty$, $\check{a}_{i,j} \in \mathbf{L}_{p_2}$, $\check{b}_{l,j} \in \mathbf{L}_{p_1}$ and $\check{c} \in \mathbf{L}_{p_0}$ such that $p_2 > 2q$ or $p_1 = \infty$ and $a_{i,j} = \check{a}_{i,j} + \hat{a}_{i,j}$, $b_{l,j} = \check{b}_{l,j} + \hat{b}_{l,j}$ for all $i,j = 1,\dots,n$, $l = 1,2$ and $c_- = \check{c} + \hat{c}$. If $a_{i,j} \in \mathbf{L}_\infty$ for all $i,j = 1\dots,n$ then choose $\check{a}_{i,j} = 0, p_2 = \infty, m = 2$.

If $a_{i,j} \in \mathbf{L}_\infty$ for all $i,j = 1,\dots,n$ then we obtain a classical Gaussian estimate.

Proof. Since the form $\mathcal{E}_{\rho,\psi}$ satisfies condition 1.3 Theorem 1.4 holds with $\delta_{\rho,\psi}$ corresponding to (7). To estimate $\delta_{\rho,\psi}$ we choose functions $\hat{a}_{i,j}, \hat{b}_{l,j}, \hat{c} \in \mathbf{L}_\infty$, $\check{a}_{i,j} \in \mathbf{L}_{p_2}$, $\check{b}_{l,j} \in \mathbf{L}_{p_1}$ and $\check{c} \in \mathbf{L}_{p_0}$ such that $a_{i,j} = \check{a}_{i,j} + \hat{a}_{i,j}$, $b_{l,j} = \check{b}_{l,j} + \hat{b}_{l,j}$ and $c_- = \check{c} + \hat{c}$. If $a_{i,j} \in \mathbf{L}_\infty$ then choose $\check{a}_{i,j} = 0, p_2 = \infty, m = 2$. With the help of these decompositions and Lemma 2.3 (ii),(iv) we obtain

$$[b_{1,j,\rho,\psi}]_\theta \leq [b_{1,j}]_{\theta/2} + |\rho|\sum_{k=1}^{n}\|\hat{a}_{j,k}\|_\infty + |\rho|\sum_{k=1}^{n}\|\check{a}_{j,k}\|_{p_2}{}^{\frac{p_2}{p_2-2q}}\left(\frac{2n\,|\rho|}{\theta}\right)^{\frac{2q}{p_2-2q}}$$
$$\leq [b_{1,j}]_{\theta/2} + |\rho|\sum_{k=1}^{n}\|\hat{a}_{j,k}\|_\infty + |\rho|\sum_{k=1}^{n}\|\check{a}_{j,k}\|_{p_2}{}^{\frac{p_2}{p_2-2q}}\left(\frac{2n\,|\rho|}{\theta}\right)^{\frac{2q}{p_2-2q}}$$

$$[b_{1,j,\rho,\psi}]_\theta^2 \leq 3\,[b_{1,j}]_\theta^2 + 3(1+|\rho|^m)\left(\sum_{k=1}^{n}\|\hat{a}_{j,k}\|_\infty\right)^2 +$$
$$+ 3(1+|\rho|^m)\left(\sum_{k=1}^{n}\|\check{a}_{j,k}\|_{p_2}{}^{\frac{p_2}{p_2-2q}}\left(\frac{2n}{\theta}\right)^{\frac{2q}{p_2-2q}}\right)^2$$

$$[b_{2,j,\rho,\psi}]_\theta^2 \leq 3\,[b_{2,j}]_\theta^2 + 3(1+|\rho|^m)\left(\sum_{k=1}^{n}\|\hat{a}_{k,j}\|_\infty\right)^2 +$$
$$+ 3(1+|\rho|^m)\left(\sum_{k=1}^{n}\|\check{a}_{k,j}\|_{p_2}{}^{\frac{p_2}{p_2-2q}}\left(\frac{2n}{\theta}\right)^{\frac{2q}{p_2-2q}}\right)^2$$

$$[c_{\rho,\psi}{}_{-}]_\theta \le [c_-]_{\theta/3} + \rho^2 \sum_{i,j=1}^n \|\hat{a}_{i,j}\|_\infty + \rho^2 \sum_{i,j=1}^n \|\check{a}_{i,j}\|_{p_2}^{\frac{p_2}{p_2-q}} \left(\frac{3n^2\rho^2}{\theta}\right)^{\frac{q}{p_2-q}} +$$
$$+ |\rho| \sum_{j=1}^n \left\|\hat{b}_{1,j} - \hat{b}_{2,j}\right\|_\infty + |\rho| \sum_{j=1}^n \left\|\check{b}_{1,j} - \check{b}_{2,j}\right\|_{p1}^{\frac{p_1}{p_1-q}} \left(\frac{3n\,|\rho|}{\theta}\right)^{\frac{q}{p_1-q}}$$
$$\le [c_-]_{\theta/3} + (1+|\rho|^m) \sum_{i,j=1}^n \|\hat{a}_{i,j}\|_\infty + (1+|\rho|^m) \sum_{i,j=1}^n \|\check{a}_{i,j}\|_{p_2}^{\frac{p_2}{p_2-q}} \left(\frac{3n^2}{\theta}\right)^{\frac{q}{p_2-q}} +$$
$$+ (1+|\rho|^m) \sum_{j=1}^n \left\|\hat{b}_{1,j} - \hat{b}_{2,j}\right\|_\infty + (1+|\rho|^m) \sum_{j=1}^n \left\|\check{b}_{1,j} - \check{b}_{2,j}\right\|_{p1}^{\frac{p_1}{p_1-q}} \left(\frac{3n}{\theta}\right)^{\frac{q}{p_1-q}}.$$

Together with (7) we obtain the estimate $\delta_{\rho,\psi}(p) \le (\hat{\delta}_1 + |\rho|^m\,\hat{\delta}_2) f(p)$. Then, by Theorem 1.4 and the values of the constants given in Remark 4.7,

$$\|T_{\rho,\psi}(t)\|_{\mathcal{L}(\mathbf{L}_1,\mathbf{L}_\infty)} \le Ct^{-n/2} \exp((\delta_2\rho^m + \delta_1)t)$$

□

Proof of Theorem 1.6. Let $\rho \in \mathbf{R}, \psi \in \Psi$, k (respectively $k_{\rho,\psi}$) be the kernel associated with T (respectively $T_{\rho,\psi}$). Then $k(t,x,y) = e^{-\rho(\psi(x)-\psi(y))} k_{\rho,\psi}(x,y,t) \ge 0$ for all $t > 0$ and almost all $x, y \in \Omega$, since the semigroup T is positive (Theorem 1.2). By Theorem 5.3 we obtain

$$|k(t,x,y)| \le Ct^{-n/2} \exp((\delta_2\,|\rho|^m + \delta_1)t - \rho(\psi(x) - \psi(y)))$$

Replacing ρ by $-\rho$ it follows that

$$|k(t,x,y)| \le Ct^{-n/2} \exp((\delta_2\,|\rho|^m + \delta_1)t - \rho\,|\psi(x) - \psi(y)|)$$

Since $\sup_{\psi\in\Psi} |\psi(x) - \psi(y)| = |x - y|$, we have

$$|k(t,x,y)| \le Ct^{-n/2} \exp((\delta_2\,|\rho|^m + \delta_1)t - \rho\,|x - y|)$$

Choosing $\rho^{m-1} = (\delta_2 mt)^{-1}\,|x-y|$ proves the claim with $\omega_2 = (\delta_2 m)^{-\frac{m}{m-1}}(1 - 1/m), \omega_1 = \delta_1$. □

6. Applications

Here we can use the results in the literature about kernel-estimates. Firstly we get p-independence of the spectrum $\sigma(A)$ using

Theorem 6.1 (Kunstmann [14, Theorem 1.1]). *Let Q be an integral operator on $\boldsymbol{L}_2(\Omega)$ given by a kernel k which satisfies*

$$|k(x,y)| \le g(x-y) \quad \forall x, y \in \Omega$$

where g is a function satisfying

$$x \mapsto \exp(\rho(|x|))g(x) \in \boldsymbol{L}_1(\boldsymbol{R}^n) \cap \boldsymbol{L}_{p_0}(\boldsymbol{R}^n)$$

for some $p_0 \in (1,\infty]$ and some Ω-admissible function ρ (i.e. $\rho : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is non-decreasing, subadditive, $\lim_{t\to 0}\rho(t) = 0$ and $x \mapsto \exp(-\rho(|x|)) \in \boldsymbol{L}_1(\tilde{\Omega})$ where $\tilde{\Omega} = \{x - y; x, y \in \Omega\}$). Then Q extends to consistent operators Q_p on all $\boldsymbol{L}_p(\Omega), p \in [1,\infty]$ and the spectrum $\sigma(Q_p)$ does not depend on p.

Corollary 6.2. *Let $m > 1$ and $d \in \boldsymbol{R}$ such that $n > m(d-1)$. Let $R_2 \in \mathcal{L}(\boldsymbol{L}_2)$ be given by the formula*

$$R_2 f(x) = \int_\Omega k(x,y) f(y)\, dx \qquad \forall f \in \boldsymbol{L}_2(\Omega)$$

with some measurable kernel $k : \Omega \times \Omega \to \boldsymbol{C}$. Assume

$$|k(x,y)| \leq g_{\mu,b}(x-y) = C \int_0^\infty t^{-d} e^{-\mu t - b(\frac{|x-y|^m}{t})^{\frac{1}{m-1}}}\, dt$$

for some $\mu, b, C > 0$. Then, there is a consistent family $(R_p \in \mathcal{L}(\boldsymbol{L}_p))_{p\in[1,\infty]}$ and the spectrum $\sigma(R_p) = \sigma(R_2)$ is independent of $p \in [1,\infty]$.

Proof. Using the continuous Minkowski inequality and the substitution $x = yt^{1/m}$, we obtain $\|g_{\mu,b}\|_p < \infty$ if $p^{-1} > \frac{m}{n}(d-1)$. Observe that the range of p does not depend on b. For any $c \in (0,b)$ and $a > 0$ we have

$$\sup_{s\geq 0} as - c(s^m/t)^{1/(m-1)} = t\left(\frac{a}{m}\right)^m \left(\frac{m-1}{c}\right)^{m-1} =: t\gamma(a,m,c).$$

Thus, choosing $a > 0$ and $c \in (0,b)$ such that $\gamma(a,m,c) < \mu$, we have

$$\left|e^{a|x|} g_\mu(x)\right| \leq g_{\mu-\gamma(a,m,c),b-c}(x)$$

for all $x \in \mathbf{R}^n$. Now choose $\rho(r) = ar$ and $p_0 > 1$ such that $p_0^{-1} > \frac{m}{n}(d-1)$. Then the assumptions of Theorem 6.1 are given. Therefore R_2 extends to a consistent family $(R_p)_{p\in[1,\infty]}$ and the spectrum of R_p is independent of $p \in [1,\infty]$. ☐

Lemma 6.3. *Let $n \in \boldsymbol{N}$, $M_i \subset \{z \in \boldsymbol{C}; \operatorname{Re} z < w\}$ for some $w \in \boldsymbol{R}$ and $N_{i,\lambda} = \{(\lambda - z)^n; z \in M_i\}$ where $i = 1,2$. If $N_{1,\lambda} = N_{2,\lambda}$ for all $\lambda > w$ then $M_1 = M_2$.*

Proof. First let $B(x,r) = \{y \in \mathbf{C}; |x-y| < r\}$, $S_{i,\lambda} = M_i \cap B\left(\lambda, \frac{\lambda-w}{\cos\pi/n}\right)$ and $T_{i,\lambda} = \{(\lambda-z)^n; z \in S_{i,\lambda}\}$. Then $T_{i,\lambda} = N_{i,\lambda} \cap B\left(0, \left(\frac{\lambda-w}{\cos\pi/n}\right)^n\right)$, $\lambda - S_{i,\lambda} = (\lambda - M_i) \cap B\left(0, \frac{\lambda-w}{\cos\pi/n}\right)$ is a subset of $\Sigma_{\pi/n} := \left\{re^{\mathrm{i}\varphi}; r > 0, |\varphi| < \pi/n\right\}$ and $T_{1,\lambda} = T_{2,\lambda}$. Of course the mapping $z \mapsto z^n : \Sigma_{\pi/n} \to \mathbf{C}$ is injective. Thus, $\lambda - S_{1,\lambda} = \lambda - S_{2,\lambda}$ and then $S_{1,\lambda} = S_{2,\lambda}$, too, for all $\lambda > w$.

Now, we finish the proof by the equations

$$\begin{aligned} M_1 &= M_1 \cap \bigcup_{\lambda>w} B\left(\lambda, \frac{\lambda-w}{\cos\pi/n}\right) = \bigcup_{\lambda>w} M_1 \cap B\left(\lambda, \frac{\lambda-w}{\cos\pi/n}\right) = \bigcup_{\lambda>w} S_{1,\lambda} \\ &= \bigcup_{\lambda>w} S_{2,\lambda} = \bigcup_{\lambda>w} M_2 \cap B\left(\lambda, \frac{\lambda-w}{\cos\pi/n}\right) = M_2 \cap \bigcup_{\lambda>w} B\left(\lambda, \frac{\lambda-w}{\cos\pi/n}\right) = M_2 \end{aligned}$$

since $\mathbf{C} = \bigcup_{\lambda>w} B\left(\lambda, \frac{\lambda-w}{\cos\pi/n}\right)$. ☐

Proof of Theorem 1.7. If T generates a C_0-semigroup on $\mathbf{L}_1$ let $P = [1,\infty)$, otherwise $P = (1,\infty)$.

Using the resolvent identity and Theorem 1.6 we find

$$(R(\lambda, A_p))^{n+1} f(x) = \frac{(-1)^n}{n!} \frac{d^n}{d\lambda^n} R(\lambda, A_p) f(x) = \int_\Omega \int_0^\infty \frac{1}{n!} t^n e^{-\lambda t} k(t,x,y)\, dt f(y)\, dy$$

for all $f \in \mathbf{L}_p$ and $\lambda > \omega_1$. Applying Corollary 6.2 we obtain $\sigma((R(\lambda, A_2))^{n+1}) = \sigma((R(\lambda, A_p))^{n+1})$ for all $\lambda > \omega_1$ and $p \in P$.

Using the spectral mapping theorem for the resolvent, i.e.

$$\sigma(R(\lambda, A)) \setminus \{0\} = \frac{1}{\lambda - \sigma(A)} = \left\{ \frac{1}{\lambda - \mu}; \mu \in \sigma(A) \right\}$$

and for bounded operators S and polynomials g $(\sigma(g(S)) = g(\sigma(S)))$, we obtain

$$\left\{1/z; z \in \sigma((R(\lambda, A_p))^{n+1}) \setminus \{0\}\right\} = \left\{(\lambda - \mu)^{n+1}; \mu \in \sigma(A_p)\right\}$$

for all $p \in P$ and $\lambda > \omega_1$. Applying Lemma 6.3 to $N_{p,\lambda} := \{(\lambda - z)^{n+1}; z \in \sigma(A_p)\}$ finishes the proof. $\square$

Remark 6.4. If $a_{i,j} \in \mathbf{L}_\infty$ then by Theorem 1.6 the kernel associated with the semigroup satisfies a classical Gaussian estimate. Thus the semigroup extends to a C_0-semigroup on $\mathbf{L}_1$.

Another way to show that we have a C_0-semigroup on $\mathbf{L}_1$ uses $\mathbf{L}_1$-contractivity of $S(t) = e^{-kt}T(t)$. The semigroup S is $\mathbf{L}_1$-contractive if $k + c - \sum_{j=1}^{n} \partial_j b_{1,j} \geq 0$ as a distribution. In fact, by Lemma 3.8 $\min\{u_+, 1\} \in V$ whenever $u \in V$. Now, a direct calculation shows $\mathcal{E}(u - \min\{u_+, 1\}, \min\{u_+, 1\}) \geq 0$ for all $u \in \mathcal{D}$. This remains true for all $u \in V$ as $\mathcal{D}$ is dense in V. By Ma and Röckner [15, Proposition I.4.3 and Theorem I.4.4] T is $\mathbf{L}_1$-contractive.

The same arguments show that T is quasi $\mathbf{L}_\infty$-contractive if there is an $k \in \mathbf{R}$ such that $k + c - \sum_{j=1}^{n} \partial_j b_{2,j} \geq 0$ as a distribution. In general, $\mathcal{E}$ is not quasi $\mathbf{L}_\infty$-contractive, see Ouhabaz [18, Remark 4.3.a] even though Condition 1.5 is satisfied.

Secondly, we have maximal $\mathbf{L}_p$-$\mathbf{L}_q$ regularity, if Condition 1.5 is satisfied and $a_{i,j} \in \mathbf{L}_\infty$ for all $i, j = 1, \ldots, n$. Since we get classical Gaussian estimates by Theorem 1.6, we can use the result of Hieber and Pruess [12, Theorem 3.1] to prove Theorem 1.8. Moreover, there exists a constant $M > 0$ such that

$$\int_0^\infty \|u(t)\|_q^p \, dt + \int_0^\infty \|u'(t)\|_q^p \, dt + \int_0^\infty \|A_q u(t)\|_q^p \, dt \leq M \int_0^\infty \|f(t)\|_q^p \, dt$$

for all $f \in \mathbf{L}_p(\mathbf{R}^+, \mathbf{L}_q(\Omega))$.

Thirdly, we can prove Theorem 1.9 getting an analytic mapping on $\mathbf{L}_1$ in an open sector.

Proof of Theorem 1.9. This proof uses Theorem 1.6 and a technique of Arendt [3]. Let $0 < (m-1)\theta_1 < (m-1)\theta_2 < \theta_3 < \theta_4 < \theta$. Replacing T by $(e^{-wt}T(t))_{t \geq 0}$ for some $w > \omega_1$, we can assume that

$$\|T(z)\|_{\mathcal{L}(\mathbf{L}_2)} \leq M_1 \quad \text{for all} \quad z \in \Sigma_{\theta_4}$$

$$0 \leq k(t, x, y) \leq C t^{-n/2} e^{-b(|x-y|^m/t)^{\frac{1}{m-1}}} \quad \text{for all} \quad t > 0, x, y \in \Omega$$

By Corollary 4.8 and duality we know, if w is sufficiently large, that

$$\|T(t)\|_{\mathcal{L}(\mathbf{L}_2, \mathbf{L}_\infty)} \leq M_2 t^{-n/4} \quad \text{and} \quad \|T(t)\|_{\mathcal{L}(\mathbf{L}_1, \mathbf{L}_2)} \leq M_3 t^{-n/4} \quad \text{for all} \quad t > 0.$$

Now, choose $\delta \in (0,1)$ such that $\delta t+\mathbf{i}s \in \Sigma_{\theta_4}$ whenever $t+\mathbf{i}s \in \Sigma_{\theta_3}$. For any $z = t+\mathbf{i}s \in \Sigma_{\theta_3}$ we have

$$\begin{aligned}\|T(z)\|_{\mathcal{L}(\mathbf{L}_1,\mathbf{L}_\infty)} &\leq \|T((1-\delta)t/2)\|_{\mathcal{L}(\mathbf{L}_1,\mathbf{L}_2)} \|T(\delta t+\mathbf{i}s)\|_{\mathcal{L}(\mathbf{L}_2)} \|T((1-\delta)t/2)\|_{\mathcal{L}(\mathbf{L}_2,\mathbf{L}_\infty)} \\ &\leq M_1M_2M_3t^{-n/2}\left(\frac{1-\delta}{2}\right)^{-n/2} =: M_4(\operatorname{Re} z)^{-n/2}.\end{aligned}$$

Applying [3, Theorem 4.2] obtain the analyticity of $T : \Sigma_{\theta_3} \to \mathcal{L}(\mathbf{L}_1, \mathbf{L}_\infty) \sim \mathbf{L}_\infty(\Omega \times \Omega)$. By [3, Lemma 4.1] there exists $K : \Sigma_{\theta_3} \times \Omega \times \Omega \to \mathbf{C}$ such that $K(\cdot, x, y)$ is analytic for all $x, y \in \Omega$, $K(z, \cdot, \cdot) \in \mathbf{L}_\infty(\Omega \times \Omega)$ for all $z \in \Sigma_{\theta_3}$ and $T(z)f(x) = \int_\Omega K(z, x, y)f(y)\, dy$ for all $f \in \mathbf{L}_1 \cap \mathbf{L}_2$. As $\mathbf{L}_\infty(\Omega \times \Omega)$ is isometric isomorph to $\mathcal{L}(\mathbf{L}_1, \mathbf{L}_\infty)$ by $k \mapsto (f \mapsto \int_\Omega k(\cdot, y)f(y)\, dy)$, we obtain $|K(z, x, y)| \leq M_4(\operatorname{Re} z)^{-n/2}$ for all $z \in \Sigma_{\theta_3}$ and almost all $x, y \in \Omega$ and $K(t, x, y) = k(t, x, y)$ for all $t > 0$ and almost all $x, y \in \Omega$.

Applying [3, Lemma 4.4] to $H(z, x, y) = K(z^{m-1}, x, y)z^{(m-1)n/2}$ we obtain the estimate $|K(z, x, y)| \leq M_0\, |z|^{-n/2}\, e^{-\beta(|x-y|^m/|z|)^{\frac{1}{m-1}}}$ for all $z \in \Sigma_{\theta_1}$ and almost all $x, y \in \Omega$ where $M_0 = \sup\{M_4, C\}$ and $\beta = \frac{sin(\gamma-\theta_1)}{\sin\gamma}b$.

Thus, $\sup_{z\in\Sigma_{\theta_1},|z|>r} \|T_p(z)\|_p < \infty$ for all $r > 0, p \in [1, \infty)$. Since the operators T_p are consistent, we can finish the proof by applying [3, Theorem 4.2] again. □

We cannot prove the boundedness of the semigroup T on $\mathbf{L}_1$ near 0 using only a pseudo Gaussian estimate of order $m \neq 2$.

References

[1] Robert A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] W. Arendt and A. F. M. ter Elst. Gaussian estimates for second order elliptic operators with boundary conditions. *J. Oper. Theory*, 38(1):87–130, 1997.

[3] Wolfgang Arendt. Semigroup properties by Gaussian estimates. *Sūrikaisekikenkyūsho Kōkyūroku*, 1009:162–180, 1997. Nonlinear evolution equations and their applications, RIMS Kyoto.

[4] D. G. Aronson. Non-negative solutions of linear parabolic equations. *Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat., III. Ser.*, 22:607–694, 1968.

[5] Pascal Auscher, Louise Barthélemy, Philippe Bénilan, and El-Maati Ouhabaz. Absence de la $\mathbf{L}^\infty$-contractivité pour les semi-groups associés aux opérateurs elliptique complexes sous forme divergence.

[6] Daniel Daners. Heat kernel estimates for operators with boundary condition. To appear in Mathematische Nachrichten.

[7] E. Brian Davies. Explicit constants for Gaussian upper bounds on heat kernels. *Am. J. Math.*, 109:319–333, 1987.

[8] E. Brian Davies. *Heat kernels and spectral theory.*, volume 92 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1989.

[9] E. Brian Davies. *Heat Kernels and Spectral Theory.* Number 92 in Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1989.

[10] E. B. Fabes and D. W. Stroock. A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. *Arch. Ration. Mech. Anal.*, 96:327–338, 1986.

[11] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order. 2nd ed.*, volume 224 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 2nd ed. edition, 1983.

[12] Matthias Hieber and Jan Pruess. Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations. *Commun. Partial Differ. Equations*, 22(9-10):1647–1669, 1997.

[13] Tosio Kato. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the corr. print. of the 2nd ed. 1980.

[14] Peer Christian Kunstmann. Kernel estimates and $\mathbf{L}^p$ spectral independence of differential and integral operators. To appear. Also available at `http://ma1serv.mathematik.uni-karlsruhe.de/page/bullbo.html`, 1999.

[15] Zhi-Ming Ma and Michael Röckner. *Introduction to the theory of (non-symmetric) Dirichlet forms.* Springer-Verlag, Berlin, 1992.

[16] Vladimir G. Maz'ya. *Sobolev spaces.* Springer-Verlag, Berlin, 1985.

[17] J. Nash. Continuity of solutions of parabolic and elliptic equations. *Am. J. Math.*, 80:931–954, 1958.

[18] El-Maati Ouhabaz. L^∞-contractivity of semigroups generated by sectorial forms. *J. Lond. Math. Soc., II. Ser.*, 46(2):529–542, 1992.

[19] Derek W. Robinson. *Elliptic operators and Lie groups.* Oxford Science Publications. Clarendon Press, Oxford, 1991.

[20] Guido Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier*, 15(1):189–257, 1965.

University of Ulm, Abteilung Angewandte Analysis, D-89069 Ulm, Germany

E-mail address: `S.Karrmann@gmx.net`

URL: `http://www.mathematik.uni-ulm.de/m5/sk`