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Numerical Finance Reading Course

Sheet 1 (April 22nd, 2009)

Discussion: Generation of Random Numbers (Sections 2.1, 2.2, 2.4, 2.5)

There are two main steps in the generation of pseudo-random numbers with arbitrary distributions:

1. Generation of uniformly distributed numbers.
   - What methods are there?
   - Why is the choice of parameters in linear congruential generators crucial?
   - How can a generator be tested?
2. Transformation of uniformly distributed numbers to obtain other distributions.
   - What general methods are there?
   - How can normally distributed random variables be generated?

Exercise 1: Linear Congruential Generators

There are many different implementations of linear congruential generators, i.e. generators of the form

$$y_{n+1} = (ay_n + b) \bmod M.$$

We want to compare the following two examples:

a) RANDU: popular generator implemented by IBM in 1970.

$$a = 2^{16} + 3 = 65539,\ b = 0,\ y_0 \text{ odd and } M = 2^{31} = 2147483648.$$

b) UNIX rand(): standard Unix random number generator.

$$a = 1103515245,\ b = 12345 \text{ and } M = 2^{31}.$$

Implement a linear congruential generator. For both examples, simulate 10000 uniformly distributed 3-dimensional pseudo-random vectors using for example $y_0 = 1$ and compare the performance of the generators by visualizing these samples in a 3D plot. Which generator would you prefer?

Hints:

- Obtain 3-dimensional vectors by setting $u_1 = \left(\frac{y_1}{M}, \frac{y_2}{M}, \frac{y_3}{M}\right)^T$, $u_2 = \left(\frac{y_4}{M}, \frac{y_5}{M}, \frac{y_6}{M}\right)^T$, etc.
- In C/C++, use `long long int` to avoid floating point exceptions.
- 3D vectors can be plotted with GNUPLOT using `splot "file"`, where the file is of the form

  ```
  u11 u12 u13
  u21 u22 u23
  ...
  ```

  Be sure that the terminal type is wxt (`set terminal wxt`), so that you can rotate the plot.

Exercise 2: Multivariate Normal Distribution

a) Let $X \in \mathbb{R}^n$ be a random vector with expectation vector $\mu_X \in \mathbb{R}^n$ and covariance matrix $\Sigma_X \in \mathbb{R}^{n\times n}$. Prove that for the expectation vector and the covariance matrix of a linear transformation $Z = AX \in \mathbb{R}^m$ with $A \in \mathbb{R}^{m\times n}$ it holds:

$$\mu_Z = A\mu_X,$$
$$\Sigma_Z = A\Sigma_X A^T.$$

b) Derive an algorithm for the generation of multivariate normal random vectors $Z \sim \mathcal{N}(\mu, C)$ $(Z, \mu \in \mathbb{R}^n, C \in \mathbb{R}^{n\times n})$. Recall that standard normally distributed pseudo-random numbers $X \sim \mathcal{N}(0, 1)$ can be obtained for example with the Box-Muller algorithm.