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Summer Term 2009

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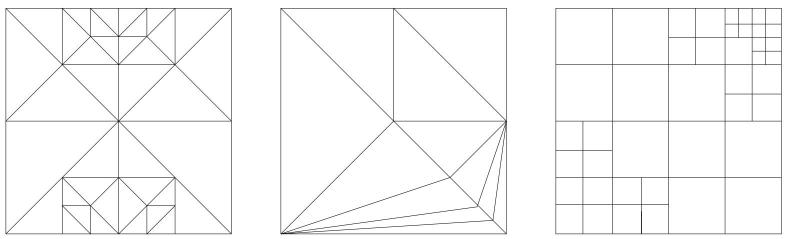
Numerical Finance Reading Course

Sheet 10 (July 2nd, 2009)

Discussion: Elliptic PDEs - Finite Elements (Sections 6.5- 6.7)

- What is the Courant Finite Element? How are the corresponding entries in the stiffness matrix derived?
- Are these grids admissible / regular / (quasi-) uniform ?



- Why should one use reference elements as described in Remark 6.5.4?
- What do we know about the approximation error? Compare with the results for Finite Differences.

Exercise 1: Characterization Theorem (Remark 6.3.5 (ii))

Prove the Characterization Theorem:

$$J(v) := \frac{1}{2}a(v,v) - (f,v)_0 \text{ has its minimum in } u.$$

$$\iff$$

$$a(u,v) = (f,v) \qquad \text{for all } v \in H_0^1(\Omega).$$

Exercise 2: Solving the Poisson Equation with FEM

Implement the Finite Element method for the problem given in Exercise 2 on Sheet 9. Plot the approximation and the exact solution for one N. Obtain the convergence rate by plotting the approximation error for different N. You can use the supremum error

$$\|u - u_h\|_\infty = \sup_{x\in[0,1]} |u(x) - u_h(x)|,$$

which can be approximated using a fine grid with $M \gg N$ intervals.

Exercise 3: Simulation of a Cox-Ingersoll-Ross process

A Cox-Ingersoll-Ross (CIR) process has the form

$$\mathrm{d}r(t) = \alpha(b - r(t))\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W_t$$

for some α, $b > 0$ and is often used to model short rates.
There are different ways to simulate such a process. We will consider the following two methods.

a) **Euler-Maruyama:**
It should be no problem to apply the Euler-Maruyama scheme to this process. However, unlike the exact solution, the Euler approximation r_{τ_n} may become negative for some τ_n. One can easily replace $\sqrt{r(t)}$ by $\sqrt{(r(t))^+}$ to avoid any numerical problems, but the question remains whether this yields an accurate representation of $r(t)$.

b) **Exact Simulation:**
It can be shown that the distribution of $r(t)$ given $r(u)$, $u < t$, is a scaled noncentral chi-square distribution. Knowing $r(0)$, one can hence simulate any $r(t)$, $t > 0$, exactly if one can generate noncentral chi-square random variables.
More precisely, we have for given $r(u)$ and $t > u$

$$r(t) = c_{t,u}\ \chi_d'^2\big(\lambda_{t,u}\, r(u)\big),$$

where $c_{t,u} = \frac{\sigma^2(1-e^{-\alpha(t-u)})}{4\alpha}$, $\lambda_{t,u} = \frac{e^{-\alpha(t-u)}}{c_{t,u}}$ and $\chi_d'^2(l)$ denotes a noncentral chi-square random variable with d degrees of freedom and non-centrality parameter l.

One can show that for any $d > 1$ it holds

$$\chi_d'^2(l) = \chi_1'^2(l) + \chi_{d-1}^2,$$

where χ_{d-1}^2 is an ordinary chi-square random variable. (This can easily be seen for $d \in \mathbb{N}$ considering that for $Z_1, \ldots, Z_d \sim \mathcal{N}(0,1)$ the sum $\sum_{i=1}^d (Z_i + a_i)^2$ is chi-square distributed with noncentrality parameter $l = \sum_{i=1}^d a_i^2$.)

This yields the following algorithm for the generation of a time discrete approximation of $r(t)$.

FOR $i = 1, \ldots, n$ *DO*

(i) Set $c = \frac{\sigma^2(1-e^{-\alpha(\tau_i - \tau_{i-1})})}{4\alpha}$, $\lambda = \frac{e^{-\alpha(\tau_i-\tau_{i-1})}}{c}$

(ii) Generate $Z \sim \mathcal{N}(0,1)$, $X \sim \chi_{d-1}^2$.

(iii) Set $r(\tau_i) = c \cdot \Big((Z + \sqrt{\lambda})^2 + X\Big)$

END

Now all that is left is the generation of chi-square random variables. Note that the chi-square distribution χ_d^2 is a special case of the gamma distribution $\Gamma(a, \beta)$ with

$a = \frac{d}{2}$, $\beta = 2$. Since $\beta X \sim \Gamma(a, \beta)$ if $X \sim \Gamma(a, 1)$, we only need to be able to generate random variables from the distribution $\Gamma(a, 1)$. This can be done with the following algorithm.

INPUT: a
INIT: $\bar{a} = a - 1$, $b = \left(a - \frac{1}{6a}\right)/\bar{a}$, $m = 2/\bar{a}$, $s = m + 2$.

REPEAT

- *(i) Generate* u_1, $u_2 \sim U[0, 1]$.
- *(ii) Set* $V = bu_2/u_1$.
- *(iii) IF* $mu_1 - s + V + 1/V \le 0$
  THEN accept,
  ELSEIF $m \log(u_1) - \log(V) + V - 1 \le 0$
  THEN accept.

UNTIL accept

RETURN $\bar{a}V$

As you can see, this method requires some additional computational effort compared to the Euler scheme, but is simulates the exact distribution and thus avoids any discretization errors.

Compare both methods by calculating the empirical density of $r(T)$ for $T = 1$, i.e. repeatedly simulating $r(T)$ and plotting a histogram, and compare it to the exact density

$$f_{r(T)}(x) = \frac{1}{c_{T,0}} \, f_{\chi'^2_d(\lambda_{T,0}\, r(0))}\left(\frac{x}{c_{T,0}}\right)$$

(you can use e.g. R or MATLAB for the evaluation of your data). Use for example the parameters $\alpha = 0.2$, $\sigma = 0.1$, $b = 0.05$ and $r(0) = 0.04$.