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Numerical Finance Reading Course

Sheet 11 - Sample Solution

Exercise 1: Binomial Tree for American Options
Implement the binomial method to find the price of an American option with inital value $S_0 = 20$, strike $K = 20$, maturity $T = 2$, volatility $\sigma = 0.2$ and riskless interest rate $r = 0.05$.

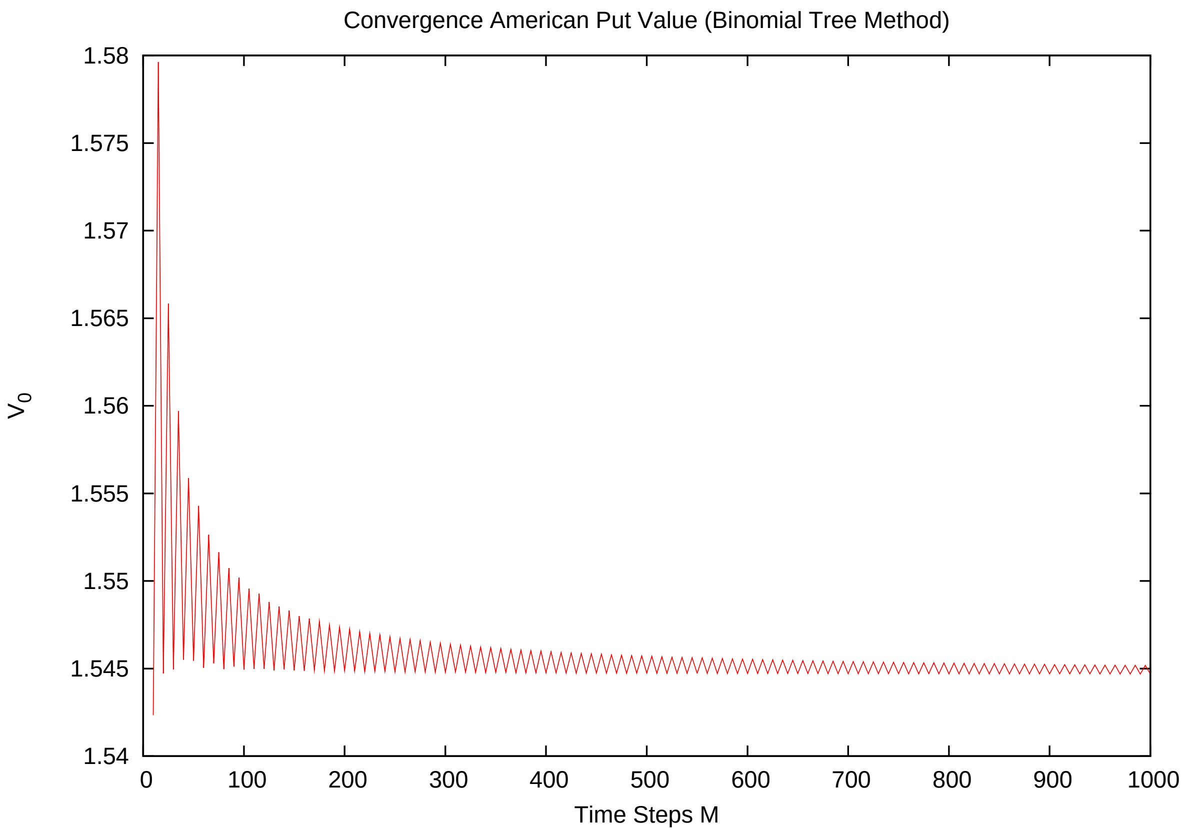
a) Plot the option value for different time steps, e.g. $M = 5, 10, ..., 1000$. What do you see? Can you explain that behaviour?

b) For a fixed M (say $M = 50$), plot the tree, i.e. the approximation of $V(S_t, t)$. Compare with the payoff function. Can you see the exercise region? Modify your implementation to calculate the exercise region and add it to your plot.

Solution:

```
#include <iostream>
#include <fstream>
#include <cmath>
#include <cstdlib>

using namespace std;

int main(){
  // Input:
  int S = 20;
  int K = 20;
  double r = 0.05;
  double sigma = 0.2;
  double T =2.;

  // Initialization:
  for(int M = 50; M <= 50; M += 5){
    double deltaT = T/(double)M;
    double alpha = exp(r*deltaT);
    double alpha_inv = exp(-r*deltaT);
    double beta = 0.5*(alpha_inv + alpha*exp(sigma*sigma*deltaT));
    double u = beta + sqrt(beta*beta - 1);
    double d = 1./u;
    double p = (alpha - d)/(u - d);

    double* V[M+1];
    for(int i = 0; i < M+1; i++){
      V[i] = (double*)malloc((i+1)*sizeof(double));
    }
    int* Sf[M+1];
    for(int i = 0; i < M+1; i++){
      Sf[i] = (int*)calloc((i+1), sizeof(int));
    }

    // Forward Phase:
```

```
    V[0][0] = S;
    for(int i = 1; i <= M; i++){
      for(int j = 0; j <= i; j++){
        V[i][j] = S*pow(u,j)*pow(d,i-j);
      }
    }


    // Evaluation: (Put)
    for(int j = 0; j <= M; j++){
      V[M][j] = (K - V[M][j] > 0) ? (K - V[M][j]) : 0;
    }


    // Backward Phase with test for early exercise:
    double val_curr;
    double val_exp;
    for(int i = M-1; i >= 0; i--){
      for(int j=0; j<= i; j++){
        val_curr = (K - V[i][j] > 0) ? (K - V[i][j]) : 0;
        val_exp = alpha_inv*( p*V[i+1][j+1] + (1.-p)*V[i+1][j]);
        if(val_curr > val_exp){
          V[i][j] = val_curr;
          Sf[i][j] = 1;
        }
        else{
          V[i][j] = val_exp;
        }
      }
    }


    //cout << M << " " << V[0][0] << endl;

    // Print Tree
    ofstream tree("tree.txt");
    if(tree.is_open()){
      for(int i = 0; i <= M ; i++){
        for(int j = 0; j <= i; j++){
          tree << i*deltaT << " " << S*pow(u,j)*pow(d,i-j) << " " << V[i][j] << endl;
        }
        tree << endl;
      }
      tree.close();
    }
    else{ cerr << "Unable to open file tree.txt" << endl;}

    // Print Sf
    ofstream time("time.txt");
    if(time.is_open()){
      for(int i = 0; i <= M ; i++){
        for(int j = 0; j <= i; j++){
          if(Sf[i][j] > 0){
            time << i*deltaT << " " << S*pow(u,j)*pow(d,i-j) << " " << V[i][j] << endl;
          }
        }
        time << endl;
      }
      time.close();
    }
    else{ cerr << "Unable to open file time.txt" << endl;}
  }

  return 0;
}
```



Exercise 2: Iterative Methods

Prove that for a $(n \times n)$- matrix G with $\|G\| < 1$, a vector $c \in \mathbb{R}^n$ and an arbitrary starting point $x^0 \in \mathbb{R}^n$ the following error bounds hold for the iterative method

$$x^{k+1} = Gx^k + c.$$

a) $\|x^k - x\| \leq \|G\|^k \|x^0 - x\|$,

b) $\|x^k - x\| \leq \frac{\|G\|^k}{1-\|G\|} \|x^1 - x^0\|$,

where x denotes the fixpoint of the iteration, i.e., $x = Gx + c$.

Solution:

a) Iteration over k:

- $k = 1$: $\|x^1 - x\| = \|Gx^0 + c - Gx - c\| \leq \|G\| \|x^1 - x\|$.
- $k \to k + 1$: Assume the induction hypothesis for k. Then

$$\begin{aligned} \|x^{k+1} - x\| &= \|Gx^k + c - Gx - c\| \leq \|G\| \|x^k - x\| \\ &\overset{I.H.}{\leq} \|G\| \|G\|^k \|x^0 - x\| = \|G\|^{k+1} \|x^0 - x\| \end{aligned}$$

b) Again iteration over k:

- $k = 1$:

$$\begin{aligned}
&& \|x^1 - x\| &\overset{(a)}{\leq} \|G\|\|x^1 - x\| = \|G\|\|x^1 - x^0 + x^0 - x\| \\
&& &\leq \|G\|\Big(\|x^1 - x^0\| + \|x^0 - x\|\Big) \\
\iff && \|x^1 - x\|\Big(1 - \|G\|\Big) &\leq \|G\|\|x^1 - x^0\| \\
\iff && \|x^1 - x\| &\leq \frac{\|G\|}{1 - \|G\|}\|x^1 - x^0\|.
\end{aligned}$$

- $k \to k + 1$: Assume the induction hypothesis for k. Then

$$\begin{aligned}
\|x^{k+1} - x\| &\leq \|G\|\|x^k - x\| \\
&\overset{I.H.}{\leq} \|G\|\frac{\|G\|^k}{1 - \|G\|}\|x^1 - x^0\| \\
&= \frac{\|G\|^{k+1}}{1 - \|G\|}\|x^1 - x^0\|.
\end{aligned}$$