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Numerical Finance Reading Course

Sheet 12 (July 16th, 2009)

Discussion: Projected SOR & Exotic Options (Sections 7.3.2, 8)

- How does the SOR method have to be changed to solve complementary problems of the form $(Au - f)^T(u - g) = 0$, $Au \geq f$, $u \geq g$?
- What are the main steps in the proof of the convergence theorem?
- What particular problem has to be solved to obtain prices for Asian options? Why is this difficult?
- Can the Black-Scholes equation be convection dominated? (Hint: Look at the coefficient form in Chapter 6.7)
- What is the idea of the SUPG method?

Exercise 1: Projected SOR Methods for an American Put

We want to price an American put (without dividends) using the projected SOR method. To this end, we transfer the Black-Scholes (in)equality to the heat (in)equality, as we have done before for European options (see Chapter 4, exercise sheet 6).

Hence, we set again $\tau = \frac{1}{2}\sigma^2(T-t)$, $x = \ln(\frac{K}{S})$ and $q = \frac{2r}{\sigma^2}$, so that

$$u(x,\tau) = \frac{1}{K}\exp\left\{\frac{1}{2}(q-1)x + \left(\frac{1}{4}(q-1)^2 + q\right)\tau\right\} V(Ke^x, T - \frac{2\tau}{\sigma^2})$$

and the relation $V_P(S,t) \geq (K-S)^+$ transforms to

$$u(x,\tau) \geq \exp\left\{\frac{1}{2}(q-1)x + \left(\frac{1}{4}(q-1)^2 + q\right)\tau\right\} (1-e^x)^+ =: g(x,\tau).$$

The pricing problem is then given by

$$\begin{cases} (u_\tau - u_{xx})(u-g) &= 0, \\ u_\tau - u_{xx} &\geq 0, \\ u &\geq g, \end{cases} \qquad \text{on } (x_{min}, x_{max}) \times \left[0, \frac{\sigma^2}{2}T\right],$$

with initial value

$$u_0(x) = g(x,0)$$

and boundary conditions

$$\lim_{x\to-\infty} u(x,\tau) = \lim_{x\to-\infty} g(x,\tau) \approx g(x_{min},\tau),$$
$$\lim_{x\to\infty} u(x,\tau) = \lim_{x\to-\infty} g(x,\tau) \approx 0.$$

To apply a SOR method, we have to rewrite $u_\tau - u_{xx} \geq 0$ in the form $Au \geq b$. Recall that for the heat equation (here with right-hand side $f = 0$) we used the six-point scheme

$$\frac{1}{\Delta t}(u_i^{k+1} - u_i^k) = D^- D^+(\sigma u_i^{k+1} + (1-\sigma)u_i^k) \qquad \text{for } \sigma \in [0,1].$$

Unlike before, we will now use the Crank-Nicolson scheme ($\sigma = \frac{1}{2}$). With $\gamma := \frac{\Delta t}{h^2}$, this gives the inequality

$$(1+\gamma)u_i^{k+1} - \frac{\gamma}{2}(u_{i+1}^{k+1} + u_{i-1}^{k+1}) \geq (1-\gamma)u_i^k + \frac{\gamma}{2}(u_{i+1}^k + u_{i-1}^k) \qquad \text{for } i = 1,\dots,N-1,$$

which corresponds to the system $Au^{k+1} \geq b^k$ with

$$A = \begin{pmatrix} 1-\gamma & -\frac{\gamma}{2} & & \\ -\frac{\gamma}{2} & 1-\gamma & -\frac{\gamma}{2} & \\ & \ddots & \ddots & \ddots \\ & & -\frac{\gamma}{2} & 1-\gamma \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)},$$

$$b^k = \begin{pmatrix} (1-\gamma)u_1^k + \frac{\gamma}{2}(u_2^k + u_0^k) \\ \\ \vdots \\ \\ (1-\gamma)u_{N-1}^k + \frac{\gamma}{2}(u_N^k + u_{N-2}^k) \end{pmatrix} + \begin{pmatrix} \frac{\gamma}{2}u_0^{k+1} \\ 0 \\ \vdots \\ 0 \\ \frac{\gamma}{2}u_N^{k+1} \end{pmatrix} \in \mathbb{R}^{N+1},$$

where u_0^k, u_N^k are given by the boundary conditions for all k.

Implement the above Crank-Nicolson scheme, where you solve for each time step k the problem

$$\begin{cases} (Au^{k+1} - b^k)^T(u^{k+1} - g^{k+1}) & = 0, \\ Au^{k+1} - b^k & \geq 0, \\ u^{k+1} & \geq g^{k+1}, \end{cases}$$

using the projected Gauß-Seidel method to obtain the prices $V(S,t)$ of an American Put with $K = 12$, $r = 0.04$, $\sigma = 0.4$ and $S \in (0.00001, 20)$ for $t \in [0,T]$. Plot the solution and compare with the values of the corresponding European Put (exercise 2 on sheet 6).