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Numerical Finance Reading Course

Sheet 3 - Sample Solution

Exercise 1: Hardy-Krause Variation

Calculate the Hardy-Krause variation $V(f)$ for the function

$$f(x_1, x_2) = \frac{3}{1 + x_1 + 2x_2}.$$

Solution:

The Hardy-Krause variation is defined as

$$V(f) = \sum_{k=1}^{d} \sum_{I \in J_k^{(d)}} V^{(k)}(f_I).$$

Here $d = 2$ and hence

$$J_1^{(2)} = \{(1), (2)\} \quad \text{and} \quad J_2^{(2)} = \{(1, 2)\}.$$

It follows that

$$f_{(1)} = \frac{3}{3 + x_1}, f_{(2)} = \frac{3}{2 + 2x_2}, f_{(1,2)} = \frac{3}{1 + x_1 + 2x_2}.$$

The calculation of the Vitaly variations $V^{(k)}(f_I)$ then yields

$$V^{(1)}(f_{(1)}) = \int_0^1 \left| \frac{\partial}{\partial x_1} f_{(1)}(x_1) \right| \mathrm{d}x_1 = \left| - \frac{3}{3 + x_1} \right|_0^1 = \frac{1}{4},$$

$$V^{(1)}(f_{(2)}) = \int_0^1 \left| \frac{\partial}{\partial x_2} f_{(2)}(x_2) \right| \mathrm{d}x_2 = \left| - \frac{3}{2 + 2x_2} \right|_0^1 = \frac{3}{4},$$

$$V^{(2)}(f_{(1,2)}) = \int_{[0,1]^2} \left| \frac{\partial^2}{\partial x_1 \partial x_2} f_{(1,2)}(x_1, x_2) \right| \mathrm{d}x_1 \mathrm{d}x_2 = \frac{5}{4}.$$

Hence, the variation in the sense of Hardy-Krause is

$$V(f) = V^{(1)}(f_{(1)}) + V^{(1)}(f_{(2)}) + V^{(2)}(f_{(1,2)}) = \frac{1}{4} + \frac{3}{4} + \frac{5}{4} = \frac{9}{4}.$$

Exercise 2: Halton Sequence

a) Implement the radical inverse function $\phi_b(i)$, using the idea that

$$i = \sum_{k=0}^{j} d_k b^k = (d_k b^{k-1} + \cdots + d_1)b + d_0.$$

You should not need to use more than one loop in your algorithm.

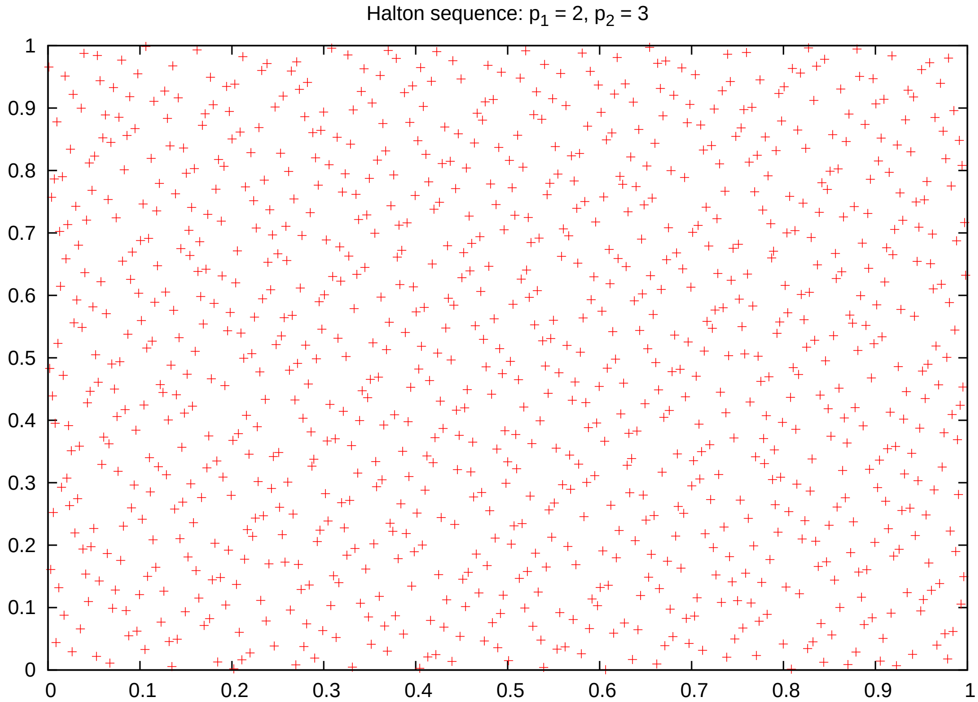
b) For a d-dimensional Halton sequence

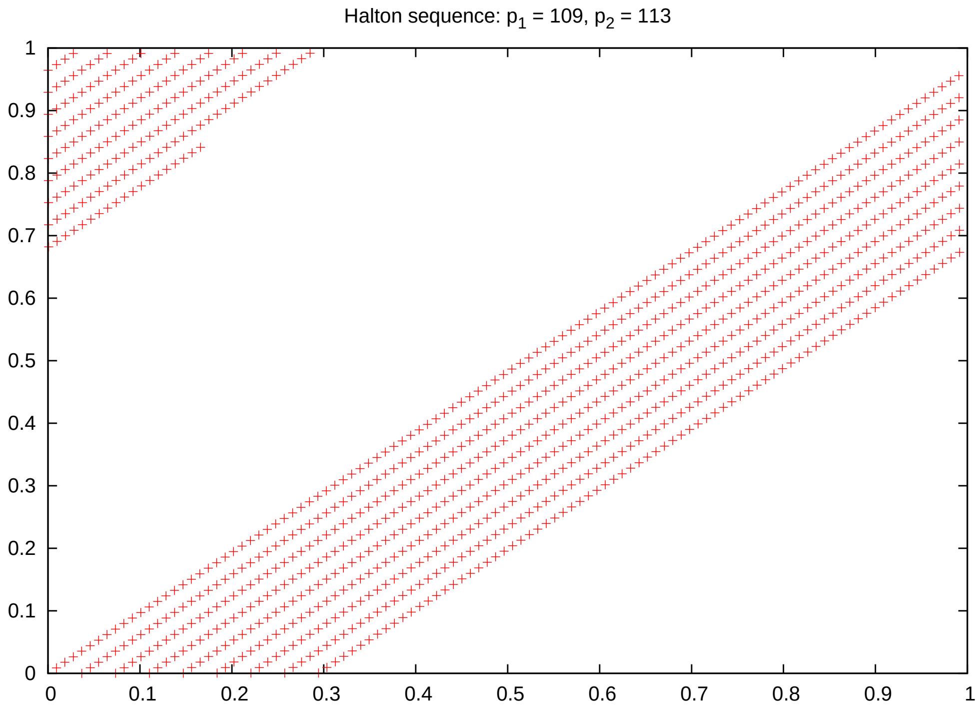
$$x_i = (\phi_{p_1}(i), \ldots, \phi_{p_d}(i)), \qquad i = 1, 2, \ldots,$$

one usually takes $p_i, \ldots, p_d$ to be the first d prime numbers. Generate 1000 points of the 2-dimensional Halton sequence (i.e. $p_1 = 2$, $p_2 = 3$). Also generate 1000 Halton points using $p_1 = 109$, $p_2 = 113$ (the 29th and 30th prime number, respectively). Compare the two point sets by plotting them (Gnuplot: `plot "data"`). What does this tell you about the behaviour of the Halton sequence in higher dimensions? (Hint: the second sequence corresponds to the projection of a 30-dimensional sequence onto the last two coordinates).

Solution:

```
#include <iostream>
#include <fstream>

using namespace std;

const double primes[50] = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,
  43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113,
  127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197,
  199, 211, 223, 227, 229};

double radicalInverse(int n, int b){

        double z = 0;
        int d;
        int denom = 1;
        do{
                d = n % b;
                n = n / b;
                z = b*z + d;
                denom = denom * b;
        }while(n > 0);

        z = (double)z / (double)denom;
        return z;
}

int main(){

        int number = 1000;

        for(int i = 1; i < number; i++){
                cout << radicalInverse(i, primes[0]) << " " << radicalInverse(i, primes[1]) << endl;
        }

        return 0;
}
```


Halton sequence: p_1 = 2, p_2 = 3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Halton sequence: p_1 = 109, p_2 = 113
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1