Prof. Dr. Karsten Urban
Kristina Steih
Summer Term 2009

Ulm University
Institute for Numerical Mathematics

Numerical Finance Reading Course

Sheet 6 (May 28, 2009)

Discussion: Finite Difference Methods (Sections 4.3/4.4)

- Explain the idea of Finite Difference Methods.
- What does one know about consistency, stability and convergence for FD methods? What assumptions are necessary?
- Explain Figure 4.2.

Exercise 1: Finite Differences

a) Prove the LU-decomposition in Section 4.3.1.

b) Why does one use $(D^-D^+u)(x)$ to approximate u_{xx} and not for example $(D^+D^+u)(x)$? Determine $(D^+D^+u)(x)$ and its approximation order and compare with $(D^-D^+u)(x)$.

Exercise 2: Recursion Method

Solve the one-dimensional Poisson problem

$$\begin{cases} -u''(x) = f(x) & , x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

using Finite Differences for the following right hand sides. Plot the convergence rates and compare them with the theoretical results for the convergence order.

a) $f(x) = 9\pi^2 \cos(3\pi(x+\frac{1}{2}))$. The exact solution is $u(x) = \cos(3\pi(x+\frac{1}{2}))$.

b) $f(x) = \begin{cases} 1, & x \in (0, \frac{1}{2}], \\ -1, & x \in (\frac{1}{2}, 1). \end{cases}$. The exact solution is $u(x) = \begin{cases} -\frac{1}{2}x^2 + \frac{1}{4}x, & x \in (0, \frac{1}{2}], \\ \frac{1}{2}x^2 - \frac{3}{4}x + \frac{1}{4}, & x \in (\frac{1}{2}, 1). \end{cases}$

Exercise 3: Explicit Euler Method

To solve the Black-Scholes Equation with Finite Differences, one can transform it to a heat equation

$$\begin{cases} \frac{\partial}{\partial \tau} u(x,\tau) - \frac{\partial^2}{\partial x^2} u(x,\tau) = 0 \text{ in } (x_{min}, x_{max}) \times (0,T) \\ \quad u(x,0) = u_0(x),\ x \in (x_{min}, x_{max}) \\ \quad u(x_{min}, t) = u_1(t), u(x_{max}, t) = u_2(t), t \in (0,T), \end{cases}$$

using

$$S = Ke^x, \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \quad V(S,t) = V\big(Ke^x, T - \frac{\tau}{\frac{1}{2}\sigma^2}\big) =: v(x,\tau)$$

$$\text{and} \quad v(x,\tau) := K \exp\left\{ -\frac{1}{2}(q-1)x - \Big(\frac{1}{4}(q-1)^2 + q\Big)\tau \right\} u(x,\tau), \quad q := \frac{2r}{\sigma^2}.$$

Note that the change of variable for t transforms the original backward PDE into a forward one. The initial condition for a European Put is then

$$u_0(x) = \max\left\{ e^{\frac{x}{2}(q-1)} - e^{\frac{x}{2}(q+1)} \ , \ 0 \right\}.$$

The boundary conditions of a European Put are transformed to

$$u(x,\tau) \to u_1(x,\tau) := \exp\left\{ \frac{1}{2}(q-1)x + \frac{1}{4}(q-1)^2\tau \right\} \qquad \text{for } x \to -\infty,$$

$$u(x,\tau) \to u_2(x,\tau) := 0 \qquad \text{for } x \to \infty.$$

However, instead of the infinite interval $x \in (-\infty, \infty)$, the truncated domain $x \in (x_{min}, x_{max})$ is used, where the choice of x_{min}, x_{max} depends on assumptions on the range of S.

Implement the Explicit Euler method to solve the Black Scholes PDE using the above transformation for a European Put with $K = 12$, $r = 0.04$, $\sigma = 0.4$ and $S \in (0.00001, 20)$. Plot the solution $V(S_i, t_j)$, $i = 1, \ldots, N$, $j = 1, \ldots, M$.

Hints:

- Implement the Euler scheme for $u(x,\tau)$ using the initial and boundary conditions as above. What is the range of x and τ? You can use for example $N = 200$, $M = 50$.
- Having calculated $u(x_i, \tau_j)$, obtain $V(S_i, t_j) = v(x_i, \tau_j)$, S_i and t_j as above.
- To plot the data with Gnuplot, print your solution in the following form (notice the blank line between blocks of equal t-value):

```
t_0 S_0 V(S_0,t_0)
t_0 S_1 V(S_1,t_0)
...  ...  ...
t_0 S_N V(S_N,t_0)

t_1 S_0 V(S_0,t_1)
...  ...  ...
t_1 S_N V(S_N,t_1)

....
```

  Then use `splot "data" with lines`.