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Numerical Finance Reading Course

Sheet 7 (June 10, 2009)

Discussion: Stochastic Differential Equations I (Sections 5.1-5.5)

- What is a Wiener Process (see e.g. Financial Mathematics I)?
- What are SDEs? Explain the notion of strong/weak solutions. When do strong solutions exist?
- What is the idea behind the Euler-Maruyama method? Compare with the deterministic approach.
- Explain the different approaches to measure the error of numerical solutions of SDEs.

Exercise 1: Black Scholes PDE

Let $S(t)$ denote some asset following a geometric Brownian motion, i.e., it holds

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

Let $B(t)$ denote a risk-free bond, so that

$$dB(t) = rB(t)dt,$$

and let $Y(t)$ denote a (normalized) replicating portfolio

$$Y(t) = c_1(t)B(t) + c_2(t)S(t) - V(S(t),t) \quad \text{for some } c_1(t), c_2(t).$$

Assume that $Y(t)$ is self-financing, i.e. it holds

$$dY(t) = c_1(t)dB(t) + c_2(t)dS(t) - dV(S(t),t).$$

Derive the Black Scholes PDE in the following way:

1. Calculate $dV(S(t),t)$ with the help of the Ito formula.
   Recall that the Itô formula in shorthand notation says that if

   $$dX_t = a(t)dt + b(t)dW(t),$$

   then

   $$df(X(t)) = \left(f_t(X(t)) + a(t)f_X(X(t)) + \frac{1}{2}b^2(t)f_{XX}(X(t)) \right) dt + b(t)f_X(X(t))dW(t).$$

2. Calculate $dY(t)$ as above.

3. Use the fact that $Y(t)$ is riskless to deduce that $dY(t) = Y(t)r\,dt$ and compare the two representations of $dY(t)$ to obtain the PDE.

Exercise 2: Simulating Wiener Processes

There are several methods to generate paths of a Wiener process. We will consider here the two most common ones.

a) **Random Walk**:
Making use of the fact that the increments of a Wiener process W are independently distributed with $W_t - W_s \sim N(0, t-s)$ for all $0 \le s < t \le T$, we can simulate paths in the following way. Let $0 = t_0 < t_1 < \cdots < t_N = T$, $Z_1, \dots, Z_N \sim N(0,1)$. Then define

$$\begin{aligned} W_0 &= 0, \\ W_{t_{i+1}} &= W_{t_i} + \sqrt{t_{i+1} - t_i}\, Z_{i-1} \qquad \text{for } i = 1, \dots, N. \end{aligned}$$

(Note that this corresponds to the Euler-Maruyama method for the PDE $dX_t = dW_t$, $X_0 = 0$.)

b) **Brownian Bridge**:
One can also start by generating the endpoint W_T of the process, and then fill in the other values using the distribution of W_t conditional on the already generated points. This procedure is known as Brownian Bridge. More precisely, consider $N = 2^n$ and $t_i - t_{i-1} = \frac{T}{N}$ for all $i = 1, \dots, N$. One then generates first W_T, then $W_{\frac{T}{2}}$ based on W_0 and W_T, then $W_{\frac{T}{4}}$, $W_{\frac{3T}{4}}$ based on W_0, $W_{\frac{T}{2}}$ and W_T, etc.

Suppose one has already calculated the $(j-1)$th refinement $W_0, W_{\frac{T}{2^{j-1}}}, W_{\frac{2T}{2^{j-1}}}, \dots,$ $W_{\frac{2^{j-1}T}{2^{j-1}}}$, $j < n-1$. Due to the independence of the increments, the value of W_s depends only on the two nearest values $W_{s_k} := W_{\frac{kT}{2^{j-1}}} < s < W_{\frac{(k+1)T}{2^{j-1}}} =: W_{s_{k+1}}$. In our construction, we always consider $s = \frac{1}{2}\left(\frac{kT}{2^{j-1}} + \frac{(k+1)T}{2^{j-1}}\right)$, i.e., the midpoint between two already calculated time points. In that case, the conditional distribution of W_s is given by

$$(W_s \mid W_{s_k} = w_{s_k}, W_{s_{k+1}} = w_{s_{k+1}}) \sim \mathcal{N}\left(\frac{w_{s_k} + w_{s_{k+1}}}{2}, \frac{T}{2^{j+1}}\right).$$

This yields the following construction, using $Z_1, \dots, Z_N \sim \mathcal{N}(0,1)$:

$$W^0(0) = 0, W^0(T) = \sqrt{T} Z_1,$$

$$W^1(t) = \begin{cases} \frac{1}{2}\big(W^0(0) + W^0(T)\big) + \sqrt{\frac{T}{2^2}}\, Z_2, & t = \frac{1}{2}, \\ W^0(t), & t = 0, T, \end{cases}$$

$$W^j(t) = \begin{cases} \frac{1}{2}\left(W^{j-1}\left(\frac{2kT}{2^j}\right) + W^{j-1}\left(\frac{(2k+2)T}{2^j}\right)\right) + \sqrt{\frac{T}{2^{j+1}}}\, Z, & t = \frac{(2k+1)T}{2^j}, \\ & k = 0, \dots, 2^{j-1} - 1, \\ W^{j-1}\left(\frac{kT}{2^{n-1}}\right), & t = \frac{2kT}{2^j}, \\ & k = 0, \dots, 2^{j-1}, \end{cases}$$

for $j = 2, \dots, n$. (Note that the second line in $W^j(t)$ corresponds just to the values of W that have already been calculated).

Generate and plot paths of $\{W(t), 0 \le t \le 1\}$ using both methods for different N (for example $N = 4, 8, 32, 512, 1024$).