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Numerical Finance Reading Course

Sheet 8 (June 18, 2009)

Discussion: Stochastic Differential Equations II (Sections 5.6-5.9)

- Explain strong convergence and consistency.
- Explain weak convergence and consistency.
- What do we know about the Euler method?
- How can higher order methods be obtained?
- Compare the methods and results for SDE with those for ordinary differential equations. What is different? Why?

Exercise 1: Strong Consistency of Euler-Maruyama (Prop. 5.6.9)

Show that the Euler-Maruyama scheme is strongly consistent with $c(\delta) = 0$ under the assumptions of Theorem 5.2.1.

Hints:

- ΔW_n is independent of $\mathcal{A}_{\tau_n}$.
- τ_n, τ_{n+1} are $\mathcal{A}_{\tau_n}$-measurable, i.e. $\mathbb{E}[\tau_n \mid \mathcal{A}_{\tau_n}] = \tau_n$.

Exercise 2: Option Pricing with Euler-Maruyama

European Barrier Options are options that depend on the underlying S to reach a given barrier H. For example, a Europan Up&Out Put becomes worthless if $S_t > H$ for any $t \in [0, T]$, so that

$$V_T = (K - S_T)^+ \, \mathbb{1}_{\{S_t < H \ \forall t \in [0,T]\}}.$$

The analytic solution for such a put is given by

$$\begin{aligned} P_0^{Up\&Out} &= e^{-rT}\mathbb{E}[V_T] \\ &= Ke^{-rT}\left(\Phi(-d+\sigma\sqrt{T}) - \left(\frac{H}{S}\right)^{2\lambda-2}\Phi(-y+\sigma\sqrt{T})\right) \\ &\quad - S_0\left(\Phi(-d) - \left(\frac{H}{S}\right)^{2\lambda}\Phi(-y)\right), \end{aligned}$$

where

$$\lambda = \frac{r + 0.5\sigma^2}{\sigma^2}, \quad y = \frac{\ln\left(\frac{H^2}{SK}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \quad d = \frac{\ln\left(\frac{S}{K}\right) + (r + 0.5\sigma^2)\,T}{\sigma\sqrt{T}}.$$

To estimate $P_0^{Up\&Out}$ numerically, one generates M replications of the path of S_t (using for example Euler-Maruyama), calculates the payoff $V_T^{(1)}, \ldots, V_T^{(M)}$ for each path and sets

$$\hat{P}_0^{Up\&Out} = e^{-rT}\frac{1}{M}\sum_{i=1}^{M} V_T^{(i)}.$$

Assume that S is geometric Brownian motion, i.e.,

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t,$$

and let $S_0 = 10$, $K = 12$, $H = 13$, $r = 0.04$, $\sigma = 0.4$ and $T = 2$. Calculate the price of a European Up&Out Put using for the simulation of the Brownian motion W_t

a) a random walk construction.

b) a Brownian Bridge construction.

c) a Brownian Bridge construction with Quasi-Monte Carlo numbers (drawing one QMC number for each path). Note that this construction is favorable for QMC, as the first dimensions of each point are those that determine the shape of the path. You can find instructions on how to generate Sobol numbers using the libraries *QuantLib* and *Boost* on the course homepage.

Compare the obtained prices for different M (e.g. $M = 2^2, \ldots, 2^{13}$) and plot the convergence rates (the analytic formula given above yields the price $P_0 = 2.047849$). You can use for example $N = 2^{13}$ time steps.