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Numerical Finance Reading Course

Sheet 9 (June 25, 2009)

Discussion: Elliptic PDEs - Galerkin Methods (Sections 6.1- 6.4)

- What is the problem with the classical formulation of PDEs, i.e., what do we need variational formulations for?
- What is meant by the variational formulation of a PDE? Also explain the definitions of a weak derivative and the Sobolev Space.
- What do we know about the existence of weak solutions? When do they correspond to our classical solutions?
- What is the idea behind the Galerkin method? Explain the importance of Céa's Lemma.

Exercise 1: Weak Derivatives

Compute the weak derivative of the function $u(x) = |x|$ on $[-1, 1]$.

Hint: Begin with the right hand side $(-1)(\partial\phi, u)_0$ of (6.3.1) and integrate it.

Exercise 2: Variational Formulation and Galerkin

Consider again the one-dimensional Poisson equation

$$\begin{cases} -u''(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

with $\Omega = (0, 1)$ and $f(x) = \begin{cases} 1, & x \in (0, \frac{1}{2}], \\ -1, & x \in (\frac{1}{2}, 1). \end{cases}$

a) Explicitly derive the variational formulation of this problem.

b) Consider as N-dimensional trial space S_h the space spanned by the basis $\{\psi_1, \dots, \psi_N\}$ with

$$\psi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h}, & x \in [x_{j-1}, x_j), \\ \frac{x_{j+1} - x}{h}, & x \in [x_j, x_{j+1}), \\ 0, & \text{otherwise}, \end{cases}$$

where $0 = x_0 < x_1 = \frac{1}{N+1} < \cdots < x_N = \frac{N}{N+1} < x_{N+1} = 1$ and $h = x_i - x_{i-1} = \frac{1}{N+1}$. (This basis is also called the *nodal basis*).

Calculate the entries of the stiffness matrix A_h and the right-hand side b_h for this basis (you can assume that N is odd).