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Numerical Finance Reading Course

Sheet 9 - Sample Solution

Exercise 1: Weak Derivatives

Compute the weak derivative of the function $u(x) = |x|$ on $[-1, 1]$.

Hint: Begin with the right hand side $(-1)(\partial\phi, u)_0$ of (6.3.1) and integrate it.

Solution:

Let $\phi \in C_0^\infty([-1, 1])$. Then

$$\begin{aligned} -\int_{-1}^{1} \phi'(x)u(x)\mathrm{d}x &= -\int_{-1}^{0} \phi'(x)(-x)\mathrm{d}x - \int_{0}^{1} \phi'(x)x\mathrm{d}x \\ &= \int_{-1}^{0} \phi'(x)x\mathrm{d}x - \int_{0}^{1} \phi'(x)x\mathrm{d}x \\ &= \big[x\phi(x)\big|_{-1}^{0} - \int_{-1}^{0} 1\phi(x)\mathrm{d}x - \big[x\phi(x)\big|_{0}^{1} + \int_{-1}^{0} 1\phi(x)\mathrm{d}x. \end{aligned}$$

As $\phi \in C_0^\infty([-1, 1])$, we know that $\phi(-1) = \phi(1) = 0$ and thus the term $\big[x\phi(x)\big|_{-1}^{0} - \big[x\phi(x)\big|_{0}^{1}$ vanishes. Hence

$$(-1)(\partial\phi, u)_0 = \int_{-1}^{1} \mathrm{sgn}(x)\phi(x)\mathrm{d}x = (v, \phi)_0,$$

where $v(x) := \mathrm{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0, \end{cases}$ is the sign function of x.

Exercise 2: Variational Formulation and Galerkin

Consider again the one-dimensional Poisson equation

$$\begin{cases} -u''(x) = f(x) & , \quad x \in \Omega, \\ u(x) = 0 & , \quad x \in \partial\Omega, \end{cases}$$

with $\Omega = (0, 1)$ and $f(x) = \begin{cases} 1, & x \in (0, \frac{1}{2}], \\ -1, & x \in (\frac{1}{2}, 1). \end{cases}$

a) Explicitly derive the variational formulation of this problem.

b) Consider as N-dimensional trial space S_h the space spanned by the basis $\{\psi_1, \dots, \psi_N\}$ with

$$\psi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h}, & x \in [x_{j-1}, x_j), \\ \frac{x_{j+1} - x}{h}, & x \in [x_j, x_{j+1}), \\ 0, & \text{otherwise}, \end{cases}$$

where $0 = x_0 < x_1 = \frac{1}{N+1} < \dots < x_N = \frac{N}{N+1} < x_{N+1} = 1$ and $h = x_i - x_{i-1} = \frac{1}{N+1}$. (This basis is also called the *nodal basis*).

Calculate the entries of the stiffness matrix A_h and the right-hand side b_h for this basis (you can assume that N is odd).

Solution:

a) It holds

$$\begin{aligned} & & -u''(x) &= f(x) && \text{in } \Omega \\ \Longrightarrow & & \int_0^1 -u''(x)v(x)\mathrm{d}x &= \int_0^1 f(x)v(x)\mathrm{d}x && \forall v \in H_0^1 \\ \Longleftrightarrow & & \big[-u'(x)v(x)\big|_0^1 + \int_0^1 u'(x)v'(x)\mathrm{d}x &= \int_0^1 f(x)v(x)\mathrm{d}x && \forall v \in H_0^1 \\ \Longleftrightarrow & & \underbrace{\int_0^1 u'(x)v'(x)\mathrm{d}x}_{=:\,a(u,v)} &= \underbrace{\int_0^1 f(x)v(x)\mathrm{d}x}_{=(f,v)_0} && \forall v \in H_0^1 \end{aligned}$$

b) *Stiffness Matrix*:
We have to calculate $a(\psi_i, \psi_j) = \int_0^1 \psi_i'(x)\psi_j'(x)\mathrm{d}x$ for all $i, j = 1, \dots, N$. Note that

$$\psi_j'(x) = \begin{cases} \frac{1}{h}, & x \in [x_{j-1}, x_j), \\ \frac{-1}{h}, & x \in [x_j, x_{j+1}), \\ 0, & \text{otherwise}, \end{cases}$$

From the definition of the basis functions, the supports of ψ_i' and ψ_j' only intersect if $j = i$, $j = i-1$ or $j = i+1$. Hence all entries $a(\psi_i, \psi_j)$ of A_h for $|i-j| > 1$ are zero. For the other entries, we have

$$\begin{aligned} a(\psi_i, \psi_i) &= \int_{x_{i-1}}^{x_{i+1}} \psi_i'(x)\psi_i'(x)\mathrm{d}x = \int_{x_{i-1}}^{x_i} \psi_i'(x)\psi_i'(x)\mathrm{d}x + \int_{x_i}^{x_{i+1}} \psi_i'(x)\psi_i'(x)\mathrm{d}x \\ &= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)^2 \mathrm{d}x + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)^2 \mathrm{d}x = \int_{x_{i-1}}^{x_{i+1}} \left(\frac{1}{h}\right)^2 \mathrm{d}x \\ &= \frac{2}{h}, \\ a(\psi_i, \psi_{i+1}) &= \int_{x_{i-1}}^{x_{i+1}} \psi_i'(x)\psi_{i+1}'(x)\mathrm{d}x = \int_{x_i}^{x_{i+1}} \psi_i'(x)\psi_{i+1}'(x)\mathrm{d}x \\ &= \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) \mathrm{d}x = -\frac{1}{h}, \end{aligned}$$

$$
\begin{aligned}
a(\psi_{i-1}, \psi_i) &= \int_{x_{i-1}}^{x_{i+1}} \psi'_{i-1}(x)\psi'_i(x)\mathrm{d}x = \int_{x_{i-1}}^{x_i} \psi'_{i-1}(x)\psi'_i(x)\mathrm{d}x \\
&= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right)\mathrm{d}x = -\frac{1}{h},
\end{aligned}
$$

so that the stiffness matrix A_h has the form

$$
A_h = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.
$$

Right Hand Side:
Since N is odd, $x_{\frac{N+1}{2}} = \frac{1}{2}$. The basis function $\psi_{\frac{N+1}{2}}$ is symmetric around this point, so that for the right hand side at $i = \frac{N+1}{2}$ it holds

$$
(f, \psi_i)_0 = \int_{x_{i-1}}^{x_{i+1}} f(x)\psi_i(x)\mathrm{d}x = \int_{x_{i-1}}^{x_i} \psi_i(x)\mathrm{d}x - \int_{x_i}^{x_{i+1}} \psi_i(x)\mathrm{d}x = 0.
$$

Similarly, we have

$$
\begin{aligned}
(f, \psi_i)_0 &= \int_{x_{i-1}}^{x_{i+1}} f(x)\psi_i(x)\mathrm{d}x = \int_{x_{i-1}}^{x_i} \psi_i(x)\mathrm{d}x = h \qquad \forall i < \frac{N+1}{2}, \\
(f, \psi_i)_0 &= \int_{x_{i-1}}^{x_{i+1}} f(x)\psi_i(x)\mathrm{d}x = \int_{x_{i-1}}^{x_i} -\psi_i(x)\mathrm{d}x = -h \qquad \forall i > \frac{N+1}{2}.
\end{aligned}
$$