Theorem 7.3.8 (Young, Varga) *Let A be consistently ordered and assume that $J = J(1)$ has only real eigenvalues such that $\rho(J) < 1$ (see Lemma 7.3.1). Then*

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\rho(J)^2}} \ , \ \rho(G_{\omega_{opt}}) = \omega_{opt} - 1 \ .$$

□

Remark 7.3.9 *Note that tridiagonal- and block-tridiagonal-matrices are consistently ordered which can easily be verified.*

7.3.2 Projected SOR-method for Complementary Problems

Now, we modify the above described SOR-method for solving the complementary problem

$$(Au - f)^T(u - g) = 0 \ , \ u \geq g \ , \ Au \geq f \tag{7.3.10}$$

which we have seen to be equivalent to (7.3.7), i.e.,

$$u = \max\{D^{-1}(Lu + Uu + f), g\} \ ,$$

if the matrix $A = D - L - U$ is s.p.d. We add a projection step in the SOR-method(7.3.8) (note: there we have used the decomposition $A = D + L + U$):

$$\left\{ \begin{array}{ll} \text{For} & i = 1, \ldots, N \ \text{ do} \\ & z_i^{(k+1)} = \frac{1}{a_{ii}}(Lu^{(k+1)} + Uu^{(k)} + f)_i \\ & u_i^{(k+1)} = \max\{u_i^{(k)} + \omega(z_i^{(k+1)} - u_i^{(k)}), g_i\} \end{array} \right. \tag{7.3.11}$$

which is called *projected SOR-method.*

We aim to prove that (7.3.11) converges towards the solution u of (7.3.10). We need some preparations. The following proofs are taken from [5].

Lemma 7.3.10 *The problem (7.3.10) is equivalent to*

$$u \geq g \ , \ J(u) = \min_{v \geq g} J(v) \tag{7.3.12}$$

where $J(v) := \frac{1}{2}v^T Av = f^T v$, if A is s.p.d.

Proof: Let u be a solution of (7.3.10) and let $v \geq g$, then

$$\begin{aligned} J(v) - J(u) &= \frac{1}{2}v^T Av - \frac{1}{2}u^T Au + f^T(u-v) \\ &= \frac{1}{2}(v-u)^T A(v-u) + u^T Av - u^T Au + u^T f - v^T f \\ &= \underbrace{\frac{1}{2}(v-u)^T A(v-u)}_{\geq 0 \text{ since } A \text{ is s.p.d.}} + v^T(Au-f) - u^T(Au-f) \\ &\geq (v-u)^T(Au-f) \\ &= \underbrace{(v-g)^T}_{\geq 0} \underbrace{(Au-f)}_{\geq 0 \text{ by } (7.3.10)} - \underbrace{(u-g)^T(Au-f)}_{=0 \text{ by } (7.3.10)} \geq 0 \ , \end{aligned}$$

i.e., $J(u) \leq J(v)$ for all $v \geq g$, i.e., u solves (7.3.12).
On the other hand, let $u \geq g$ solve (7.3.12). Let $v^{(k)} := u + \varepsilon\delta_k$, $\varepsilon > 0$ and denote by $\delta_k = (\delta_{1,k}, \ldots, \delta_{n,k},)^T$ the k-th canonical vector. This means that $v^{(k)} \geq u \geq g$ and

$$\begin{aligned} 0 \leq J(v^{(k)}) - J(u) &= \frac{\varepsilon^2}{2}\delta_k^T A\delta_k + \varepsilon\delta_k^T(Au-f) \\ &= \frac{\varepsilon^2}{2}a_{kk} + \varepsilon(Au-f)_k \ , \quad \forall\varepsilon > 0, \end{aligned}$$

which implies $0 \leq (Au-f)_k + \frac{\varepsilon}{2}a_{kk} \to (Au-f)_k$ as $\varepsilon \to 0$ for all k, i.e. $Au \geq f$.
Now suppose $(Au-f)_k > 0$ and $u_k \geq g_k$ for some k. Choose $\varepsilon > 0$ small enough such that $w^{(k)} := u - \varepsilon\delta^k \geq g_k$. This implies $0 \leq J(w^{(k)}) - J(u) = \frac{\varepsilon^2}{2}a_{kk} - \epsilon(Au-f)_k < 0$ for ε small enough, which finally gives $(Au-f)^T(u-g) = 0$. $\square$

Theorem 7.3.11 (Cryer) *Let* $A \in \mathbb{R}^{n\times n}$ *s.p.d.,* $b, f \in \mathbb{R}^n$*,* $1 < \omega < 2$*, then* $\{u^{(k)}\}_{k\in N}$ *defined by (7.3.11) converges towards the unique solution* u *of (7.3.10).*

Remark 7.3.12 *The latter theorem also states that the complementary problem (7.3.10) has a unique solution.*

Proof of Theorem 7.3.11:
We split the proof in two parts.

1.) **Uniqueness of solution:**
Let w_1, w_2 be two solutions of (7.3.10) and (7.3.12), respectively. Thus, we have by Lemma 7.3.10

$$\begin{aligned} 0 &= J(w_1) - J(w_2) \\ &= \tfrac{1}{2}\underbrace{(w_1-w_2)^T A(w_1-w_2)}_{\geq 0} + \quad \underbrace{w_1^T(Aw_2-f) - w_2^T(Aw_2-f)} \\ &\qquad\qquad\qquad\qquad\qquad\qquad = (w_1-w_2)^T(Aw_2-f) \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \underbrace{(w_1-g)^T}_{\geq 0}\underbrace{(Aw_2-f)}_{\geq 0} - \underbrace{(w_2-g)^T(Aw_2-f)}_{=0} \\ &\geq \tfrac{1}{2}(w_1-w_2)^T A(w_1-w_2) \geq 0, \end{aligned}$$

i.e., $0 = (w_1-w_2)^T A(w_1-w_2)$, which means that $w_1 - w_2 = 0$.

2.) **Existence of a solution:**
The idea is to show convergence of $u^{(k)}$ in (7.3.11)

a) For all i, k there exists some $\omega_k \in [0, w]$ such that for $u_i^{(k+1)} := u_i^{(k)} + \omega_{i,k}(z_i^{(k+1)} - u_i^{(k)})$ we have

- If $g_i \leq u_i^{(k)} + \omega(z_i^{(k+1)} - u_i^{(k)})$ and thus $\omega_{ik} = \omega$.
- If $g_i > u_i^{(k)} + \omega(z_i^{(k+1)} - u_i^{(k)})$and thus we have $u_i^{(k+1)} = g_i$ since $u_i^{(k)} \geq g_i$
  (because $u_i^{(k)} = \max\{\ldots, g_i\}$), we have $z_i^{(k+1)} - u_i^{(k)} < 0$ and hence
  $$\omega_{i,k} := \frac{u_i^{(k)} - gi}{u_i^{(k)} - z_i^{(k+1)}} \geq 0$$
  and $\omega_{i,k} < \omega$. This implies that $u_i^{(k)} + \omega_{i,k}(z_i^{(k+1)} - u_i^{(k)}) = u_i^{(k)} - u_i^{(k)} + g_i = g_i = u_i^{(k+1)}$.

b) Set $u^{(k,i)} = (u_1^{(k+1)}, \ldots, u_i^{(k+1)}, u_{i+1}^{(k)}, \ldots, u_N^{(k)})^T$ and

$$J_j := J(u^{(k,i)}), \quad j := N(k-1) + i$$

Note that

$$u^{(k,i,0)} = (u_1^{(k+1)}, \ldots, u_N^{(k+1)})^T := u^{(k,N+1)}$$

and

$$(u^{(k,i)} - u^{(k,i-1)}) = (u_i^{(k+1)} - u_i^{(k)})\delta_i \tag{7.3.13}$$

for $\delta_i = (\delta_{1,i}, \ldots, \delta_{N,i})^T$. By (7.3.11)

$$\begin{aligned} a_{ii}(z_i^{(k+1)} - u_i^{(k)}) &= (Lu^{(k+1)} + Uu^{(k)} + f)_i - a_{ii}u_i^{(k)} \\ &= -(Au^{(k,i)})_i + f_i \\ &= -(Au^{(k,i)} - f)_i \qquad (7.3.14) \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \Rightarrow J_j - J_{j-1} &= J(u^{(k,i)}) - J(u^{(k,i-1)}) \\ &= \tfrac{1}{2}(u^{(k,i)} - u^{(k,i-1)})^T A(u^{(k,i)} - u^{(k,i-1)}) \\ &\quad +(u^{(k,i)} - u^{(k,i-1)})^T (Au^{(k,i)} - f) \\ &= \tfrac{1}{2}a_{ii}(u_i^{(k+1)} - u_i^{(k)})^2 - a_{ii}(u_i^{(k+1)} - u_i^{(k)}) \underbrace{(z_i^{(k+1)} - u_i^{(k)})}_{=\frac{1}{\omega_{i,k}}(u_i^{(k+1)} - u_i^{(k)})} \\ &= -\tfrac{a_{ii}}{2}\left(\tfrac{2}{\omega_{ik}} - 1\right)(u_i^{(k+1)} - u_i^{(k)})^2 \\ &\le -\underbrace{\frac{a_{ii}}{2}}_{>0}\underbrace{\left(\frac{2}{\omega} - 1\right)}_{>0}\underbrace{(u_i^{(k+1)} - u_i^{(k)})^2}_{\ge 0} \qquad \text{if } \omega_{i,k} > 0 \\ &\le 0. \end{aligned}$$

If $\omega_{i,k} = 0$ we have $u_i^{(k+1)} = u_i^{(k)}$, hence $J_j = J_{j-1}$, i.e., $J_j \searrow$ $(j \to \infty)$.
On the other hand $J_j = \frac{1}{2}\underbrace{u^{(k,i)^T} A u^{(k,i)}}_{\ge 0} - \underbrace{f^T u^{(k,i)}}_{\le c}$ which implies the existence of the limit $J = \lim\limits_{j\to\infty} J_j$.

c) A standard estimate yields for any component index i

$$\begin{aligned} |u_i^{(k+1)} - u_i^{(k)}| &= \left(\frac{2}{a_{ii}(\frac{2}{\omega_{i,k}} - 1)}(J_{j-1} - J_j)\right)^{1/2} \\ &\le \left(\frac{2}{\min\limits_i a_{ii}(\frac{2}{w_{ik}}\omega - 1)}(J_{j-1} - J_j)\right)^{1/2} \to 0, \end{aligned}$$

which means that $\{u_i^{(k)}\}_k$ is a Cauchy sequence. Thus, there exists some u_i such that $\lim\limits_{k\to\infty} u_i^{(k)} = u_i$.

d) If we define

$$z_i : \quad = \quad \lim_{k\to\infty} z_i^{(k+1)} = \frac{1}{a_{ii}}(Lu + Uu + f)_i = u_i - \frac{1}{a_{ii}}(Au - f)_i$$

we get

$$u_i = \max\{u_i + \omega(z_i - u_i), g_i\} = \max\{u_i - \omega\frac{1}{a_{ii}}(Au - f)_i, g_i\}.$$

Thus, we have $\min\{\frac{\omega}{a_{ii}}(Au - f)_i, u_i - g_i\} = 0$, which is equivalent to (7.3.10), which proves the theorem. ☐

Chapter 8

Exotic Options

All nonstandard options (i.e., non American and European) are called exotic options. Nowadays there is a whole variety of such taylor-made options. The general observation for such options is that transformation methods to simple pde's like the heat equations do not work. The corresponding pde has to be solved directly which causes several numerical difficulties. Let us just collect some main types of exotic options without going into details and without claiming completeness:

- compound options
- chooser options
- binary options
- path-dependent options: barrier options, lookback options, Asian options.

Some of these options can be reduced to the Black–Scholes equation. Here, we focus on some aspects for those cases where this reduction is not possible.

8.1 Asian Options

As an example, let us consider Asian options whose characteristics is the dependency of its price from an average of values of the underlying S_t at previous times. Asian options can be of European and American style. The average can be taken in an arithmetic sense

$$\frac{1}{m}\sum_{i=1}^{n} S_{t_i}, \quad \frac{1}{T}\int\limits_0^T S_t \, dt$$

or a geometric sense

$$\left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n}, \quad \exp\left(\frac{1}{T}\int\limits_0^T \log(S_t) \, dt\right).$$

Let us denote by $\bar{S}$ one of these averages.

Definition 8.1.1 *With the average $\bar{S}$ of S_t, the payoff function of an Asian option is defined as*

$$\begin{array}{ll} (\bar{S}-K)^+, \ (K-\bar{S})^+ & \textit{average strike call / put} \\ (S_T-\bar{S})^+, \ (\bar{S}-S_T)^+ & \textit{average price call / put.} \end{array}$$

For the modelling, the average $\bar{S}$ is exposed as

$$A_t := \int\limits_0^t f(S_\theta, \theta) d\theta, \tag{8.1.1}$$

where $f(S,t)$ models the type of average. The stochastic behaviour of A_t is described in the Black–Scholes model as

$$dA_t = a_A(t)dt + b_A dW_t. \tag{8.1.2}$$

In (8.1.1) we would just have

$$a_A(t) = f(S_t, t), \quad b_A \equiv 0.$$

The advantage of (8.1.2) is that Itô's lemma can be used to derive the following SDE for the price $V = V(S, A, t)$

$$dV_t = \left(\frac{\partial}{\partial t}V + \mu S\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}V + f(S,t)\frac{\partial}{\partial A}V\right) dt + \sigma S \frac{\partial}{\partial S} dW_t, \tag{8.1.3}$$

or as a pde

$$\frac{\partial}{\partial t}V + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}V + rS\frac{\partial}{\partial S}V + f(S,t)\frac{\partial}{\partial A}V - rV = 0. \tag{8.1.4}$$

Hence, we obtain the additional term

$$f(S,t)\frac{\partial}{\partial A}V.$$

Note that no second derivative of V with respect to A is present which makes the problem convection dominated or even hyperbolic.

First one can reduce the problem to a 1D problem (details see [14], 216-219, exercise) by introducing the auxiliary variable

$$R_t := \frac{1}{S_t}\int_0^t S_\theta d\theta, \tag{8.1.5}$$

where we restrict ourselves for simplicity to the arithmetic average, i.e., $f(S,t) = S$. Then, one assumes a separation of the form

$$V(S,A,t) = S \cdot H(R,t) \tag{8.1.6}$$

for some function H. Here, R is an independent variable. Then, one gets the following problem

$$\frac{\partial}{\partial t}H + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2}{\partial R^2}H + (1 - rR)\frac{\partial}{\partial R}H = 0 \tag{8.1.7}$$

$$H = 0 \quad \text{for} \quad R \to \infty \tag{8.1.8}$$

$$\frac{\partial}{\partial t}H + \frac{\partial}{\partial R}H = 0 \quad \text{for} \quad R = 0 \tag{8.1.9}$$

$$H(R_T, T) = \left(1 - \frac{1}{T}R_T\right)^+. \tag{8.1.10}$$

The obvious advantage is the reduction of the dimension of the problem.

8.2 Convection–Diffusion Problems

We can view (8.1.7) as a particular example of a convection-diffusion problem of the form

$$\partial_t u - (au' - cu)' = f \tag{8.2.1}$$

for $u = u(t, x)$. In fact

$$\begin{aligned}(au' - cu)' &= au'' + a'u' - cu' - c'u \\ &= au'' + (a' - c)u' - c'u.\end{aligned}$$

Such equations appear in many applications e.g. transport, reaction in porous media, semiconductors and so on. The behaviur of such equations can be characterised by the **global Peclét number**

$$Pe := \frac{\|c\|_\infty \operatorname{diam}(\Omega)}{\|a\|_\infty}, \tag{8.2.2}$$

where Ω is the spatial domain, $x \in \Omega$. The global Peclét number measures the proportion of reaction/convection to diffusion. For example, one has

- $Pe \sim 25$ (groundwater transport)
- $Pe \sim 10^7$ (semiconductor)

A little bit vague a problem (8.2.1) is called **convection dominated** if

$$Pe \gg 1.$$

Example 8.2.1 *For $k > 0$ consider*

$$\begin{cases} -(ku' + u)' = 0 \quad \text{in} \quad \Omega := (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{8.2.3}$$

which has the exact solution

$$u(x) = \frac{1 - \exp(x/k)}{1 - \exp(1/k)}.$$

*The global Peclét number is given as $Pe = \frac{1}{k}$. Even for moderate values of Pe (e.g. $Pe \sim 100$), we observe a strong **boundary layer**:*

Let us describe what happens for the numerical solution. Let us consider a FDM w.r.t. an equidistant grid with step size

$$h = \frac{1}{M+1}$$

and use control differences for both terms

$$\left(-\frac{2k}{h} - 1\right) u_{i-1} + \frac{4k}{h} u_i + \left(-\frac{2k}{h} + 1\right) u_{i+1} = 0,\ 1 \le i \le M. \qquad (8.2.4)$$

Using the ansatz $u_i = \lambda'$ gives an exact solution

$$u_i = \frac{1 - \left(\frac{2k+h}{2k-h}\right)'}{1 - \left(\frac{2k+h}{2k-h}\right)^{M+1}}.$$

*For $2k < h$ (which is realsitic for $k \sim 10^{-7}$) we obtain heavy oscillations in the numerical solution. These oscillations do **not** occur in the exact solution. For $2k > h$, the oscillations disappear but one may lack convergence.*

Let us describe some of the technical problems that occur here. Let us consider the pde

$$-\varepsilon u'' + cu' + ru \qquad (8.2.5)$$

which is of the form (8.2.1). The bilinear form associated to (8.2.5) reads

$$a(u,v) = (\varepsilon u', v') + (cu', v) + (ru, v), \quad u, v \in H_0^1(\Omega) =:?? \qquad (8.2.6)$$

If we assume that

$$r - \frac{1}{2}c' \ge r_0$$

for some constant $r_0 > 0$ one can easily show (exercise)

$$a(v,v) \ge \tilde{\alpha}\|v\|_{\varepsilon}^2, \quad \tilde{\alpha} := \min\{1, r_0\}$$

where

$$\|v\|_\varepsilon := (\varepsilon|v|_1^2 + \|v\|_0^2)^{1/2} \tag{8.2.7}$$

is the ε-weighted H^1-norm. Performing the complete error analysis gives

$$\|u - u_h\|_\varepsilon \le Ch\varepsilon^{-3/2}, \quad \varepsilon \to 0^+. \tag{8.2.8}$$

Note that the factor $\varepsilon^{-3/2}$ can be extremely large.

8.3 SUPG-method

The SUPG (streamline upwind Petrov-Galerkin) method is maybe the most commonly used method for the numerical solution of convection-dominated problems. The main idea was introduced by Hughes and Brooks in 1979. In order to describe the method, we consider the model problem in 2D, namely

$$\begin{aligned} L_\varepsilon u := -\varepsilon\Delta u + c\cdot\nabla u + ru &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \Gamma = \partial\Omega, \end{aligned} \tag{8.3.1}$$

with $\varepsilon > 0$, coefficient functions $c \in C^1(\bar\Omega, \mathbb{R}^n)$, $r \in C(\bar\Omega)$ and a given right-hand side $f \in L_2(\Omega)$. As above, we assume

$$r - \frac{1}{2}\nabla\cdot c \ge r_0 > 0$$

for some constant $r_0 > 0$. The bilinear form reads

$$a(u,v) := \int_\Omega \Big[\varepsilon\nabla u\cdot\nabla v + c\cdot\nabla u\, v + r\, uv\Big]\, dx$$

for $u, v \in V := H_0^1(\Omega)$, so that the variational formulation reads

$$u \in V: \qquad a(u,v) = (f,v)_{0;\Omega}, \quad v \in V. \tag{8.3.2}$$

Using a standard finite element discretization with test and trial spaces

$$V_h := \{v_h \in V : v_h|_K \in \mathcal{P}_k(K),\ K \in \mathcal{T}_h\},$$

we obtain a standard error estimate

$$\inf_{v_h\in V_h} \|u - v_h\|_{l,K} \le c_{\text{err}}\, h_K^{k+1-l}|u|_{k+1,K}$$

if $u \in H^{k+1}(\Omega)$ for $l \in \{0,1,2\}$, all $K \in \mathcal{T}_h$ and a constant c_{err}. This is a standard finite element error estimate (Jackson inequality). Moreover, there is also an *inverse inequality*

$$\|\Delta v_h\|_{0,K} \leq \frac{c_{\text{in}}}{h_K}|v_h|_{1,K}, \qquad v_h \in V_h$$

for all $K \in \mathcal{T}_h$. Such an estimate is based upon the finite dimensionality of V_h. Note that the two constants $c_{\text{err}}, c_{\text{in}} > 0$ do not depend on u and v_h, nor on K.

The basic idea to overcome the above described stability problems is to add suitable (local) weighted residuals to the variational formulation (8.3.2). Interpreting the original problem in L_2 and restricting it to each element K gives

$$L_\varepsilon u = f \quad \text{a.e. in } K, \quad K \in \mathcal{T}_h.$$

Now we multiply this equation with test functions

$$\tau(v_h)_{|K},$$

where $\tau : L_2(\Omega) \to L_2(\Omega)$ is a suitable function to be detailed later. Moreover, we introduce scaling factors

$$\delta_K \in \mathbb{R}, \qquad K \in \mathcal{T}_h,$$

and obtain

$$\sum_{K \in \mathcal{T}_h} \delta_k(-\varepsilon\Delta u + c \cdot \nabla u + ru, \tau(v_h))_{0,K} = \sum_{K \in \mathcal{T}_h} \delta_K(f, \tau(v_h))_{0,K}.$$

This equation is added to the discrete variational problem and we obtain

$$\begin{aligned} a_h(u, v_h) &:= a(u, v_h) + \sum_{K \in \mathcal{T}_h} \delta_k(-\varepsilon\Delta u + c \cdot \nabla u + ru, \tau(v_h))_{0,K} \\ (f, v_h)_h &:= (f, v_h)_{0,\Omega} + \sum_{K \in \mathcal{T}_h} \delta_K(f, \tau(v_h))_{0,K} \end{aligned}$$

and the following new discrete problem results

$$u_h \in V_h : \qquad a_h(u_h, v_h) = (f, v_h)_h, \quad v_h \in V_h.$$

If the original and the new discrete problem have unique solutions, we obtain the error equation (Galerkin orthogonality)

$$a_h(u - u_h, v_h) = 0, \qquad v_h \in V_h.$$

Remark 8.3.1 *(a) Well-known choices for τ are*

- $\tau(v_h) := c \cdot \nabla v_h$ *(streamline-diffusion method)*
- $\tau(v_h) := -\varepsilon \Delta v_h + c \cdot \nabla v_h + r v_h$ *(Galerkin/least Squares).*

(b) One can show that the additional term adds some artificial diffusion *which is the reason for the stabilization.*

With quite some technicalities, one can prove the following error estimate.

Theorem 8.3.2 *Let the parameters be chosen as*

$$\delta_K = \begin{cases} \delta_1 \frac{h_K^2}{\varepsilon}, & \text{if } Pe_K \leq 1, \\ \delta_2 h_K, & \text{if } Pe_K > 1, \end{cases}$$

with $\delta_1, \delta_2 > 0$ are independent of K and ε is chosen in such a way that

$$0 < \delta_K \leq \frac{1}{2} \left\{ \frac{h_K^2}{\varepsilon c_{\text{inv}}^2}, \frac{r_0}{\|r\|_{0,\infty,K}} \right\}.$$

If the weak solution is in $K^{k+1}(\Omega)$, then

$$\|u - u_h\|_{\text{sd}} \leq C(\sqrt{\varepsilon} + \sqrt{h}) h^k \, |u|_{k+1,\Omega},$$

where $\| \cdot \|_{\text{sd}}$ denotes the streamline diffusion norm

$$\|u\|_{\text{sd}}^2 := \varepsilon |v|_1^2 + r_0 \|v\|_0^2 + \sum_{K \in \mathcal{T}_h} \delta_k \|c \cdot \nabla u\|_{0,K}^2.$$

□

Remark 8.3.3 *(a) The reason for the name* Petrov-Galerkin *comes from the fact that the above streamline diffusion method can also be interpreted as a variational problem where one uses different test and trial spaces. This is called a Petrov-Galerkin method.*

(b) The major drawback is that still there is a dependence on negative powers of ε. Alternatives are Finite Volume methods or Discontinuous Galerkin methods.

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