Simulation of Typical Modulated Poisson-Voronoi Cells with Applications to Telecommunication Network Modelling

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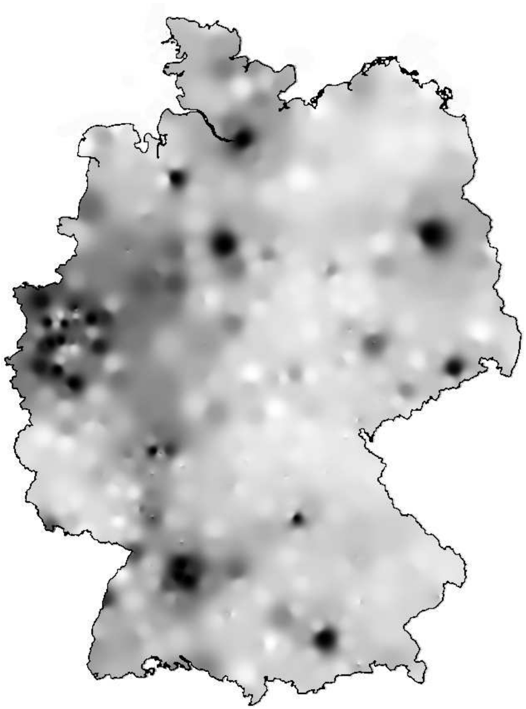
Overview

1. Motivation
2. Modulated Poisson-Voronoi Tessellations
3. Simulation of the Typical Cell
4. Numerical Results
5. Summary and Outlook

Motivation

- Consider telecommunication networks on nationwide scales => population intensities



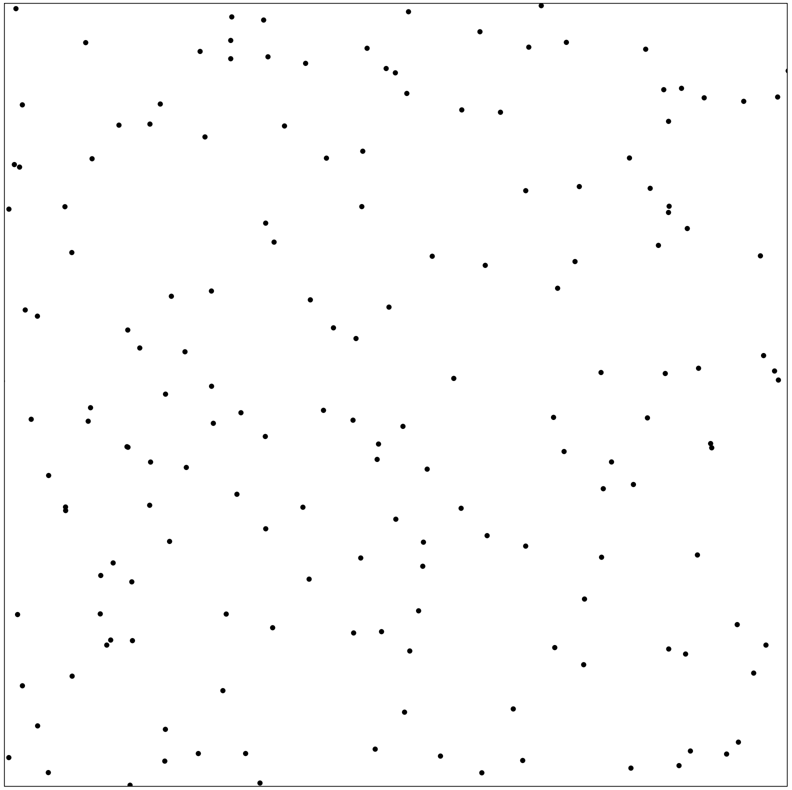
Estimation of population intensities in Germany

Motivation

- Models needed that can reflect a great variety of scenarios
- First step analysis of modulated poisson point processes w.r.t.
  - corresponding Voronoi tessellation
  - simulation of typical cell
  - cost analysis
- Voronoi cells can reflect serving zones in the access network
- Points might also be considered as locations of antennas in a mobile scenario

Modulated Poisson-Voronoi tessellations

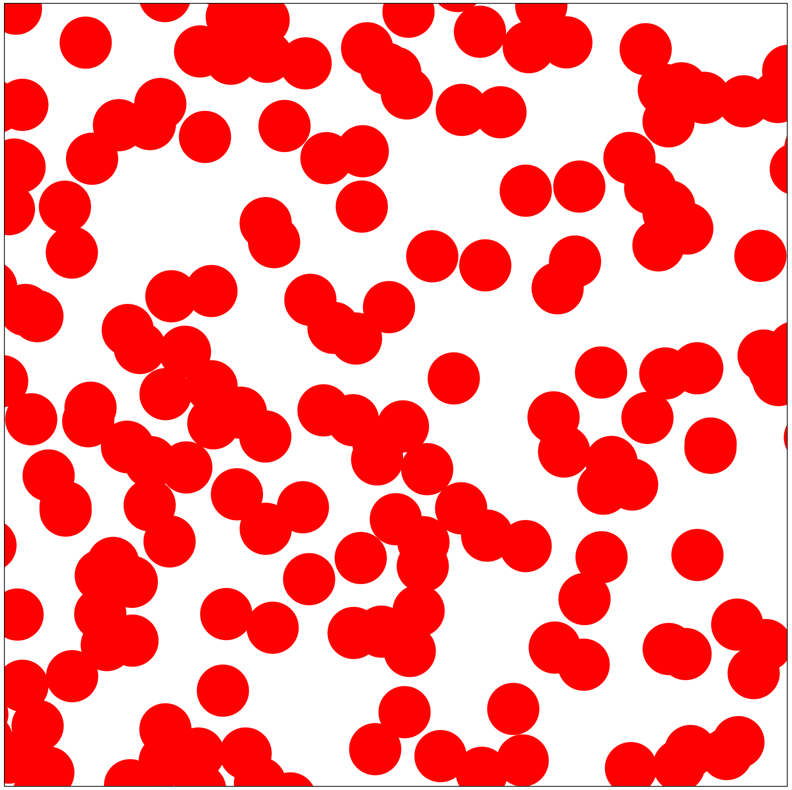
- Based on a Boolean model Ψ that has circular grains with fixed radii



Realisation of a Poisson process

Modulated Poisson-Voronoi tessellations

- Based on a Boolean model Ψ that has circular grains with fixed radii

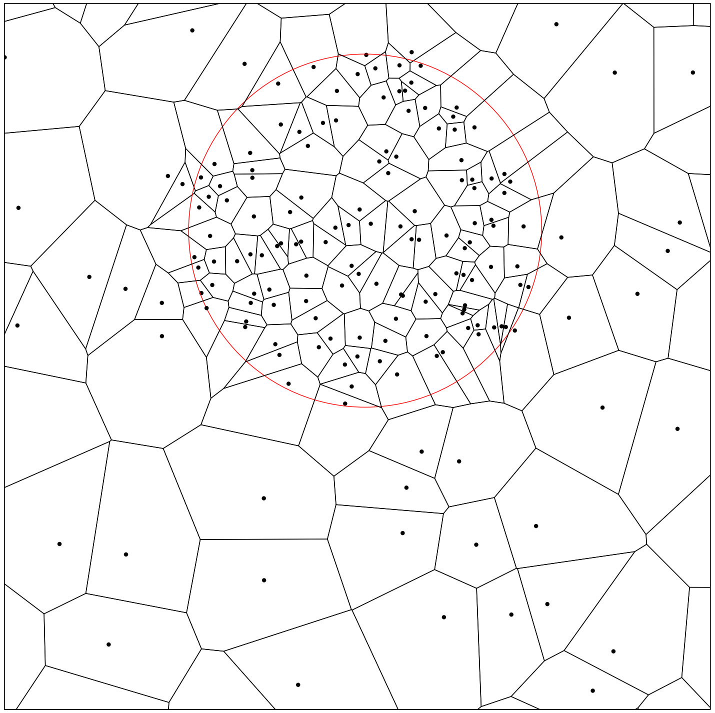


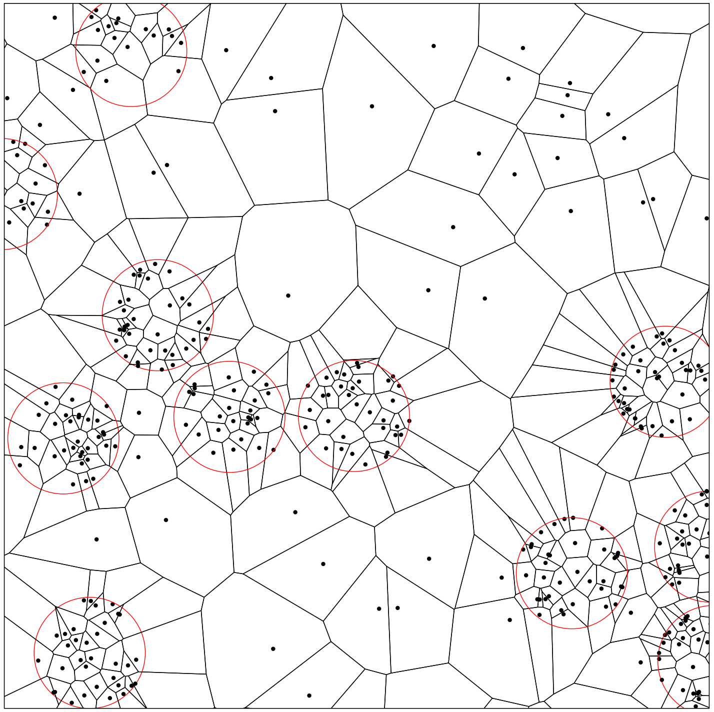
Realisation of a Boolean model Ψ

Modulated Poisson-Voronoi tessellations

- Consider a (planar) Cox point process X with (random) intensity measure $\Lambda_X(.)$ given by
  - $$\Lambda_X(dx) = \begin{cases} \lambda_1 dx & \text{if} \quad x \in \Psi \\ \lambda_2 dx & \text{if} \quad x \notin \Psi \end{cases}$$
  - X is called a Ψ-modulated Poisson process
- Consider corresponding Voronoi tessellation τ_X
  - Stationary model
  - Allows modelling instationarities

Modulated Poisson-Voronoi tessellations





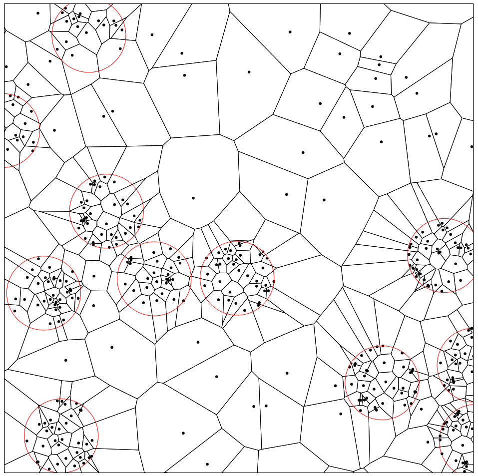
Realisations of modulated Poisson-Voronoi tessellations

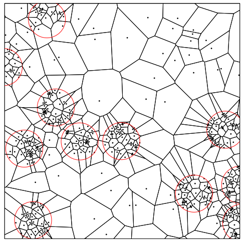
Modulated Poisson-Voronoi tessellations

- 4 model parameters
  - Intensity β of the germs of Ψ
  - Radius R of the circular grains of Ψ
  - Intensities λ_1 and λ_2 of X
- Derived characteristics (Ψ stationary)
  - Coverage probability
    $p = P(0 \in \Psi) = 1 - \exp\left(\beta\pi E(R^2)\right)$
  - Intensity of the modulated Poisson process
    $\lambda_X = p\lambda_1 + (1-p)\lambda_2$

Modulated Poisson-Voronoi tessellations
Scaling invariance

- Initial model has 4 parameters (λ_1, λ_2, β, R)
- These 4 parameters can be reduced to 3 parameters $\underline{\kappa} = (\kappa_1, \kappa_2, \kappa_3)^t$ using scaling invariance properties
  - $\kappa_1 = p$
  - $\kappa_2 = \lambda_1/\beta$
  - $\kappa_3 = \lambda_2/\beta$

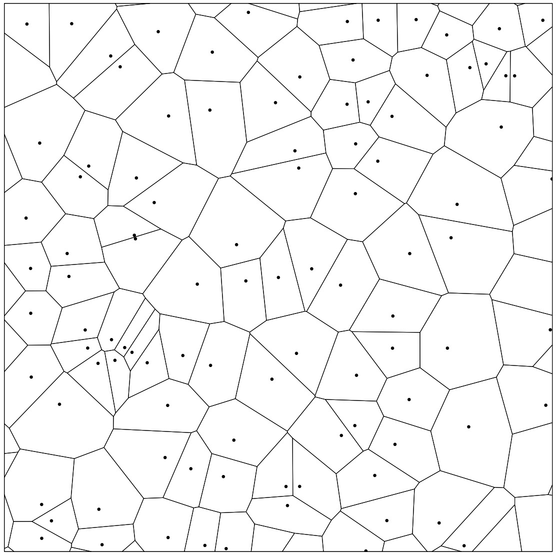




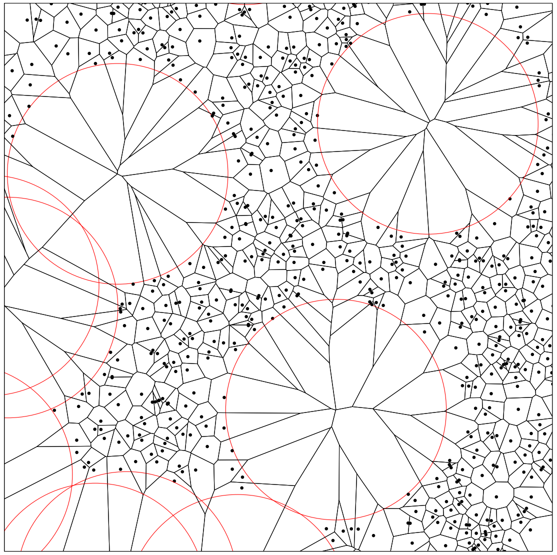
Same random structure but different scale

Modulated Poisson-Voronoi tessellations
Some special cases

- Poisson-Voronoi ($\lambda_1 = \lambda_2$ or $p = 0$ or $p = 1$)



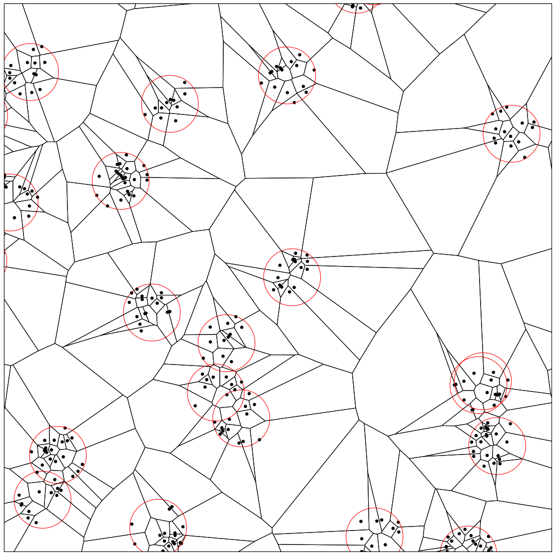
- Swiss-cheese ($\lambda_1 = 0$)



Modulated Poisson-Voronoi tessellations

Some special cases

- Inner-city ($\lambda_2 = 0$)



Simulation of Typical Cell

- Typical (Voronoi-)cell Ξ^* is drawn uniformly from all cells
- Some functionals of interest
  - Area $\nu_2(\Xi^*)$
  - Perimeter $\nu_1(\partial\Xi^*)$
  - Number of vertices $\eta(\Xi^*)$
  - Cost functional(s)
    - $c(\Xi^*) = \int_{\Xi^*} ||u|| du$
    - $c'(\Xi^*) = \lambda_X \int_{\Xi^*} ||u|| du$
- Moments and distribution

Simulation of Typical Cell

- Advantages compared to large sampling window methods
  - No edge effect problems
  - Simulation easily partitionable (=> parallelisation)
  - No memory problems
- Drawbacks
  - Simulation not clear
  - Efficient stopping criteria needed
    - Initial cell
    - Typical cell

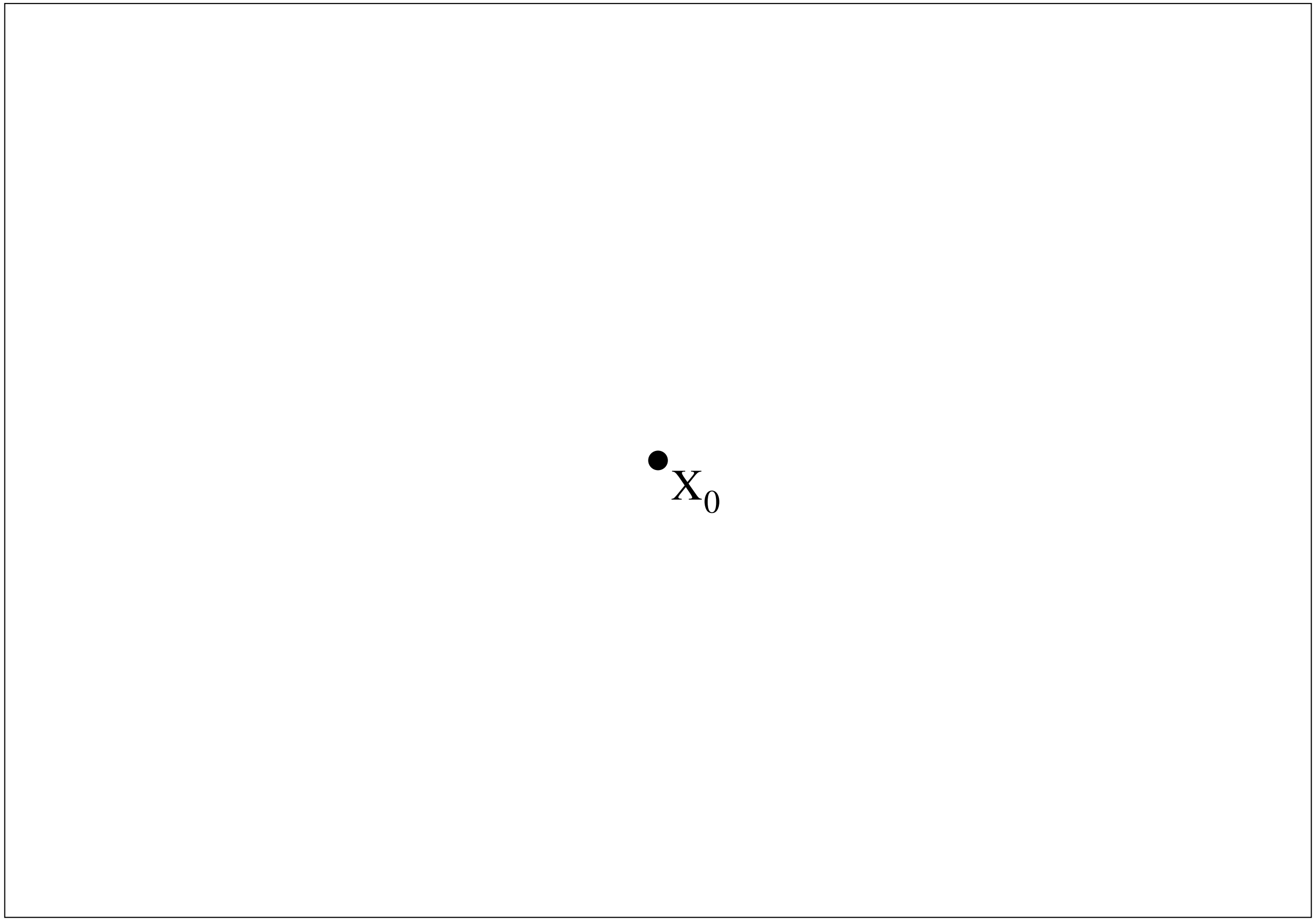
Simulation of Typical Cell
Algorithm

- Simulation based on representation of typical cell
  - $P_o = \delta_{\delta_o} * P_{Q_o}$
  - δ_{δ_o} distribution of a (deterministic) point in o
  - Q_o Palm distribution of Λ_X at o
  - P_{Q_o} distribution of a Cox process with measure Q_o
- Additional point is added to X in o
  - Coverage probability $p_c = P(o \in \Psi || o \in X) \neq p$
  - $p_c = \frac{p\lambda_1}{p\lambda_1 + (1-p)\lambda_2}$
- Alternating radial simulation of X and Ψ
  - For $X_i \in X$ it must be known if $X_i \in \Psi$
  - => Simulate Ψ_j until $||\Psi_j|| > ||X_i|| + R$

Simulation of Typical Cell
Algorithm

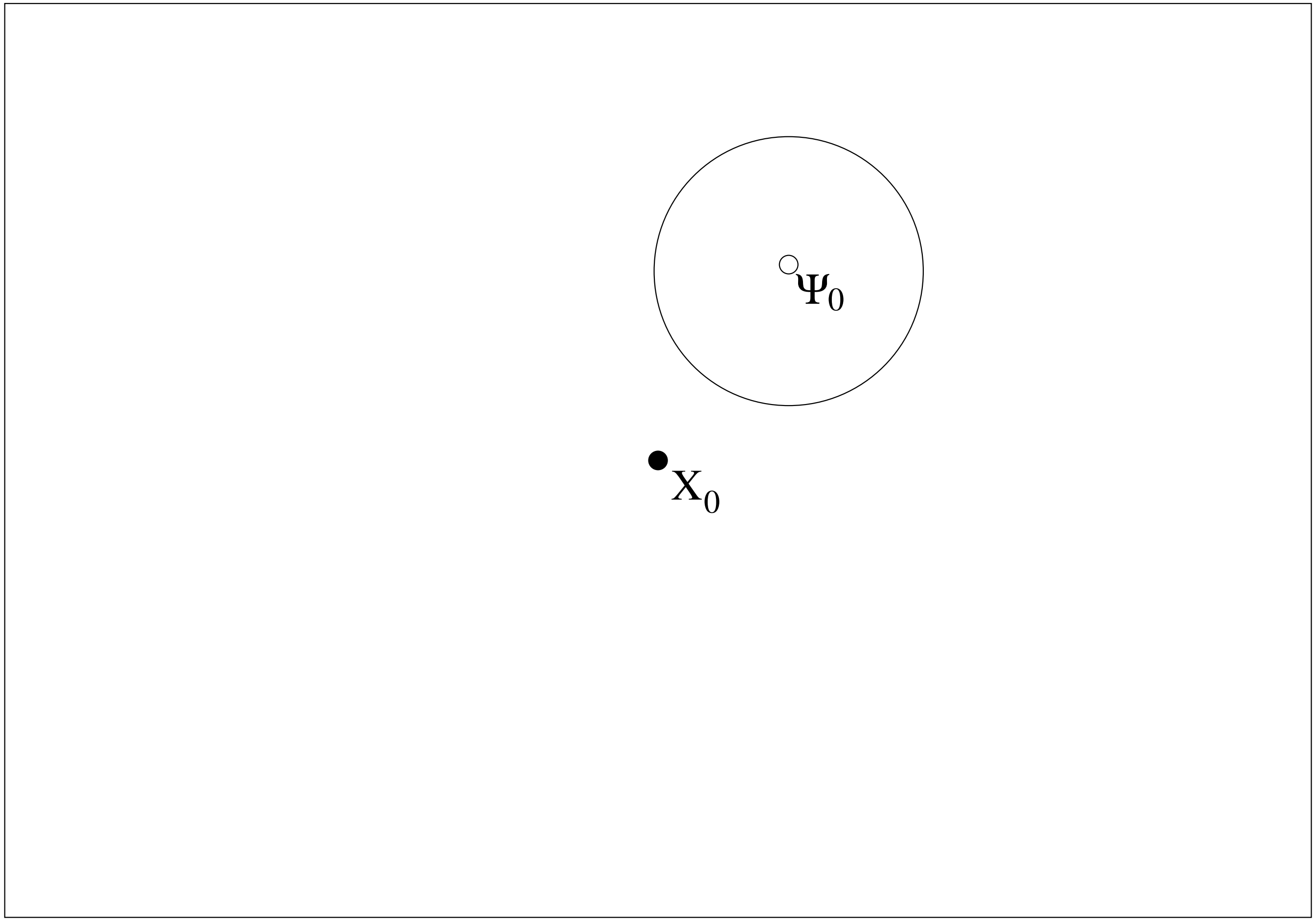
1. Put point $X_0 \in X$ in o
2. Determine if $o \in \Psi$ using p_c
3. Simulate grains Ψ_j of Ψ
   - Germ radially Poisson (intensity β)
   - Conditional to $o \in \Psi$ for Ψ_o
4. Simulate points $X_i \in X$
   - Radially Poisson simulation
   - Intensity $\lambda_{max} = \max\{\lambda_1, \lambda_2\}$
   - Thinning by λ_2/λ_1 or by λ_1/λ_2

Simulation of Typical Cell
Algorithm



Point X_0 in origin

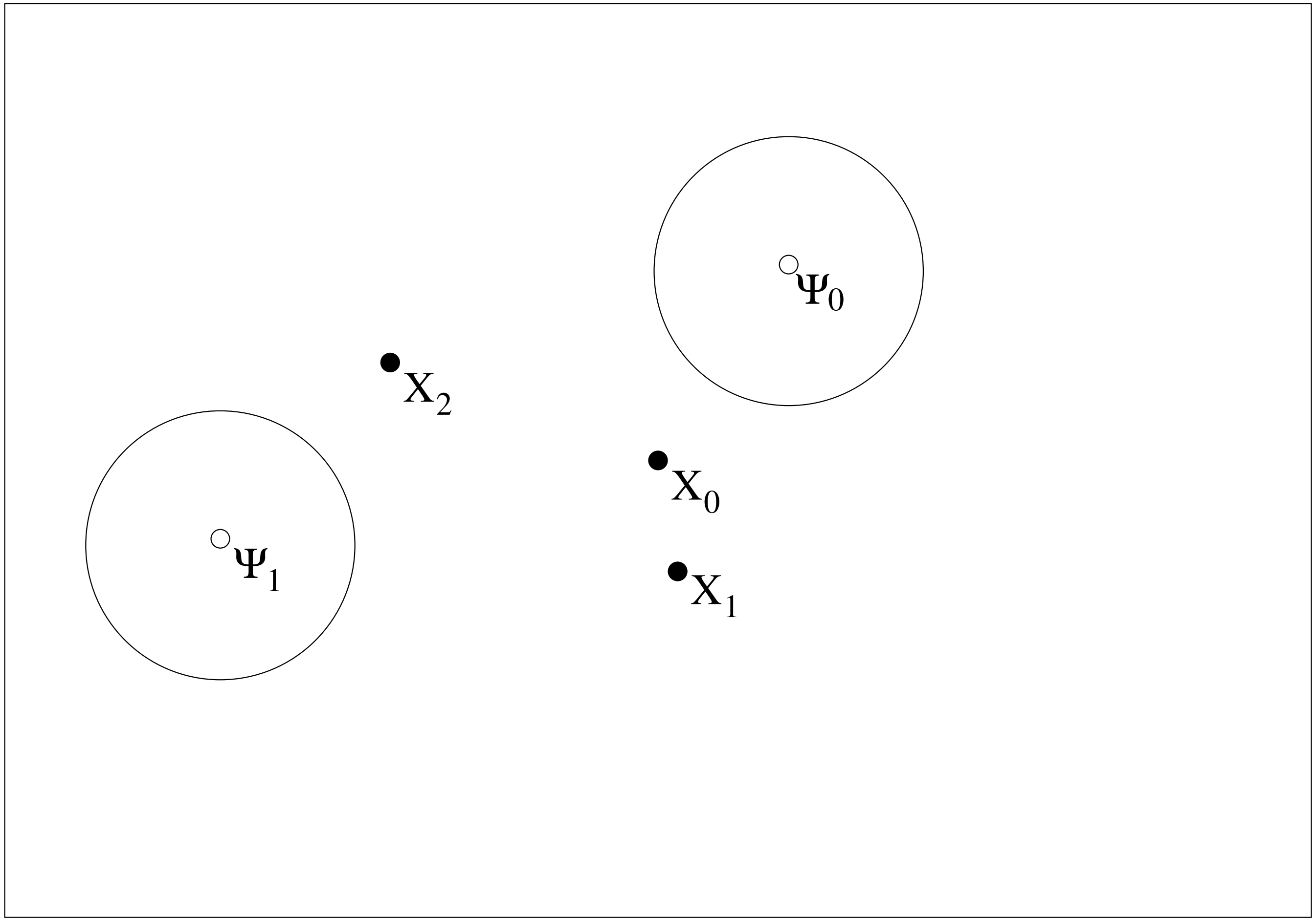
Simulation of Typical Cell
Algorithm



First grain with midpoint Ψ_0

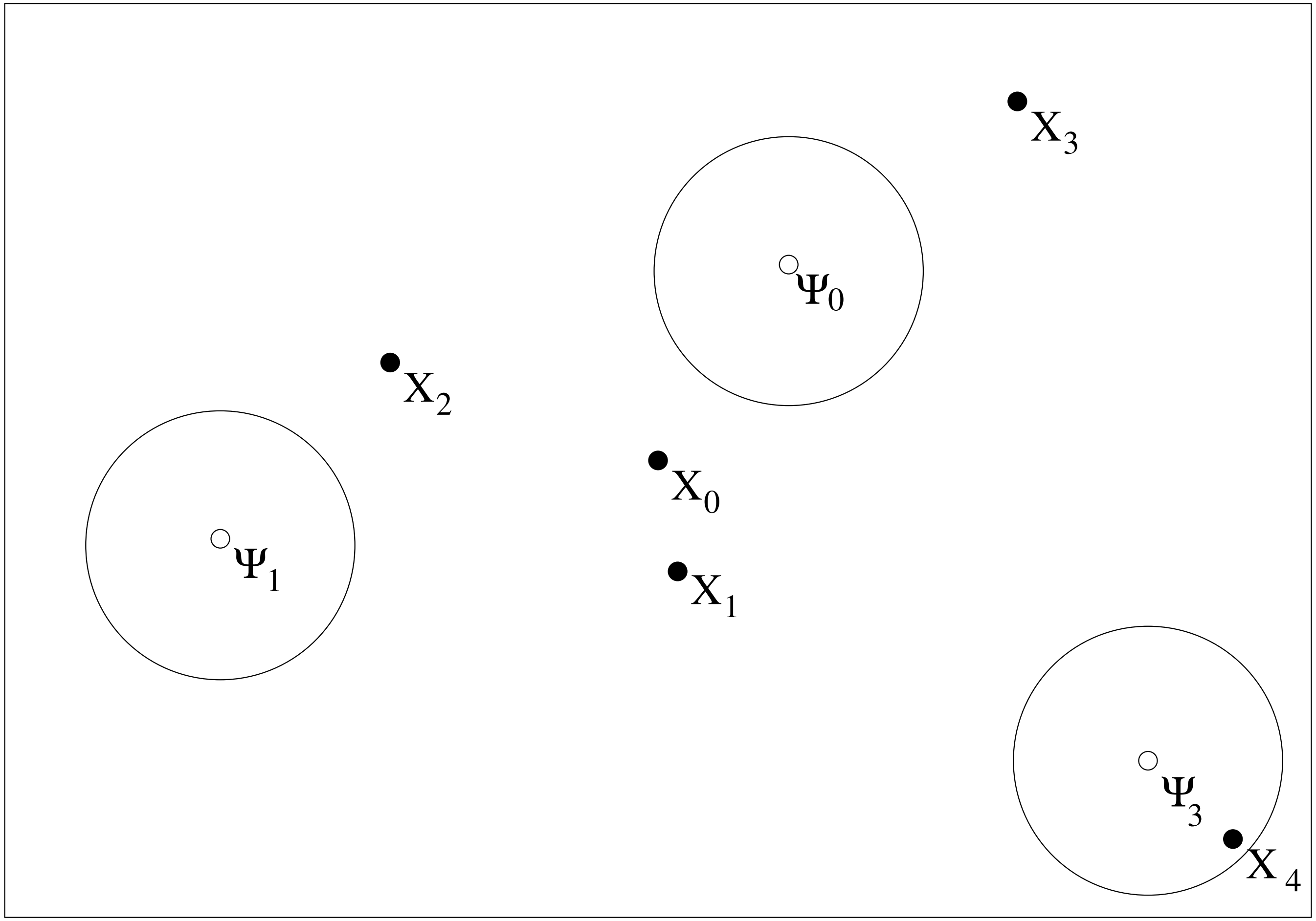
Simulation of Typical Cell

Algorithm



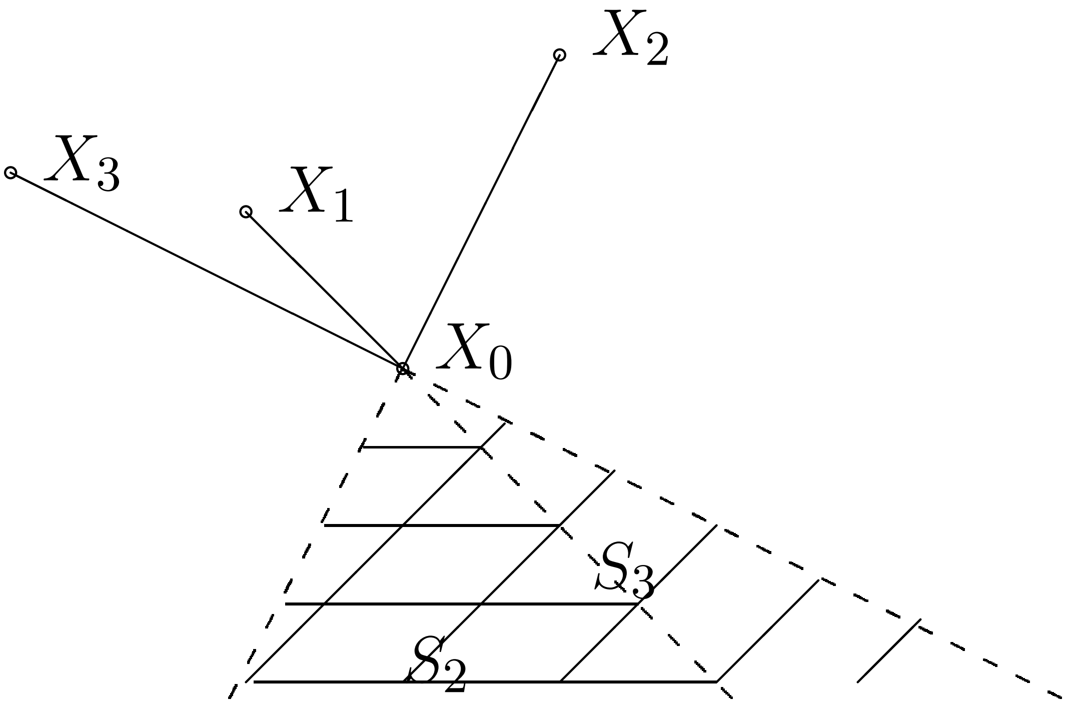
For X_2 more information about Ψ is needed

Simulation of Typical Cell
Algorithm



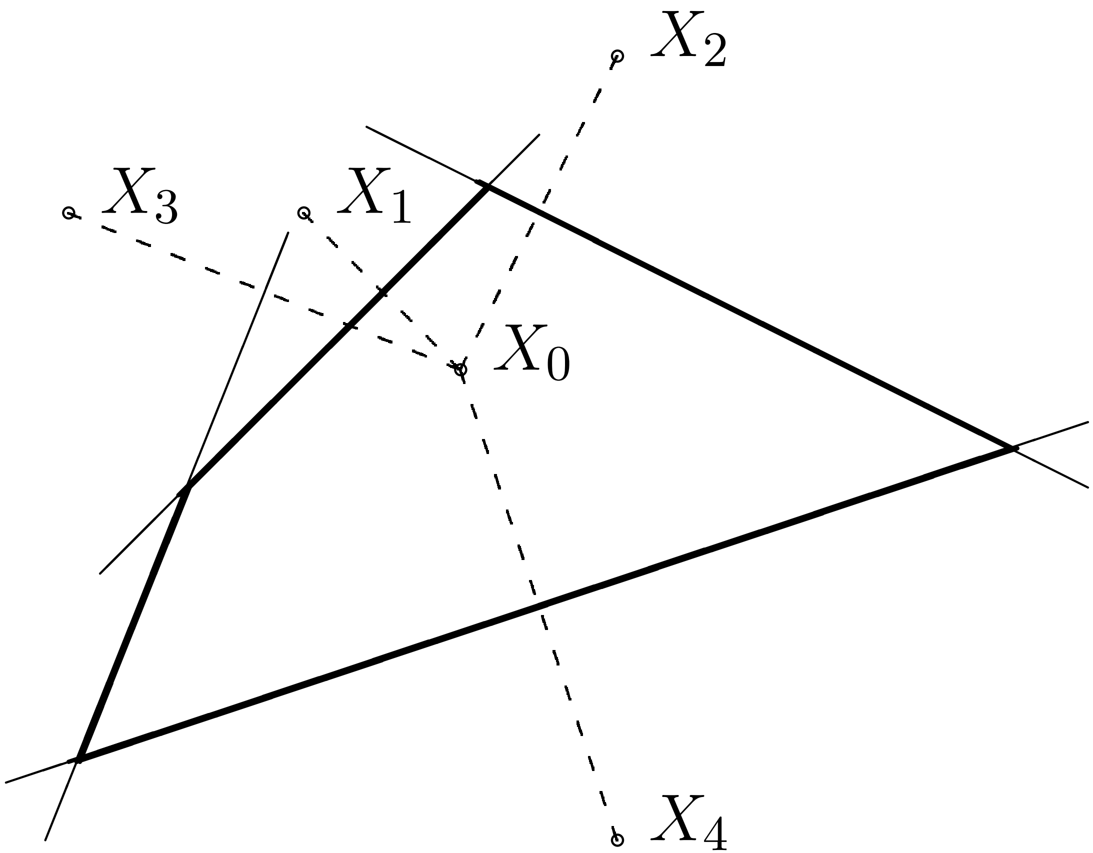
Further alternating simulation of X and Ψ

Simulation of Typical Cell
Algorithm



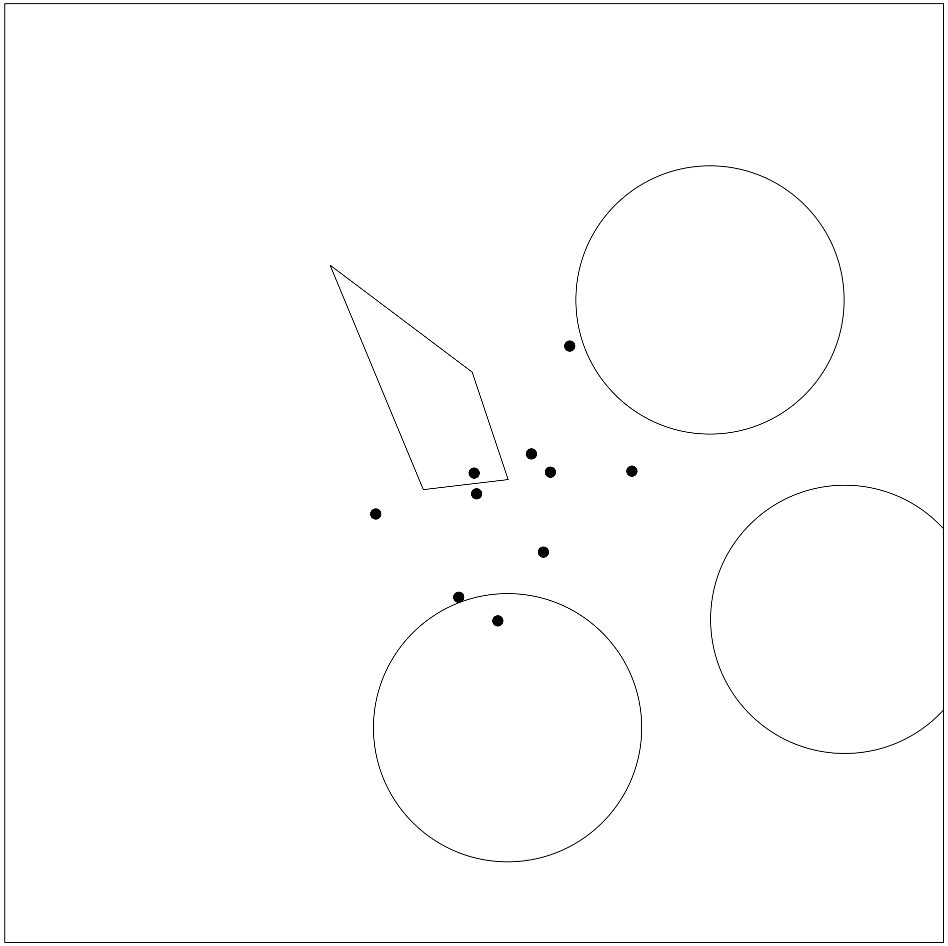
Stopping criterion for initial cell

Simulation of Typical Cell
Algorithm



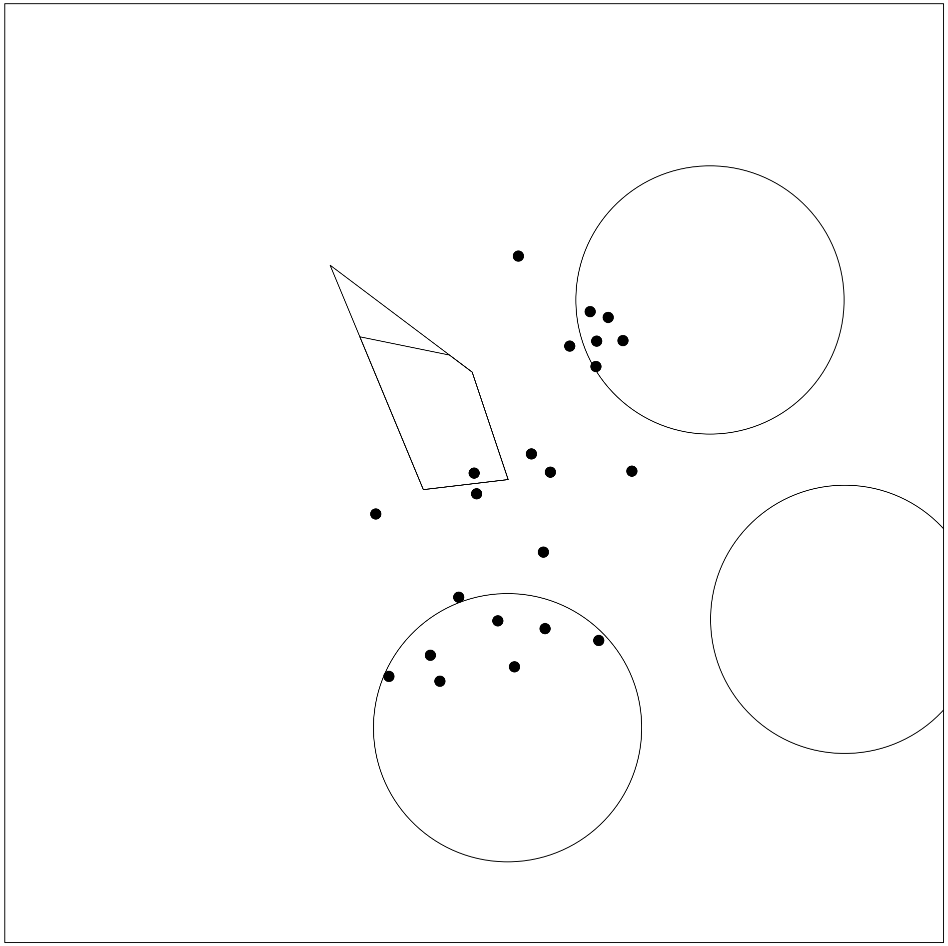
Construction of initial cell using bisectors

Simulation of Typical Cell
Alteration of Initial Cell



Initial modulated Cox-Voronoi cell

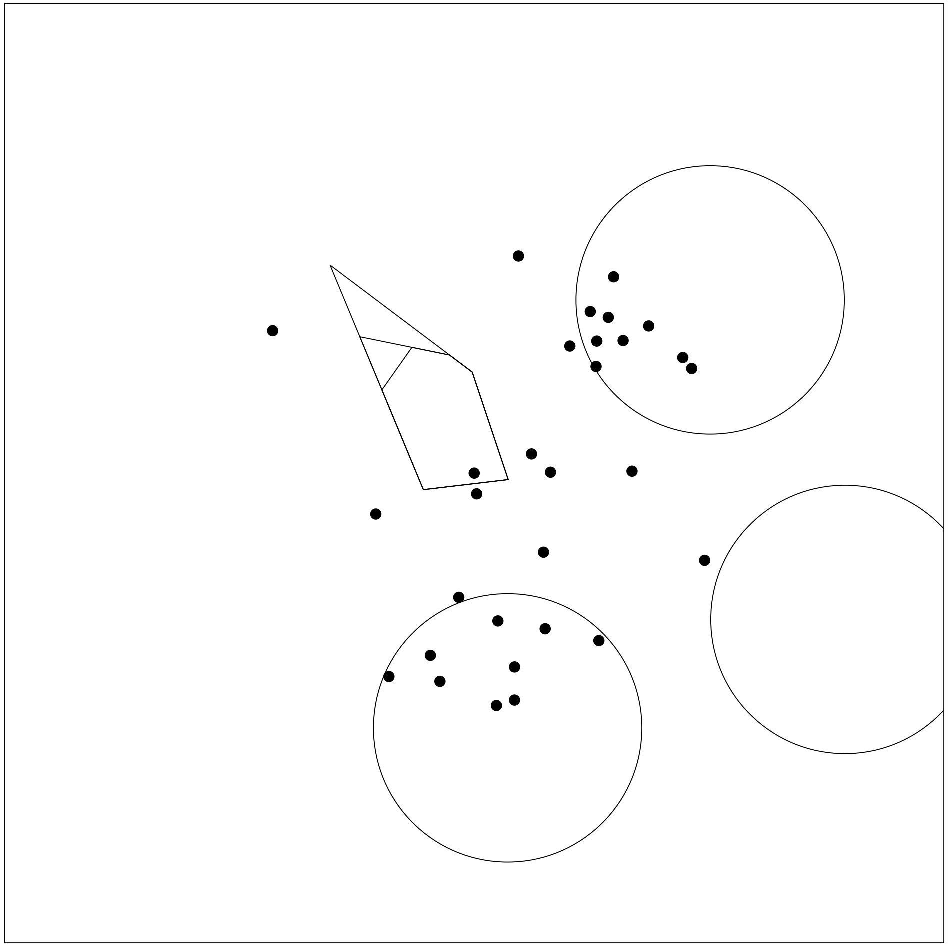
Simulation of Typical Cell
Alteration of Initial Cell



First alteration of initial cell

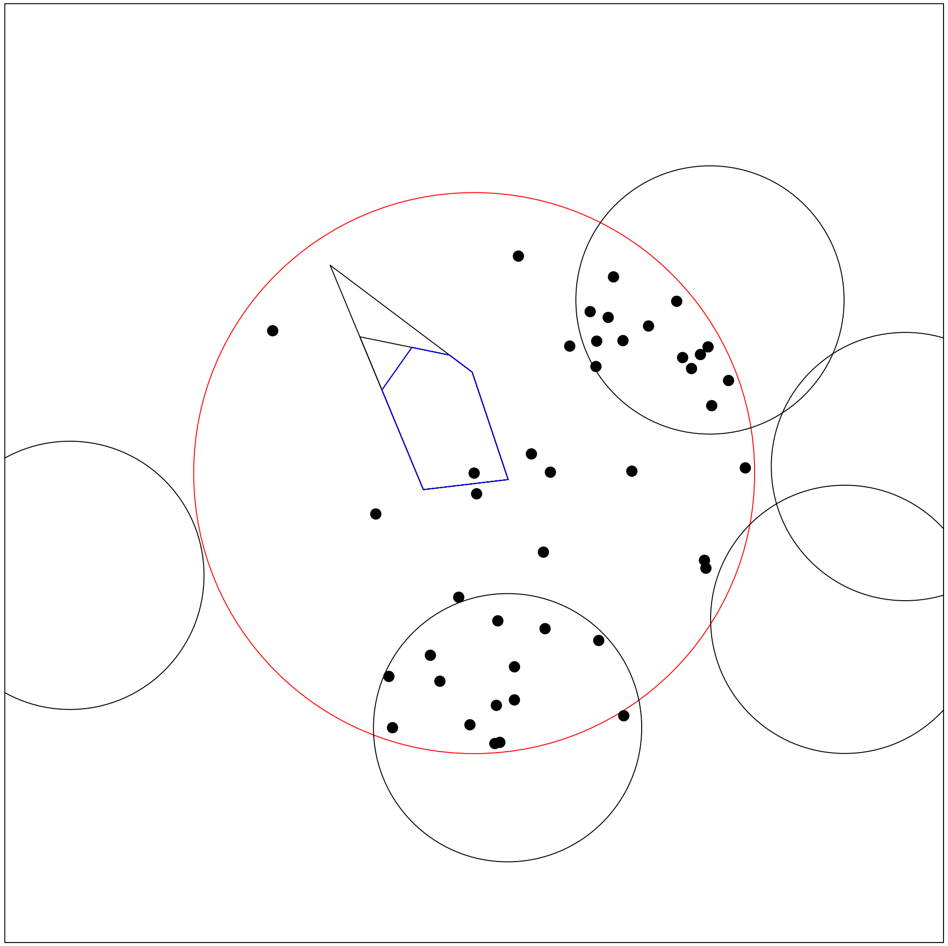
Simulation of Typical Cell

Alteration of Initial Cell



Second alteration of initial cell

Simulation of Typical Cell
Stopping Criterion



Realisation of typical cell

Simulation of Typical Cell
Extension to Random Radii

- Consider $R \sim U[r-\delta, r+\delta]$ instead of R fixed
- Algorithm has to be modified slightly w.r.t.
  - simulation of Ψ_0
  - amount of information about Ψ needed for X_i
- Similar modifications possible for other distributions of R (finite support)

Simulation of Typical Cell
Implementation Tests

- Tests in this context means testing of software with random outputs
  - Deterministic (classical) tests hardly usable
  - Random tests based on statistical test methods
- Tests based on known theoretical formulae
  - $E(\nu_2(\Xi^*)) = 1/\lambda_X$
  - $E(\eta(\Xi^*)) = 6$
  - $E(c(\Xi^*)) = \frac{1}{2\sqrt[3]{\lambda}}$ in the Poisson-Voronoi case
- Tests based on scaling invariance properties

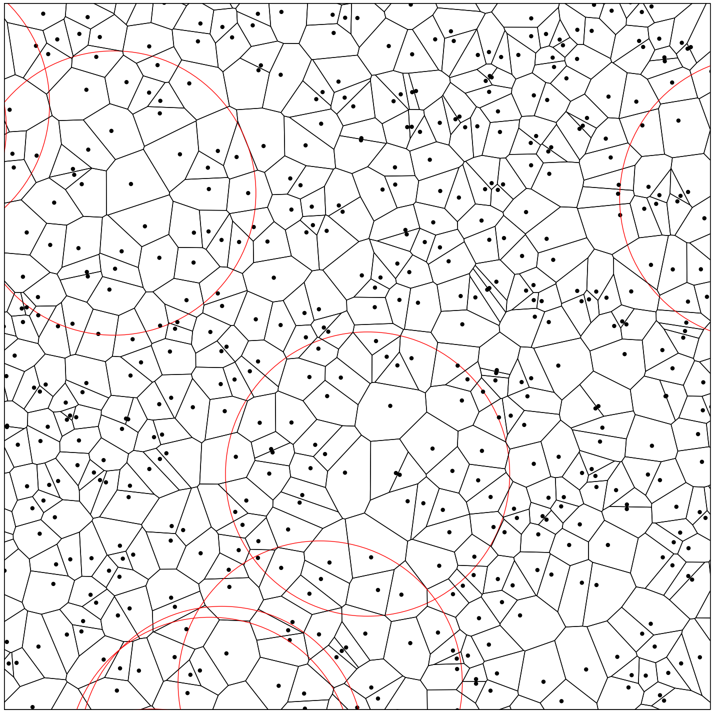
Numerical Results

- Several scenarios
  - Poisson-Voronoi
  - Swiss-cheese
  - Random radii
- Functionals considered
  - Area, perimeter, number of vertices
  - Cost functional(s)
    - $c(\Xi^*) = \int_{\Xi^*} ||u|| du$
    - $c'(\Xi^*) = \frac{1}{E(\nu_2(\Xi^*))} \int_{\Xi^*} ||u|| du = \lambda_X \int_{\Xi^*} ||u|| du$

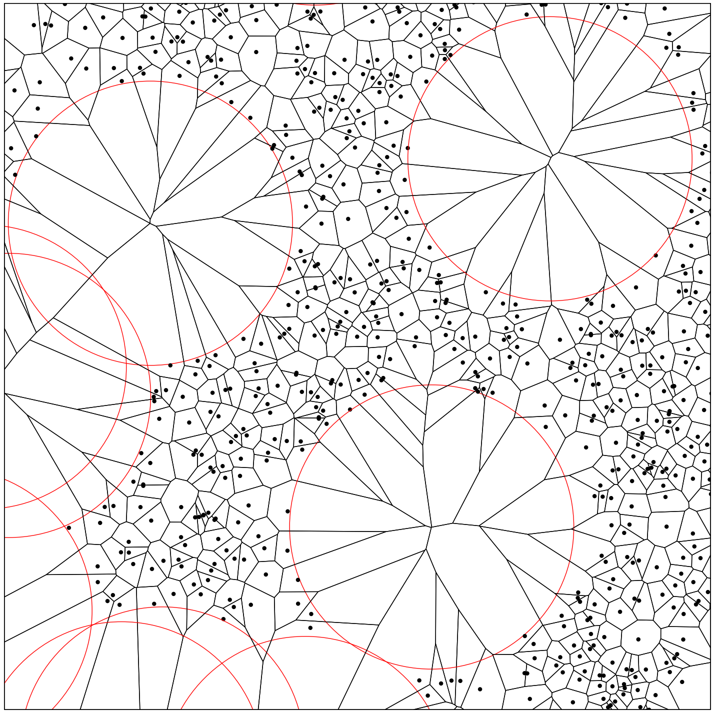
Numerical Results
Transition to Swiss Cheese

- We simulated $n = 2000000$ typical cells for each case
- Parameter values
  - $p = 0.6$, $\beta = 0.2$ => $R = 1.20761$
  - $\lambda_X = 12$
  - $\lambda_1 \to 0$
  - $\lambda_2 = \frac{\lambda_X - p\lambda_1}{1-p} = 30 - \frac{3}{2}\lambda_1$
- $\lambda_X = 12$ => $E(\nu_2(\Xi^*)) = \frac{1}{12}$ constant

Numerical Results
Transition to Swiss Cheese



$\lambda_1 = 10,\ \lambda_2 = 15$



$\lambda_1 = 0,\ \lambda_2 = 30$

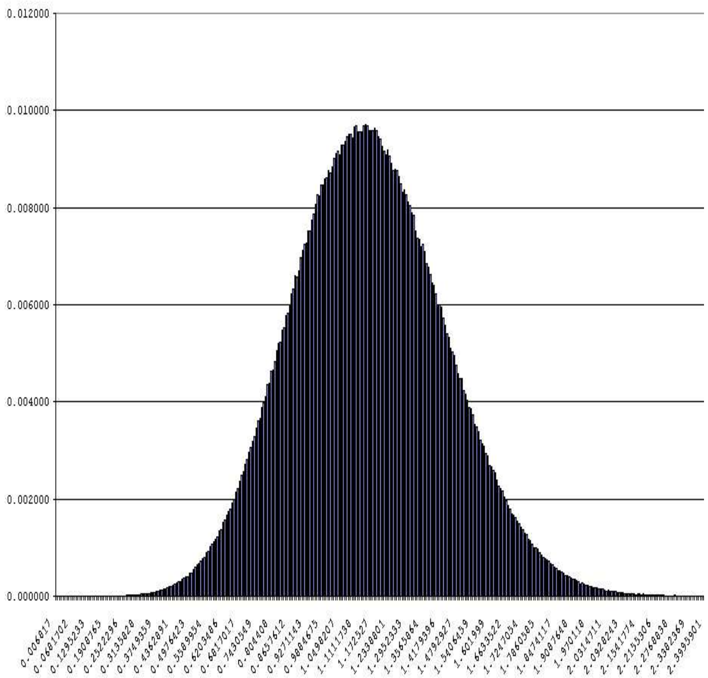
Numerical Results
Transition to Swiss Cheese

λ_1	λ_2	$\widehat{E}c'(\Xi^*)$	$\widehat{E}c(\Xi^*)$	$\widehat{E}\nu_2(\Xi^*)$	$\widehat{E}\nu_1(\partial\Xi^*)$
12	12	0.14437	0.01203	0.08337	1.15496
6	21	0.16310	0.01359	0.08335	1.11524
1	28.5	0.28536	0.02378	0.08355	1.01529
0.2	29.7	0.40872	0.03406	0.08345	1.02070
0.05	29.95	0.45953	0.03829	0.08339	1.03381
0	30	0.47946	0.03996	0.08317	1.03903

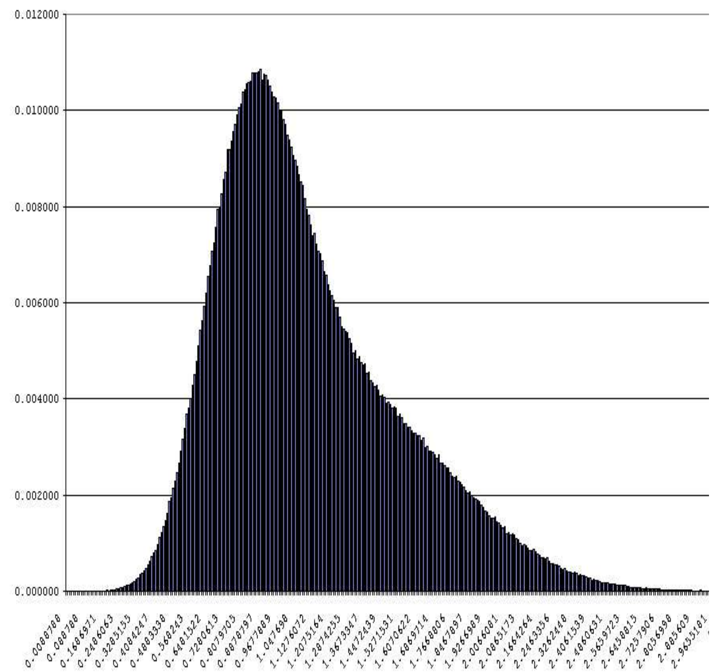
Estimates of first moments for $\lambda_1 \to 0$

Numerical Results

Transition to Swiss Cheese



$\lambda_1 = 12$, $\lambda_2 = 12$

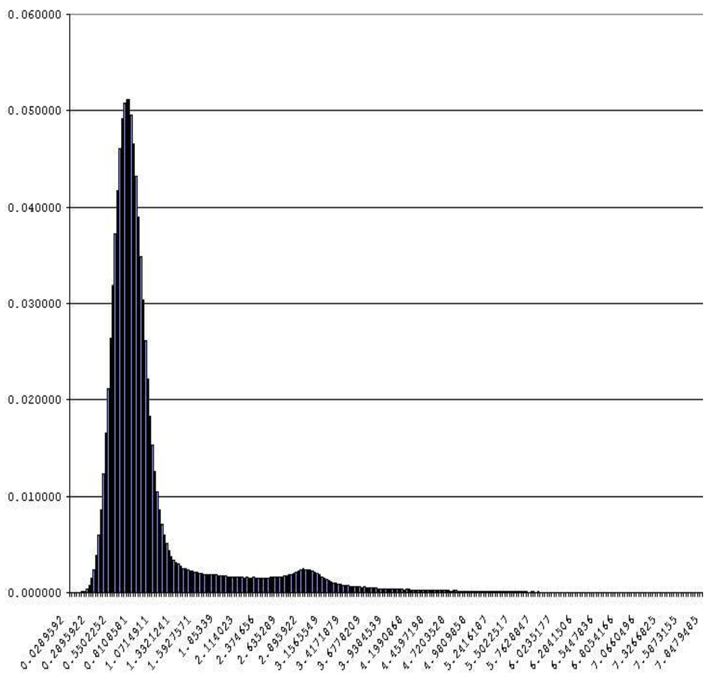


$\lambda_1 = 6$, $\lambda_2 = 21$

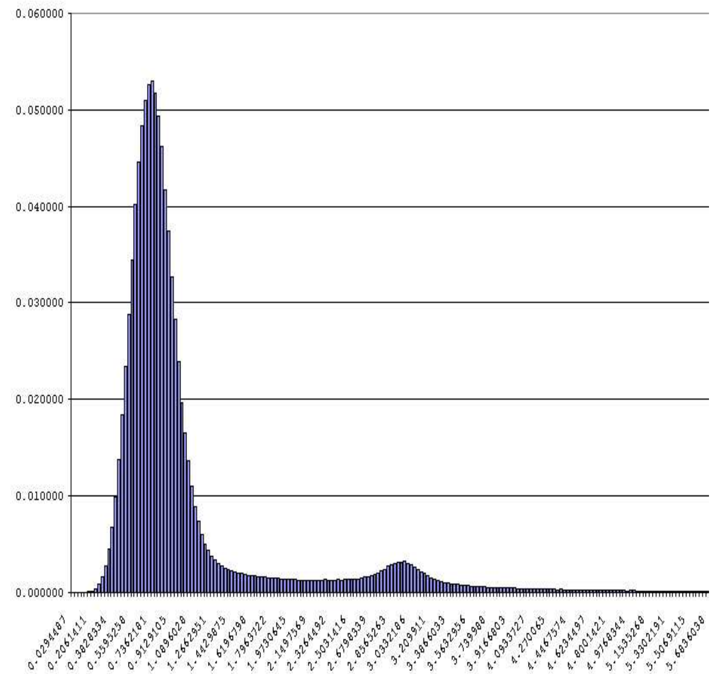
Perimeter of the typical cell

Numerical Results

Transition to Swiss Cheese



$\lambda_1 = 0.05,\ \lambda_2 = 29.95$



$\lambda_1 = 0,\ \lambda_2 = 30$

Perimeter of the typical cell

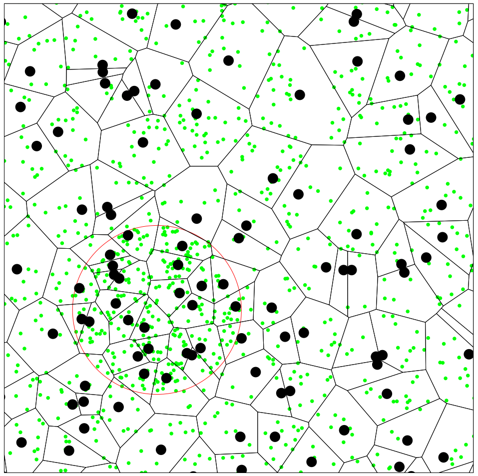
Summary and Outlook
Modifications

- First step towards modelling of nationwide scale networks
- Modifications
  - Grain shapes
  - Random grain parameter
    - Finite support
    - Other distributions
  - Germ distribution

Summary and Outlook

Extensions

- Variable population densities
  - Equipment and population placed according to two modulated Poisson process X_H and X_L with same underlying Boolean model Ψ
  - Cost functional mean distance to the nearest nuclei
    - Usage of Neveu's formula
    - $c^*_{LH} = \mathbb{E}_{X_L}(||\widetilde{X}_n - N(\widetilde{X}_n)||) = \frac{\lambda_H}{\lambda_L}\mathbb{E}_{X_H}\int_{\Xi^*}||u||\Lambda_L(du)$

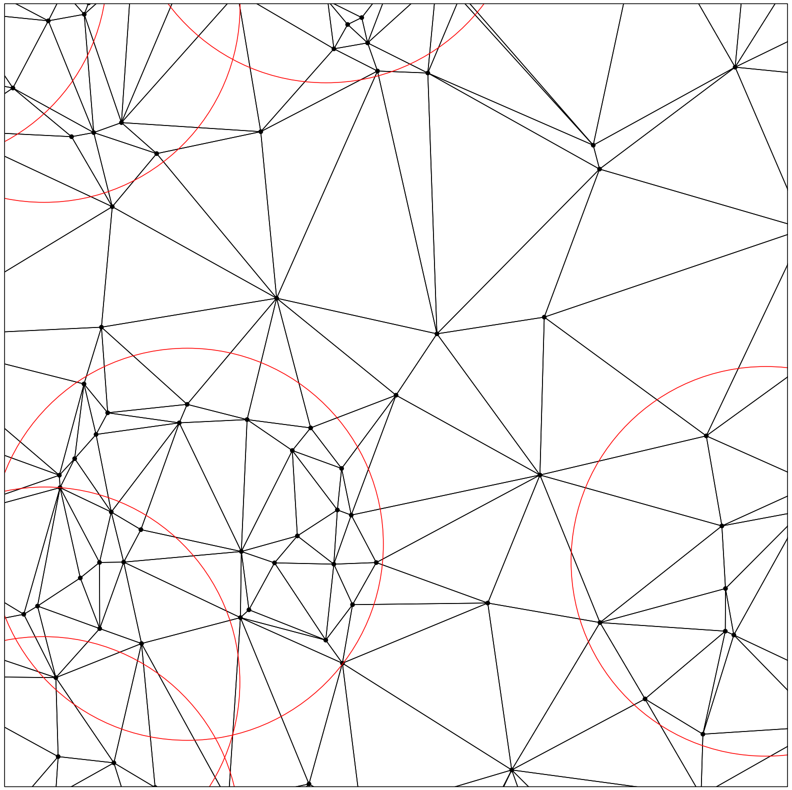


Summary and Outlook
Extensions

- Multi-modulated Poisson-Voronoi
  - Boolean models $\Psi_1, ..., \Psi_n$
  - Corresponding intensity measures $\Lambda_1, ..., \Lambda_n$
  - Larger variability
- Modulated Poisson-Voronoi connected with line-based Cox-Voronoi
  - Equipment based on line segments
  - According to linear Poisson processes
  - Segments inside Boolean model
  - Inter-city connections

Summary and Outlook
Extensions

- Modulated Poisson-Delaunay (=> connection length between neighboring Voronoi cells)



Literature

- C. Gloaguen, F. Fleischer, H. Schmidt, V. Schmidt, F. Schweiggert (2006) *Simulation of typical modulated Poisson-Voronoi cells and their application to telecommunication network modelling*. Preprint.
- C. Gloaguen, F. Fleischer, H. Schmidt, V. Schmidt (2005) Simulation of typical Cox-Voronoi cells, with a special regard to implementation tests. *Mathematical Methods of Operations Research* **62**, 357-373.
- B. Blaszczyszyn, R. Schott (2005) Approximations of functionals of some modulated Poisson-Voronoi tessellations with applications to modeling of communication networks. *Japan Journal of Industrial and Applied Mathematics* **22**(2), 179-204.