# Spatial models built on non homogeneous Poisson Point Processes

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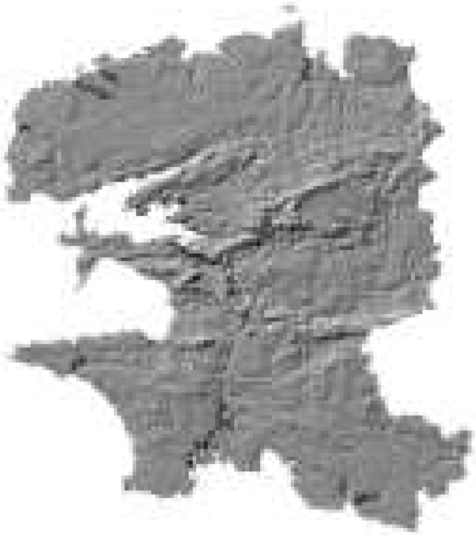
# Overview

- Motivation
- Transform to/from homogeneity
- Toolbox for real situations analysis
- Conclusion

# Motivation

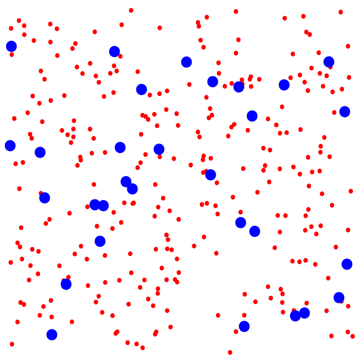


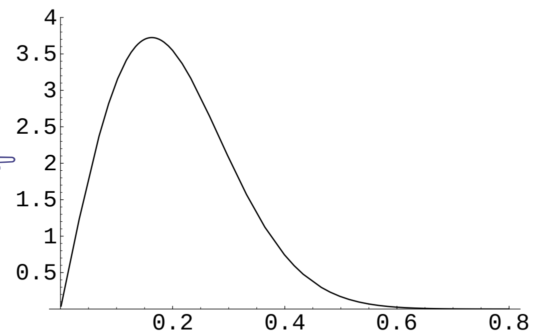
- repartition of population is non homogenous
- localization of network nodes follows accordingly
- this depends on the type of nodes and their hierarchical level

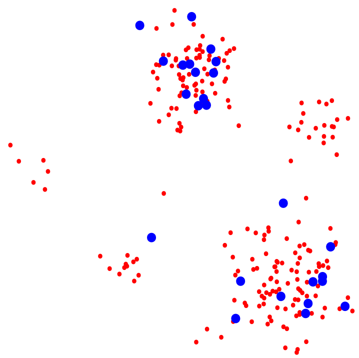
# Motivation

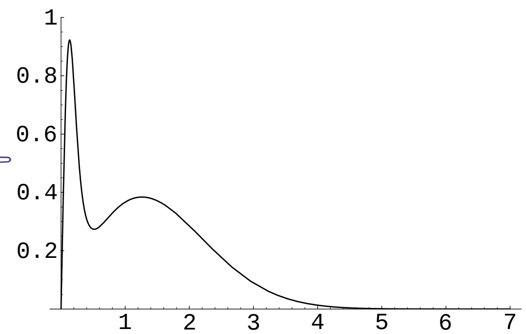
- so far, most telecommunication models assume spatial homogeneity
  - hierarchical models, Poisson Voronoï Agregated Tessellations. . .
  - road systems in the Stochastic Subscriber Line Model
- homogeneous models are useful
  - in some cases, ex : involving high level network nodes
  - as first order models to investigate faisability
- spatial non-homogeneity can be structurant

  the cost of line of sight connection from node $n$ (red) to closest node $N$ (blue) depends on the length distribution







# Motivation

- need for practical methods to deal with spatial inhomogeneity
  - considering a variety of situations (shape, location)
  - considering a great number of parameters
  - computationally rapid
  - based on parameters that can be infered from reality
- several theoretical approaches are possible
  - modulated Poisson
  - perturbation of Poisson-based models
  - transform of homogeneous planar Poisson Point Processes (PPP)
  - ...

# Transform to/from homogeneity

- an homogeneous PPP (noted $N$)
  - is defined by its constant intensity in $\mathbb{R}^2 (= E)$
    $$\beta\lambda(du)$$
- a non-homogeneous PPP
  - is defined by its non constant intensity in $\mathbb{R}^2 (= F)$
    $$\mu(dx) = p(x)\lambda(dx)$$
  - is an image-process $X(N)$ in $F = X(E)$
- the transform $X$
  - must be a bicontinuous bijection
  - must be unique $\rightarrow$ additional constraints
  - can be analytically computed in some cases

# Transform to/from homogeneity

- computation of the transform $X$
  - measures $\mu$ and $\lambda$ are images $\mu = X(\lambda) \Leftrightarrow \lambda = X^{-1}(\mu)$

$$\int_D \lambda(du) = \int_D X^{-1}(\mu)(du) = \int_{X(D)} \mu(dx)$$

  - usual change of variables with Jacobian $J_{X^{-1}}$

$$\int_D \lambda(du) = \int_{X(D)} |\text{det} J_{X^{-1}}(x)| \lambda(dx)$$

  - partial derivative equation $p(x_1, x_2) = |\text{det} J_{X^{-1}}(x_1, x_2)|$
  - specification of invariant 1D-sets ensures unicity
    - lines parallel to one axis
    - lines radiating form a center

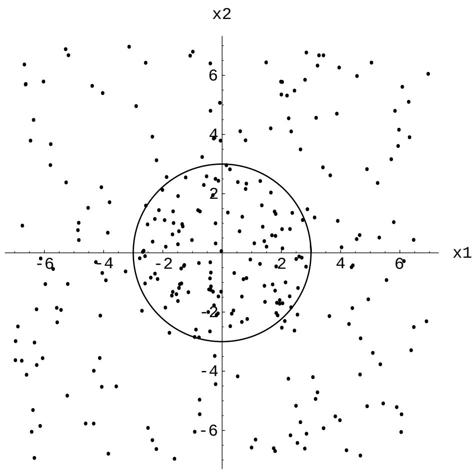
# Transform to/from homogeneity

- example : discontinuous circle, radial transform
  - non homogeneous $X(N)$ in $F$ , intensity $\mu(dx) = p(x)\lambda(dx)$

$$p(x) = \{\alpha \text{ if } x \in \text{disc } \Gamma(0, R), \text{ else } \beta\}$$

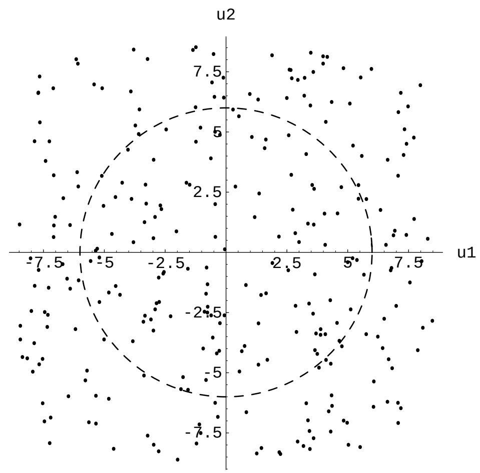
  - transformation to homogeneous $X$ in $E$

$X(N)$ in $F$




$\xrightarrow{X^{-1}}$

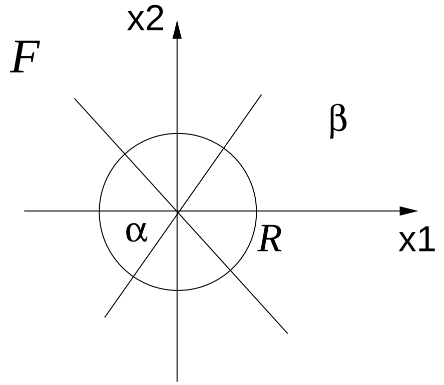
$N$ in $E$



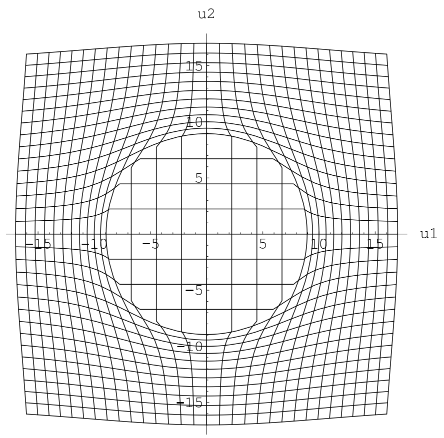
# Transform to/from homogeneity

- example : discontinuous circle, radial transform
  - radial invariance (centered in $O$) demands

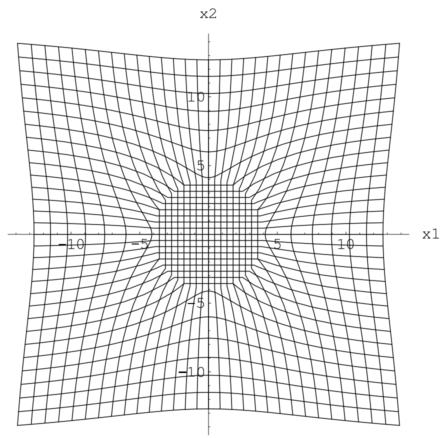
$$(u_1, u_2) = X^{-1}(x_1, x_2) = \phi(x_1, x_2)(x_1, x_2)$$




  - unique solution $\phi(x_1, x_2) = \{\sqrt{\alpha}$ if $x \in \Gamma, \sqrt{\beta + (\alpha - \beta)\frac{R^2}{x_1^2+x_2^2}}$ else $\}$

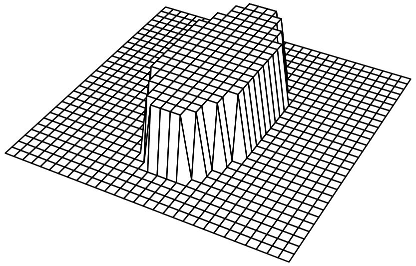



iso-$x_i$ curves in $E$, from $X^{-1}$

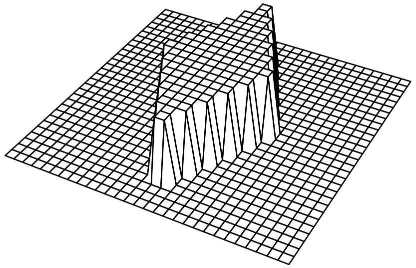



iso-$u_i$ curves in $F$, from $X$

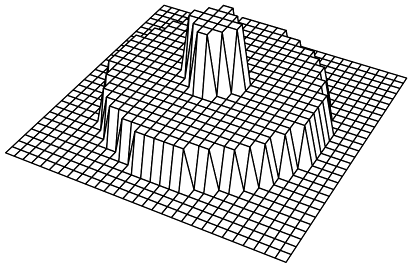
- analitycal expressions are available for simple parametric shapes
  - discontinuous densities :



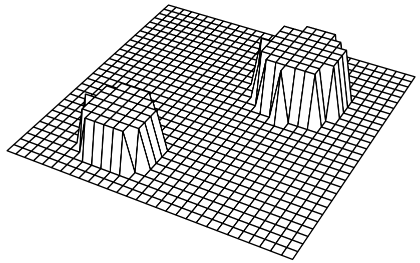
ellipsis



diamond

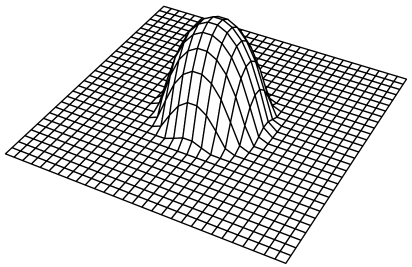


cake

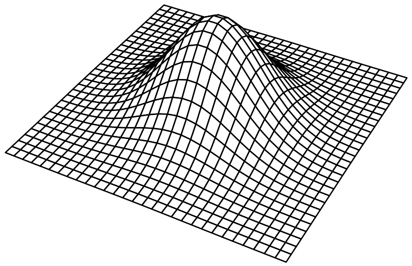


2 circles

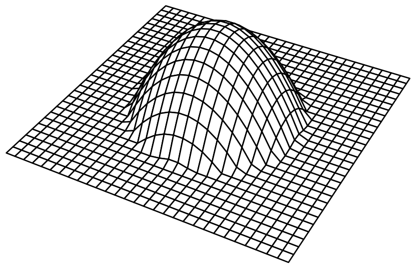
  - continuous densities :



bell



gaussian

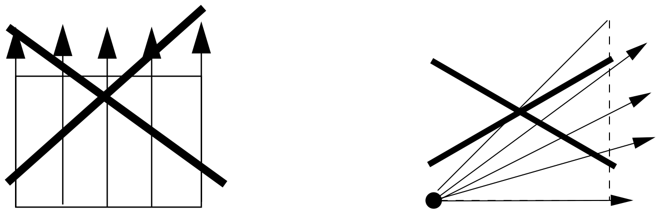


Epanechikov

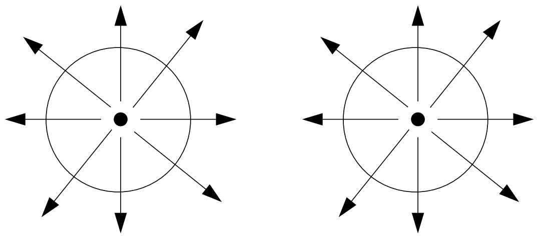
  - radial and/or parallel transform whenever possible

# Transform to/from homogeneity

- exact transforms can easily be derived in other cases
  - direct calculation for complicated single shapes (ex general polygons)
    careful choice of invariant varieties
  - modification of existing results by changing
    the intensity levels, rescaling both coordinates
    the shapes and locations, translation and/or flattening . . .
- this does not cover all our needs
  - somes cases cannot be solved exactely

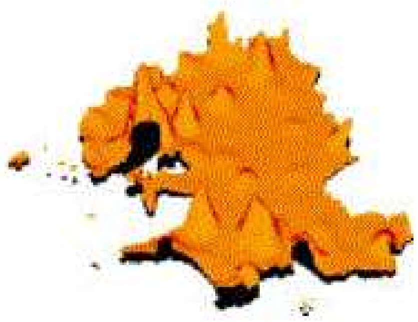


continuity is not verified



no exact transform for the set

# Toolbox for real situations analysis



the idea is to describe complex situations by mixing

approximate & exact forms

analytical & numerical results

- the "toolbox" contains
  - far field approximation

    formal definition from induced metrics

    application to model juxtaposition and superposition
  - "numerical" tools

    fast realizations of non homogeneous PPP

    transformation/construction of geometric figures
  - "analytical" tools

    formal reuse of homogeneous results

# Toolbox - far field

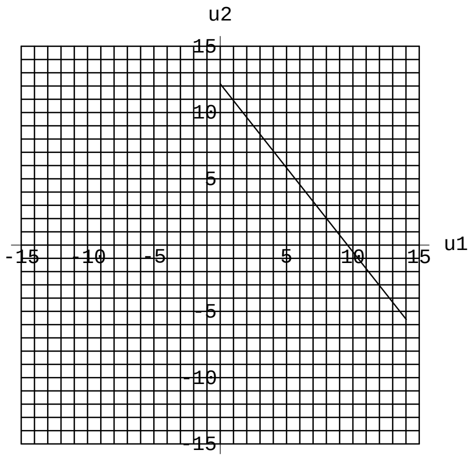
- induced metrics    recall: $N$ in $E$ intensity $\beta\lambda(du)$, $X(N)$ in $F$ intensity $p(x)\lambda(dx)$

$E$, Euclidian metric

$$ds^2 = \eta_{ij} du_i du_j$$

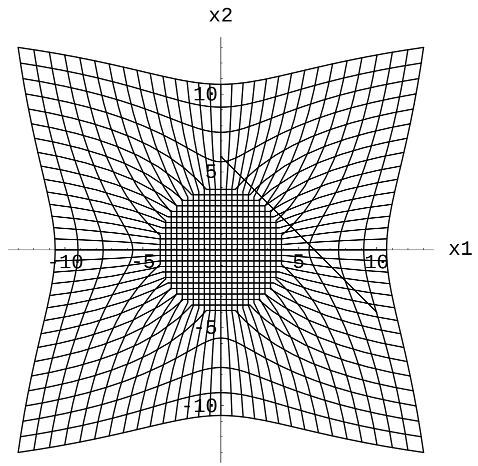
$F$, induced metric

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu$$




$$\xrightarrow{x = X(u)}$$

$$g_{\mu\nu} = \eta_{ij} \frac{\partial u_i}{\partial x_\mu} \frac{\partial u_j}{\partial x_\nu}$$



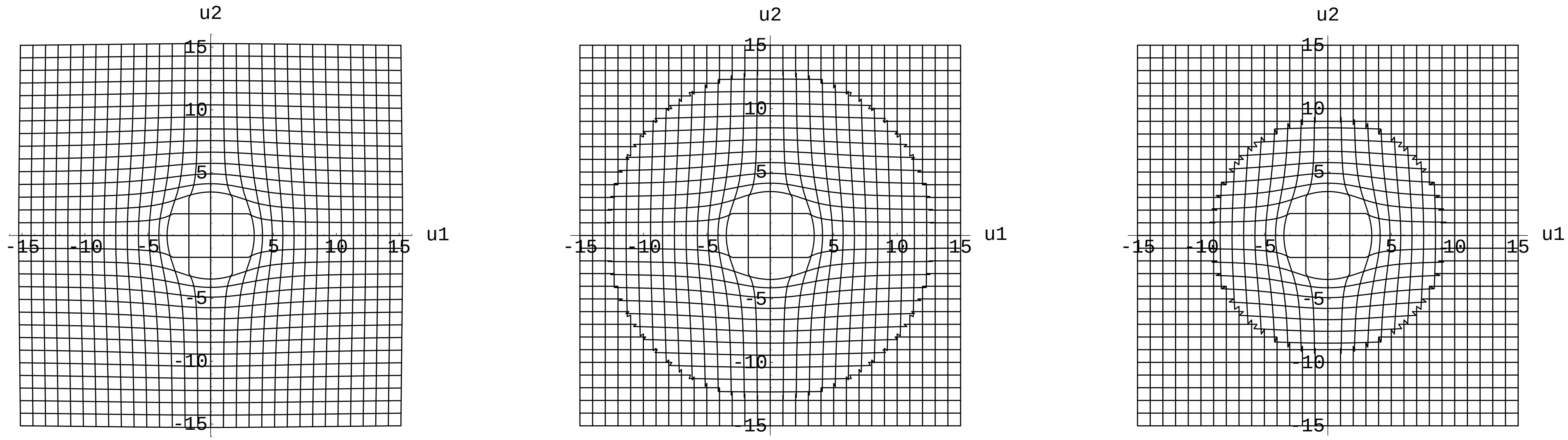

- metric $g$ is location dependant and defined from $p$ and its parameters
- metric $g$ is continous and can be written analytically
- several other metrics can be defined in $E$ and $F$

  induced from $X$ or $X^{-1}$

  considering Euclidian or polar coordinate systems

# Toolbox - far field

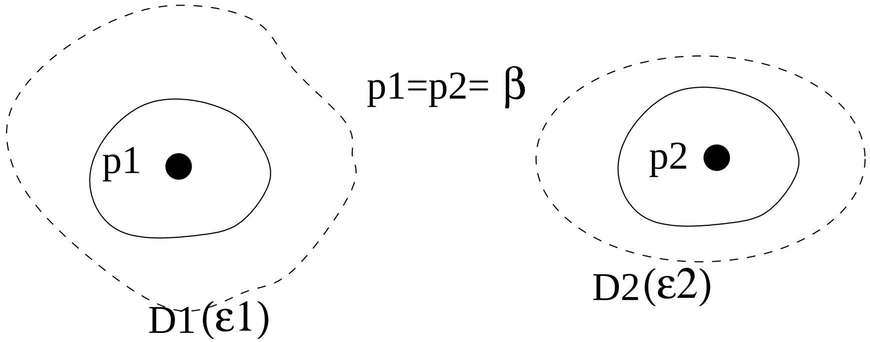
- formal definition of far field limit
  - compare natural polar metric $\delta$ and induced polar metric $m$
  - relative error $(\delta - m)/\delta < \epsilon$ farther from $O$ than $D(\epsilon)(= R\sqrt{\frac{(\alpha-\beta)}{\beta\epsilon}}$ for circle)
  - stitching together homogeneous and non-homogeneous spaces
  - example : gaussian density $\alpha = 5, \beta = 1, R = 2$, iso-$x_i$ curves in $E$



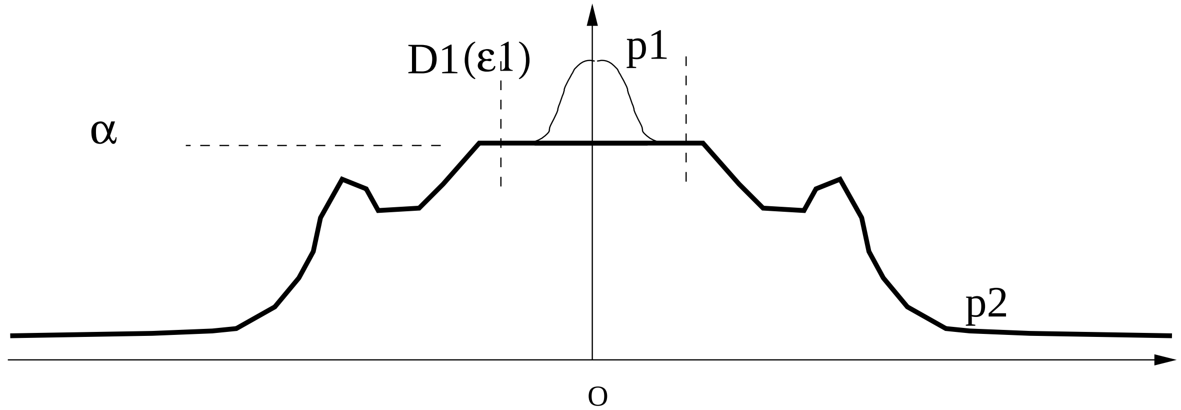

no stiching | $\epsilon = 0.05, D = 12.6$ | $\epsilon = 0.1, D = 8.9$

# Toolbox - far field

- approximate transforms using far field assumption
  - the transform affects points closer than $D(\epsilon)$
  - juxtaposition same basis intensity

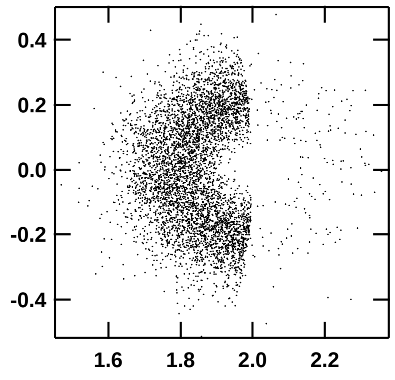


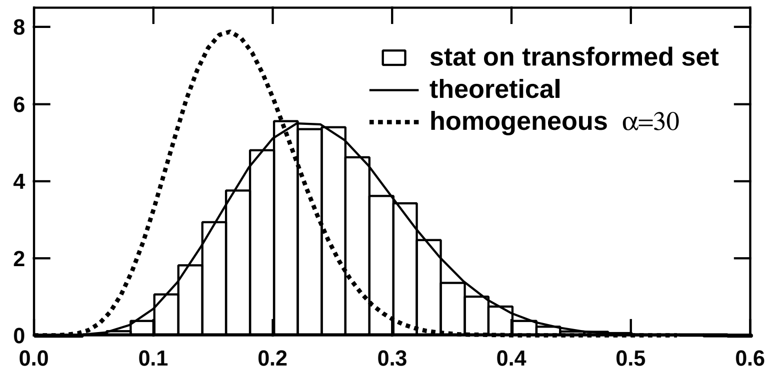

  - superposition



# Toolbox - numerical applications

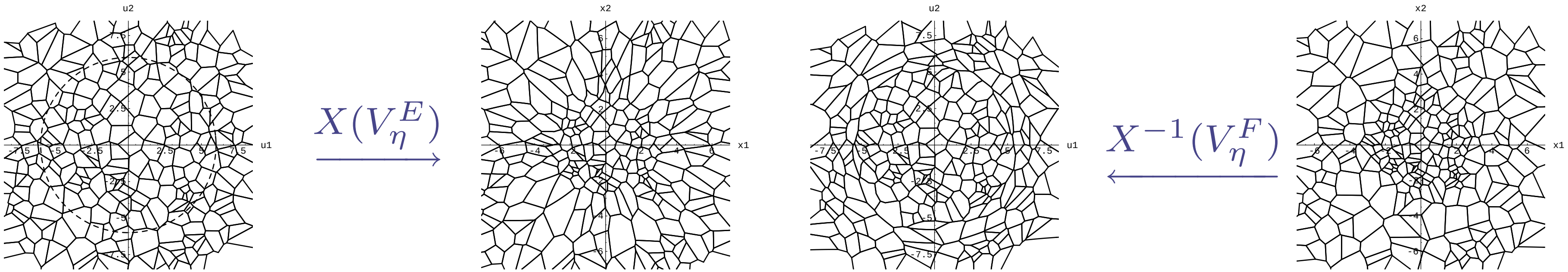
- fast generation of realizations of non-homogeneous PPP
  - generate one reference set $\mathcal{U}$ homogeneous PPP of unit intensity
    $T$ radial simulations, coordinates of $n$ points closest to O
  - compute $X(\mathcal{U})$ to build the non homogeneous set
- statistics on non-homogeneous PPP
  - example : distances between point $A$ and the 3rd closest point from $X(N)$
    observe hom. $N$ from $X^{-1}(A)$, translation $\mathcal{U}_A = \mathcal{U} + X^{-1}(A)$
    compute $X(\mathcal{U}_A)$ and order distances to $A$, extract the third item
    $\alpha = 30, \beta = 1, R = 2, A = (2, 0)$ located on the circular discontunity



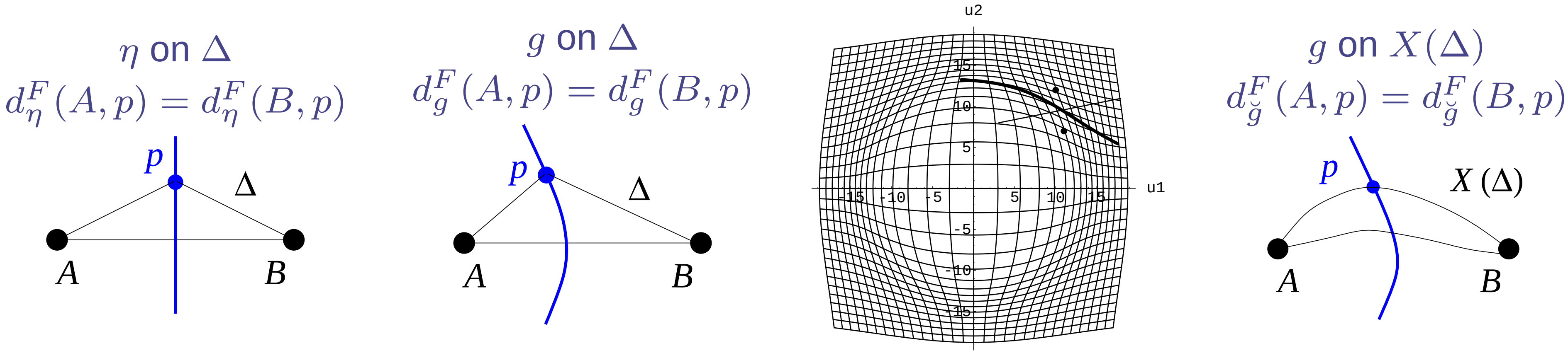



# Toolbox - numerical applications

- transformation of geometric figure (Voronoï tessellation)
  - Euclidian $\eta$ on straight $\Delta$ : nuclei from $N \to V_\eta^E$, nuclei from $X(N) \to V_\eta^F$

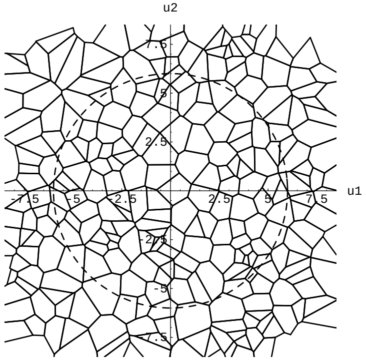



- drawing of general bisectors
  - use non euclidian metrics and/or transforms of straight lines $\Delta$ in algorithms



# Toolbox - analytical considerations

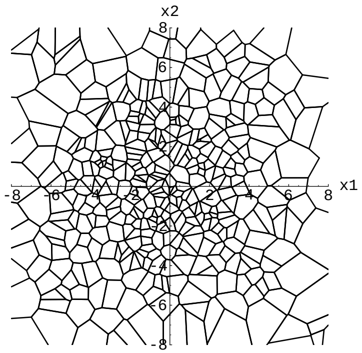
- metrics provide formal identity of writing
  - average area of Poisson Voronoï cell, nucleus in $A$




$V_\eta^E$, polar metrics $\delta_A$ centered in $A$ and $\delta_z$

$$\mathcal{A}^E(A) = \iint_E dz \sqrt{|det(\delta_A(z))|} \quad e^{-\iint_{\Gamma(z)} dz' \sqrt{|det(\delta_z(z'))|}} = \frac{1}{\beta}$$

**same rule for bisectors : keep $\delta_A$** **different nuclei location $N \to X(N)$ : change $\delta$ in $m$**




$V_\eta^F$, $\delta_A$ and induced polar metric centered $y$ $m_y$

$$\mathcal{A}^F(A) = \iint_F dy \sqrt{|det(\delta_A(y))|} \quad e^{-\iint_{\Gamma(y)} dy' \sqrt{|det(m_y(y'))|}}$$

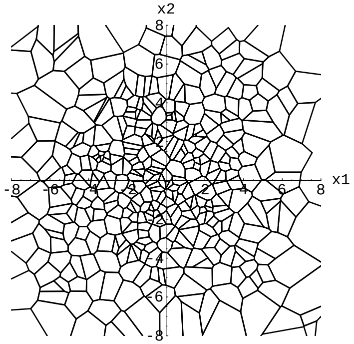
  - remarks

    not too useful for areas ($\sqrt{|det(m)|} \Leftrightarrow p$)

    re-use of sensible multiD integration codes may not be straightforward

# Toolbox - analytical considerations

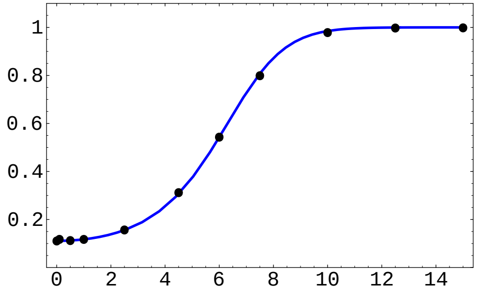
- computation and fitting of tessellation characteristics



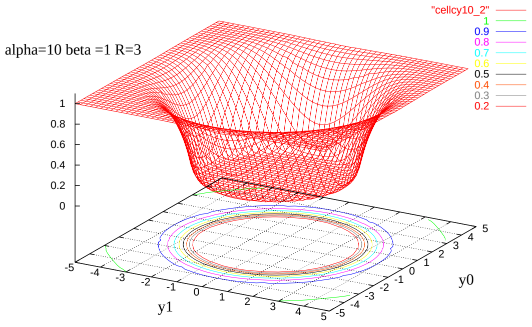

  - example : characteristics of Poisson Voronoï cells
    $V_\eta^F$ tessellation
    average area $\mathcal{A}^F(A)$ of cell nucleus in $A$
    average perimeter $\mathcal{P}^F(A)$

  - study the dependance on $A$ and on the density shape and parameters

gaussian density area




circular discontinuity




  - approximations for smooth densities : ex gaussian case $\mathcal{A}^F(A) \sim 1/p(A)$
  - build a library from fitted results making the most of symmetries

# Conclusion

- explicit transforms and induced metrics allow
  - to re-use results from homogeneous PPP : same formal writing
  - to be introduced in algorithms $\rightarrow$ radial simulation of "typical" cells...?
- practical interest for France Telecom
  - gather a set of formulas and tools adapted to spreadsheet codes
    no need of random generators to generate points of non homgeneous PPP
  - analysis of complex situations
    by mixture of approximations, fitted and exact results
- perspective
  - working on length distributions
  - analysis of tessellations characteristics
  - consider several hierachical levels of non homogeneous PPP
  - homogenization technique is not limited to PP

# Literature

- homogeneous hierarchical stochastic telecommunication models
  - F. Baccelli, M. Klein, M. Lebourges and S. Zuyev
    Stochastic geometry and architecture of communication nerworks
    Telecommun. Syst. 7 pp209-227 (1997)
  - K. Tchoumatchenko and S. Zuyev
    Aggregate and fractal tessellations
    Probab. Theory Relat. Fields 121, pp198-218 (2001)
- road systems in the Stochastic Subscriber Line Model

  just have a look on http://www.geostoch.de
- computation of the transform
  - R. Senoussi, J. Chadoeuf and D. Allard
    Weak Homogenization of point processes by space deformations
    Adv. Appl. Prob. (SGSA) 32 pp. 948-959 (2000)

Thank you for your attention !