Analysis of Mean Shortest Path Lengths in the Stochastic Subscriber Line Model

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Outline

- **Stochastic–geometric network modelling**
  - Real network data
  - Stochastic Subscriber Line Model
  - Geometry Model, Equipment Model, Topology Model

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- **Neveu's Exchange Formula for Palm Distributions**

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- **Mean shortest path lengths**
  - Simulation methods
  - Application of Neveu
- **Mean subscriber line lengths**

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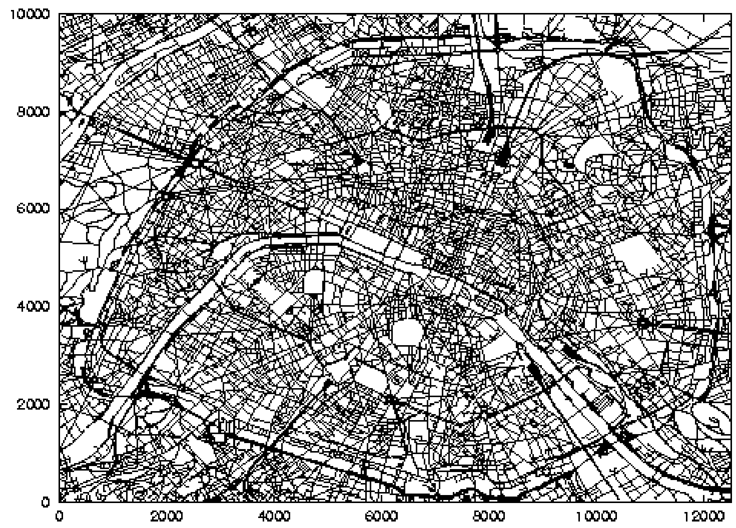
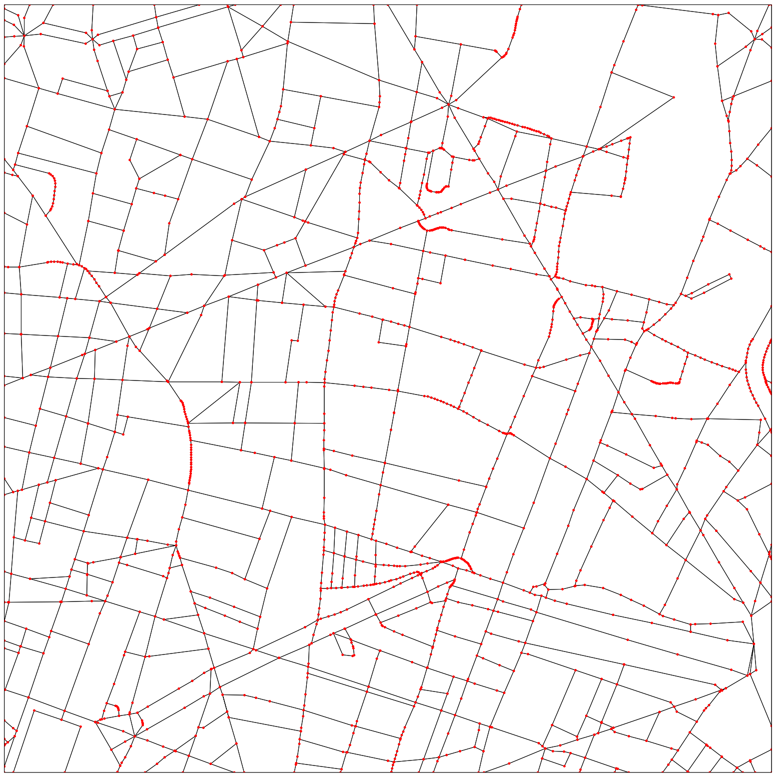
- **Neveu's Exchange Formula for Palm Distributions**
- **Mean shortest path lengths**
  - Simulation methods
  - Application of Neveu
- **Mean subscriber line lengths**
- **Numerical examples**

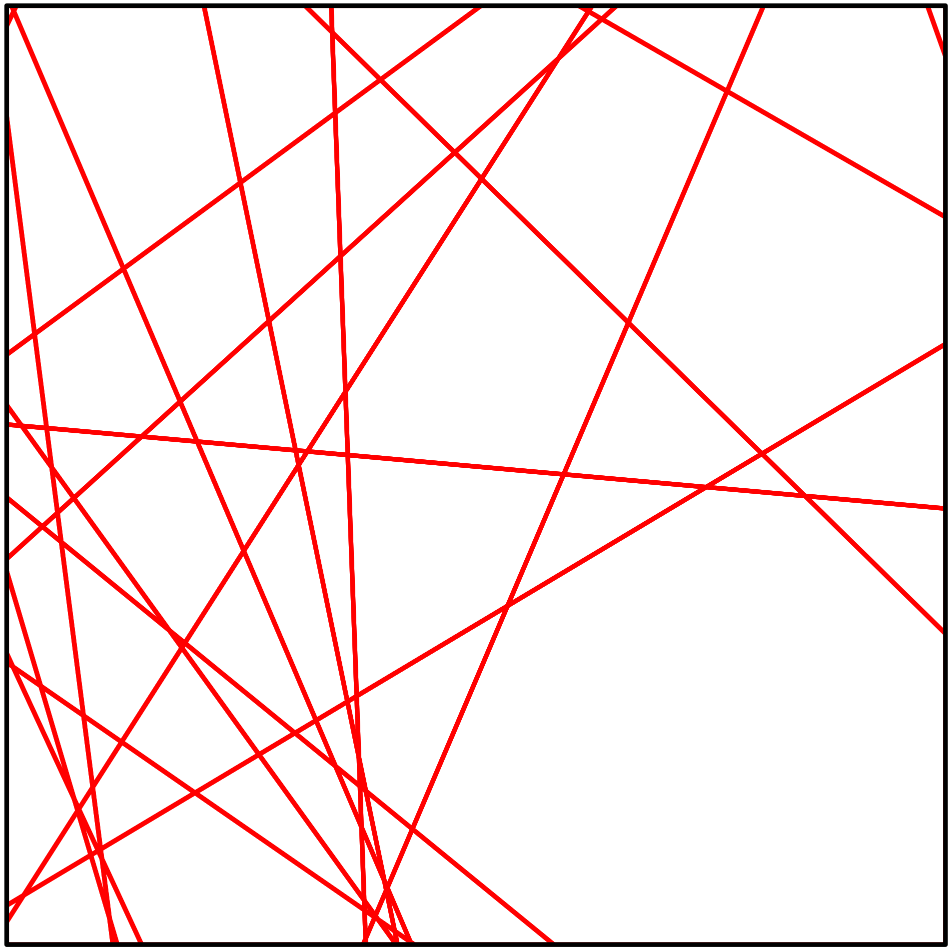
Stochastic–geometric network modelling
Real network data





Infrastructure system of Paris

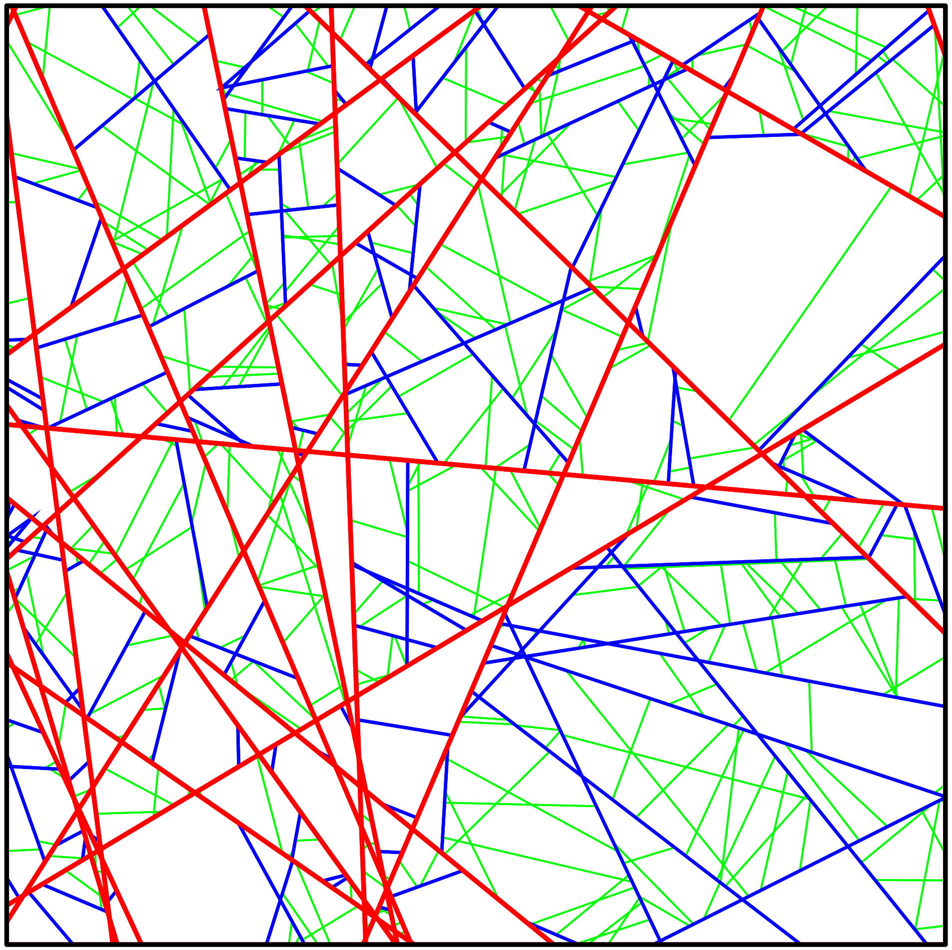
Stochastic–geometric network modelling
Stochastic Subscriber Line Model



Main roads

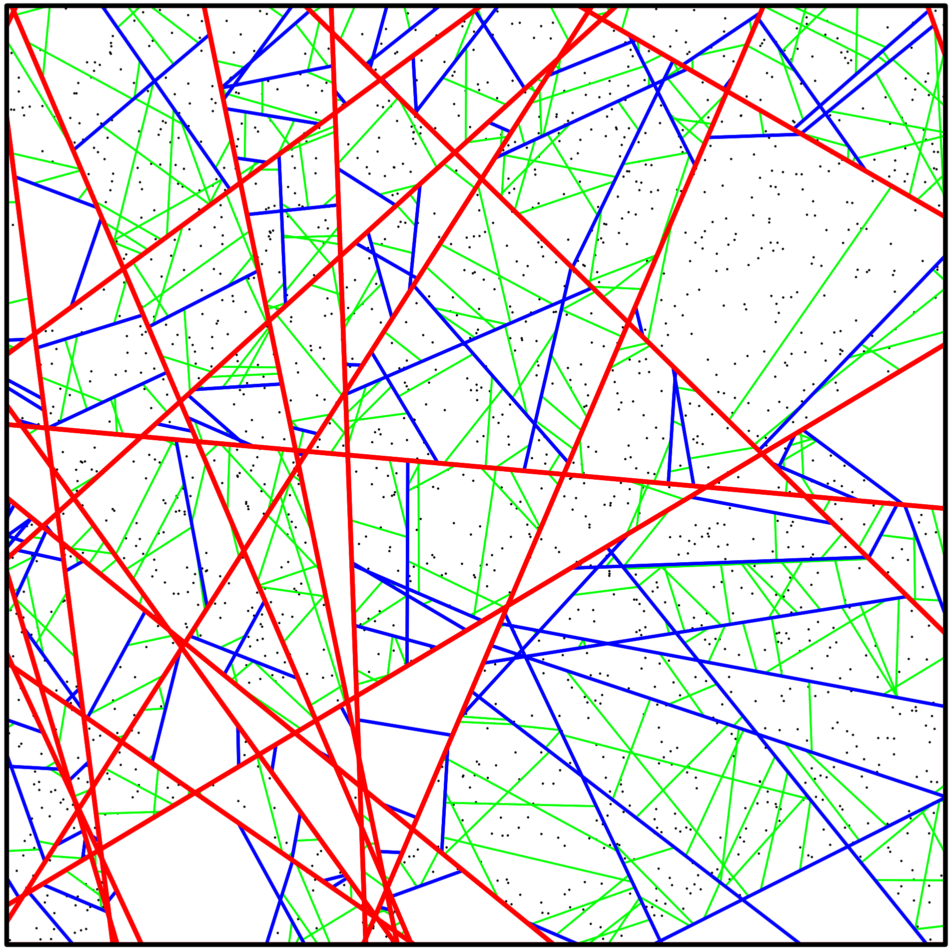
Stochastic–geometric network modelling

Stochastic Subscriber Line Model



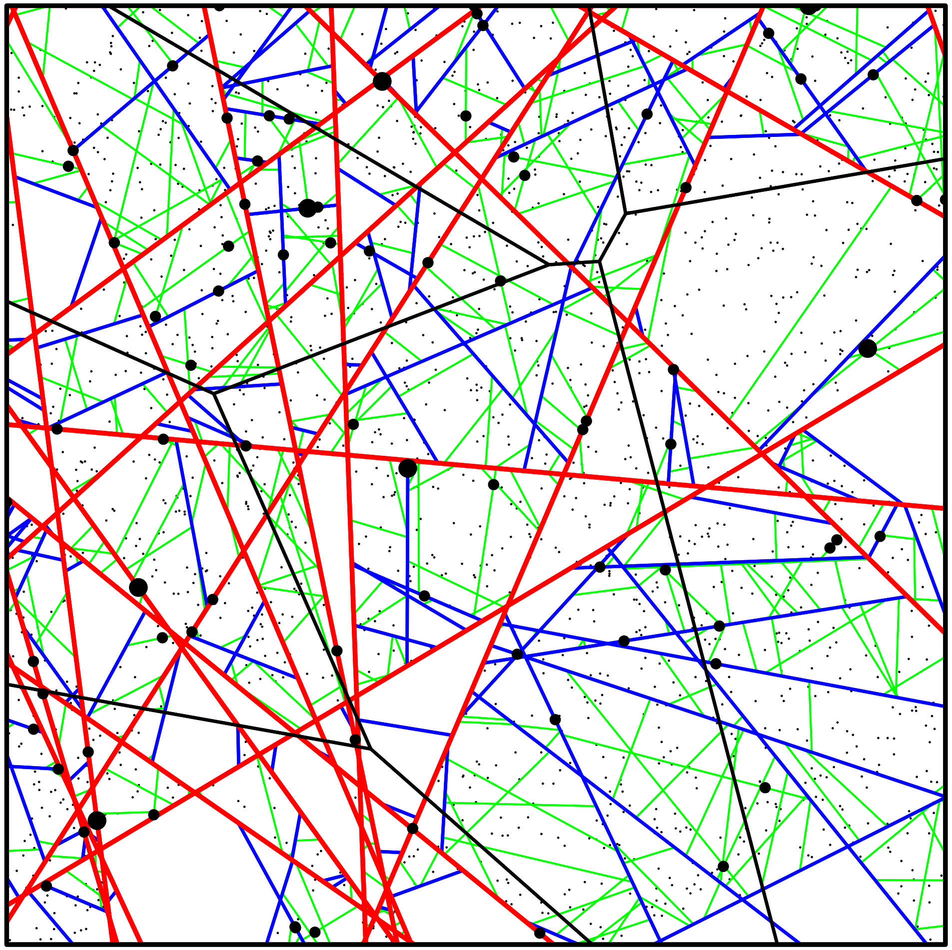
Main roads and side streets

Stochastic–geometric network modelling
Stochastic Subscriber Line Model



Location of subscribers

Stochastic–geometric network modelling
Stochastic Subscriber Line Model



Network nodes and serving zones

Stochastic–geometric network modelling

Stochastic Subscriber Line Model

Stochastic Subscriber Line Model (SSLM)

Geometry Model

Infrastructure system (roads, ...)

urban and rural

- Simple Tessellations (PLT, PVT, PDT)
- Nestings
- Multi−type nestings

Network Model

Placement of components

2−level models:

Higher level comp. (HLC) and Lower level comp. (LLC)

- WCS (wire center stations, repartiteur) and SAI (service area interface, sous−repartiteur)
- SAI and subscriber

Topology Model

Topology of connections (tree, ...)

2−level models:

- Serving zones: WCS are nucleii of Voronoi−cells
- SAI is connected to WCS of its serving zone

Neveu's formula
Random point processes

- Mapping $X : \Omega \to M(\mathbb{R}^2) = M$
  - $(\Omega, \mathcal{A}, \mathbb{P})$: probability space
  - M: Set of simple and locally finite counting measures
- Representations of X
  - **Random counting measure** $\sum_{x \in \text{supp}(X)} \delta_x$
  - Sequence $\{X_n\}_{n \geq 1}$ of **random points**
- **Intensity measure** $\Lambda : \mathcal{B}(\mathbb{R}^2) \to [0, \infty]$

$$\Lambda(B) = \mathbb{E}X(B)\,, \quad B \in \mathcal{B}(\mathbb{R}^2)$$

- In case of stationarity
  - $\Lambda(B) = \lambda\, \nu_2(B)$ for $B \in \mathcal{B}(\mathbb{R}^2)$
  - λ is called **intensity**

Neveu's formula

Random marked point processes

- Mapping $X_D : \Omega \to M(\mathbb{R}^2 \times D) = M_D$
- Representation of X_D
  - Random counting measure
  - Sequence of $\{[X_n, D_n]\}_{n\geq 1}$ random marked points

Neveu's formula
Random marked point processes

- Mapping $X_D : \Omega \to M(\mathbb{R}^2 \times D) = M_D$
- Representation of X_D
  - Random counting measure
  - Sequence of $\{[X_n, D_n]\}_{n\geq 1}$ random marked points
- Flow $\{\theta_x \, , x \in \mathbb{R}^2\}$ with $\theta_x : \Omega \to \Omega$
- Palm distribution $\mathbb{P}^*_{X_D}$ of $X_D : \Omega \to M_D$,

$$\mathbb{P}^*_{X_D}(A \times G) = \frac{1}{\lambda \nu_2(B)} \int_{\Omega} \int_{\mathbb{R}^2 \times G} \mathbf{I}_B(x)\, \mathbf{I}_A(\theta_x \omega)\, X_D(\omega, d(x,g)) \mathbb{P}(d\omega)$$

for $B \in \mathcal{B}(\mathbb{R}^2)$ and $A \in \mathcal{A}$, $G \in \mathcal{B}(D)$, $0 < \lambda < \infty$

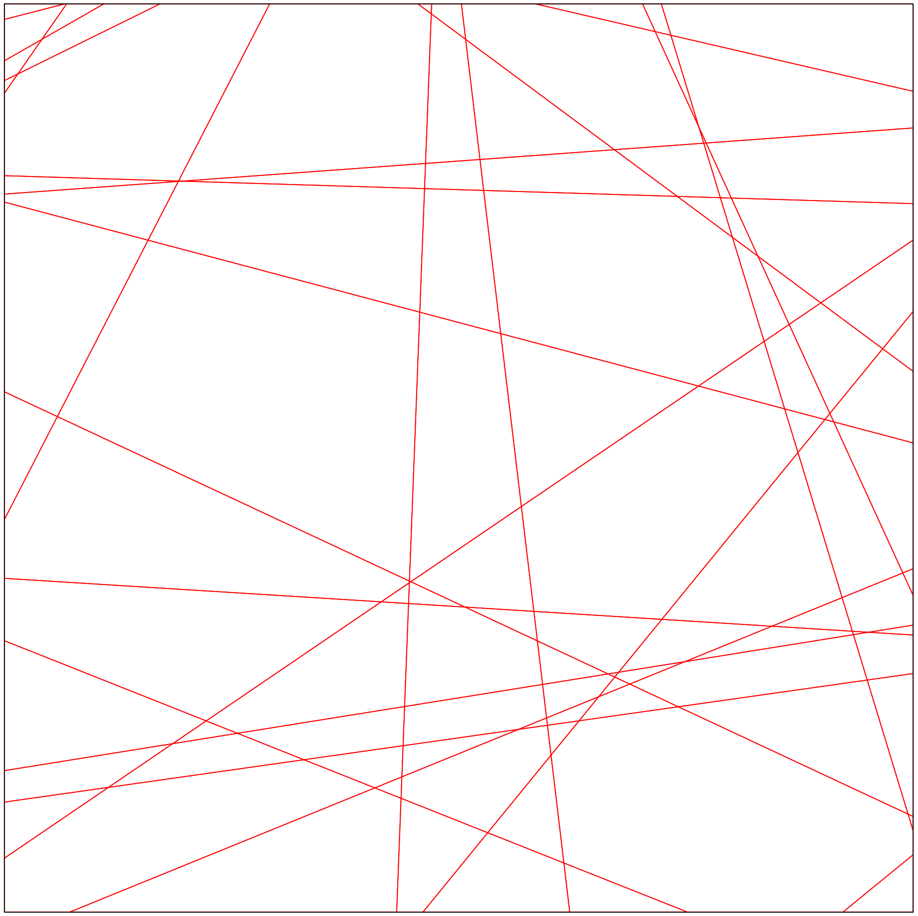
Neveu's formula

For any measurable function $f : \mathbb{R}^2 \times D \times \widetilde{D} \times \Omega \to [0, \infty)$

$$\lambda \int\limits_{\Omega \times D} \int\limits_{\mathbb{R}^2 \times \widetilde{D}} f(x, g, \widetilde{g}, \theta_x \omega) \widetilde{X}_{\widetilde{D}}(\omega, d(x, \widetilde{g})) \mathbb{P}^*_{X_D}(d(\omega, g))$$

$$= \widetilde{\lambda} \int\limits_{\Omega \times \widetilde{D}} \int\limits_{\mathbb{R}^2 \times D} f(-x, g, \widetilde{g}, \omega) X_D(\omega, d(x, g)) \mathbb{P}^*_{\widetilde{X}_{\widetilde{D}}}(d(\omega, \widetilde{g}))$$

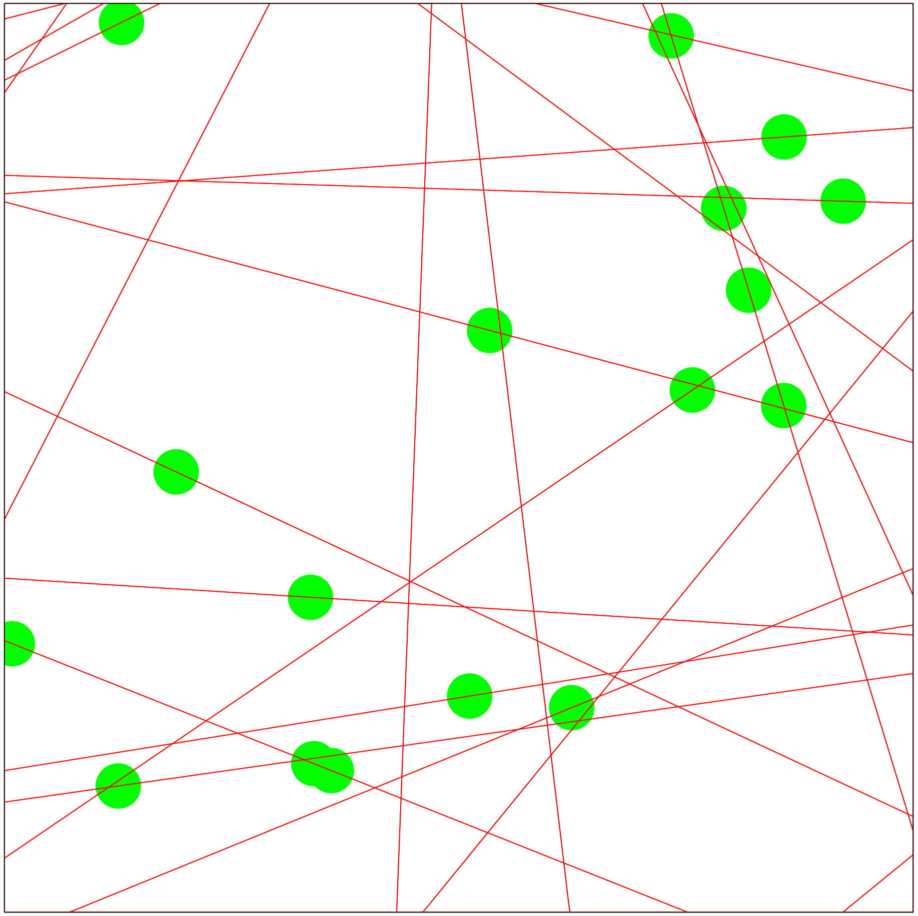
Model description
Geometry model



- Poisson line tessellation X_ℓ
- Intensity $\gamma > 0$: Mean total length of lines per unit area

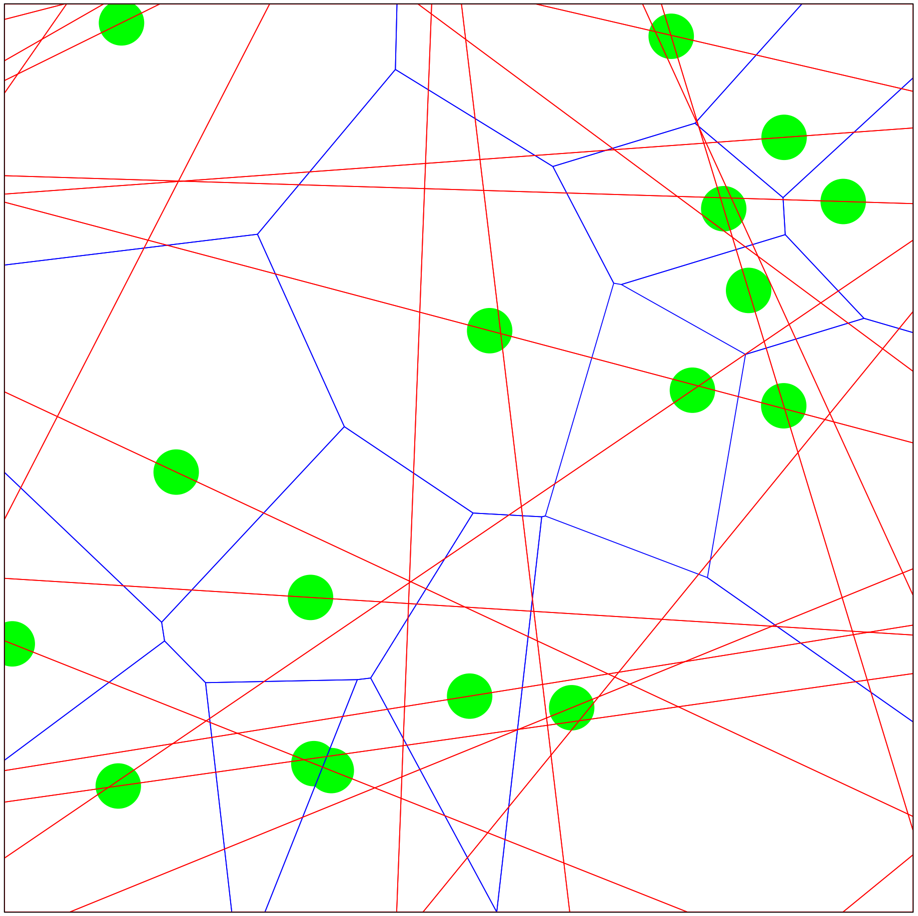
Model description

Placement of higher level components



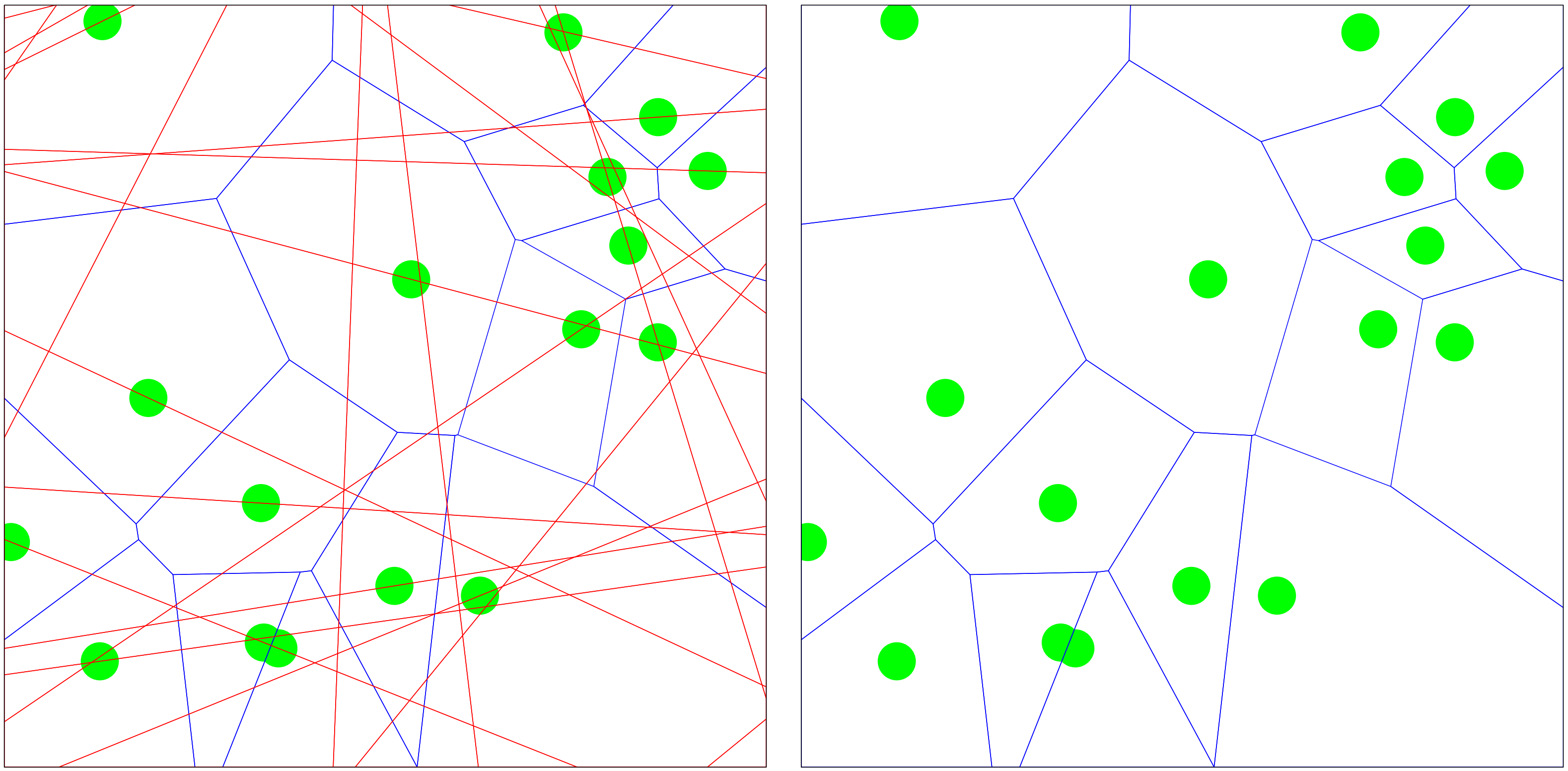
- Stationary ergodic point process $X_H = \{X_n\}_{n \geq 1}$ on the lines
- Special case: Poisson process X_H
- Linear intensity $\lambda_1 > 0$
- Planar intensity $\lambda_H = \gamma \, \lambda_1$

Model description
Serving zones



- Sequence of serving zones $\{\Xi(X_n)\}_{n\geq 1}$
- Later on:
  - Typical serving zone Ξ^*
  - Typical line system $L(\Xi^*)$

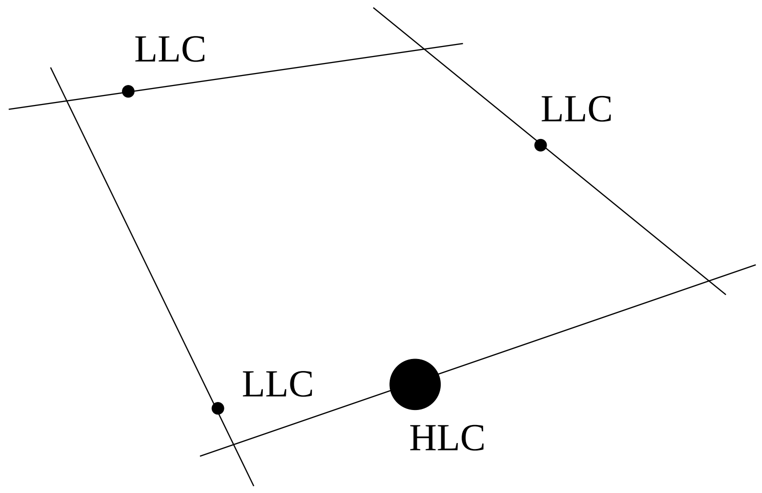
Model description
Serving zones



Cox-Voronoi tessellation (CVT) based on a Poisson line process

Mean shortest path lengths

Placement of lower level components



Linear placement on lines

- Stationary Poisson process $\{\widetilde{X}_n\}_{n\geq 1}$ on the lines
  - Linear intensity $\lambda_2 > 0$
  - Planar intensity $\lambda_L = \gamma\,\lambda_2$
- Stationary marked point process $X_L = \{[\widetilde{X}_n, c(P(\widetilde{X}_n, N(\widetilde{X}_n)))]\}_{n\geq 1}$
  - $N(\widetilde{X}_n)$ location of nearest HLC of $\widetilde{X}_n$
  - $P(\widetilde{X}_n, N(\widetilde{X}_n))$ shortest path from $\widetilde{X}_n$ to $N(\widetilde{X}_n)$
  - $c(P(\widetilde{X}_n, N(\widetilde{X}_n)))$ length (costs) of $P(\widetilde{X}_n, N(\widetilde{X}_n))$

Mean shortest path lengths
Simulation methods

- Natural Approach
  - Simulate network in (large) sampling window $W \subset \mathbb{R}^2$
  - Compute $c(P(\widetilde{X}_n, N(\widetilde{X}_n)))$ for each $\widetilde{X}_n$
  - Compute **mean shortest path length**

$$c_{LH}(W) = \frac{1}{\#\{n : \widetilde{X}_n \in W\}} \sum_{n \geq 1} \mathbb{1}_W(\widetilde{X}_n) c(P(\widetilde{X}_n, N(\widetilde{X}_n)))$$

- Disadvantages
  - W small $\Rightarrow$ edge effects significant
  - W large $\Rightarrow$ memory and runtime problems

Mean shortest path lengths
Simulation methods

- Alternative Approach
  - $\{W_i\}_{i \geq 1}$ averaging sequence of sampling windows
  - Ergodicity of X_L yields with probability 1 that

$$\lim_{i \to \infty} c_{LH}(W_i) = c^*_{LH}$$

The limit c^*_{LH} is given by ($B \in \mathcal{B}_0(\mathbb{R}^2)$)

$$\frac{1}{\lambda_L \nu_2(B)} \mathbb{E} \sum_{n \geq 1} \mathbb{I}_B(\widetilde{X}_n) c(P(\widetilde{X}_n, N(\widetilde{X}_n))) = \mathbb{E}_{X_L} c(P(o, N(o)))$$

- Disadvantages
  - Simulation not clear
  - Not very efficient

Mean shortest path lengths

- Application of Neveu
  - $$\mathbb{E}_{X_L}\, c(P(o, N(o))) = \frac{1}{\mathbb{E}_{X_H} \nu_1(L(\Xi^*))}\, \mathbb{E}_{X_H} \int\limits_{L(\Xi^*)} c(P(u, o))\, du$$
  - With
$$\frac{1}{\mathbb{E}_{X_H} \nu_1(L(\Xi^*))} = \lambda_1$$
- Independence from λ_2
- Simulation algorithm for typical cell of CVT needed

Mean shortest path lengths
Computational algorithm

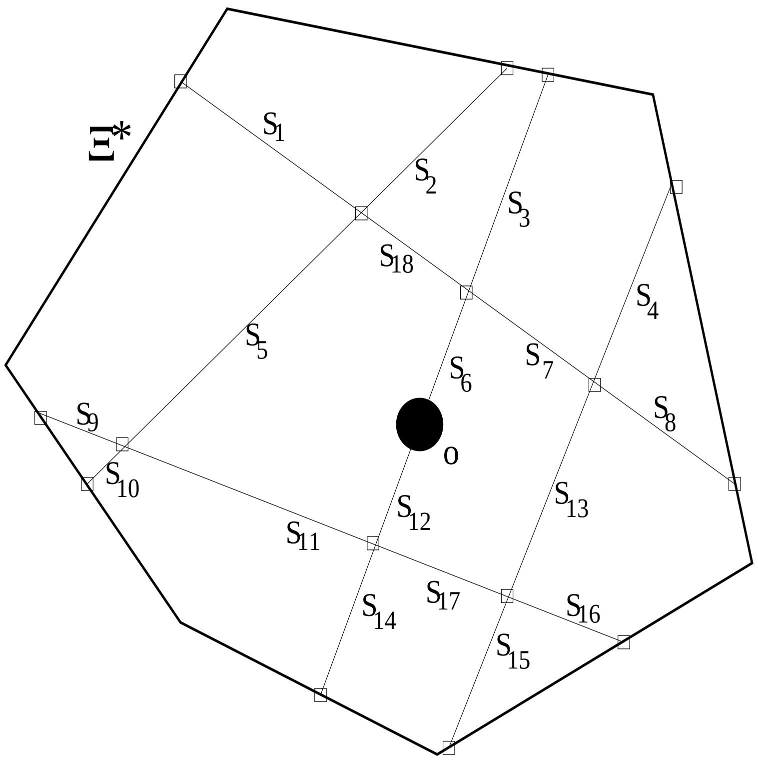
Estimators for c^*_{LH} $(k \geq 1)$,

- $$\widehat{c}_{LH}(k) = \frac{1}{\sum\limits_{i=1}^{k} \nu_1(L(\Xi_i^*))} \sum_{i=1}^{k} \sum_{j=1}^{M_i} \int_{S_i^{(j)}} c(P(u,o))\, du$$

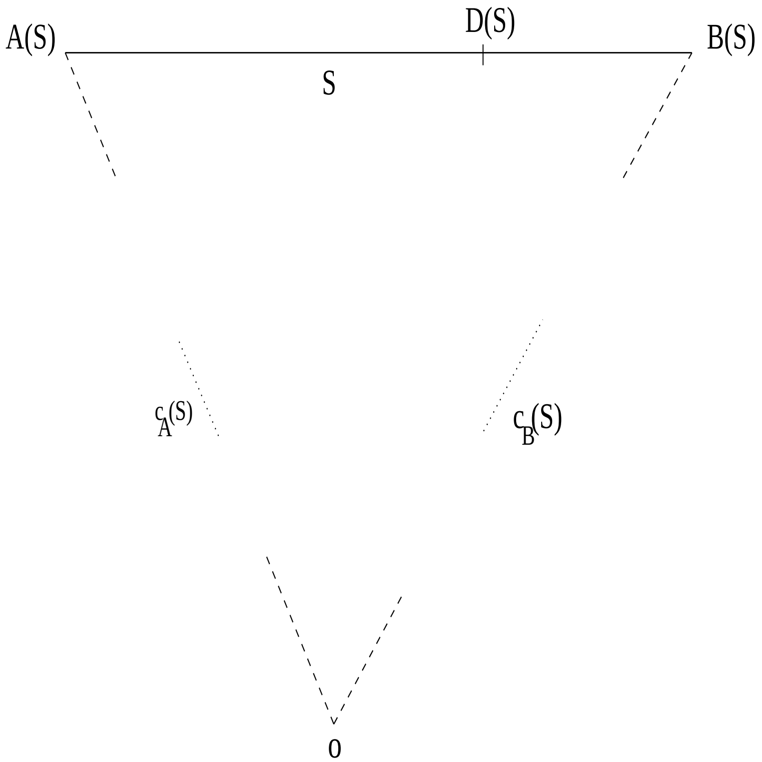
- $$\check{c}_{LH}(k) = \lambda_1 \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{M_i} \int_{S_i^{(j)}} c(P(u,o))\, du$$

- $\int_{S_i^{(j)}} c(P(u,o))\, du$ can be analytically calculated

Mean shortest path lengths
Computational algorithm



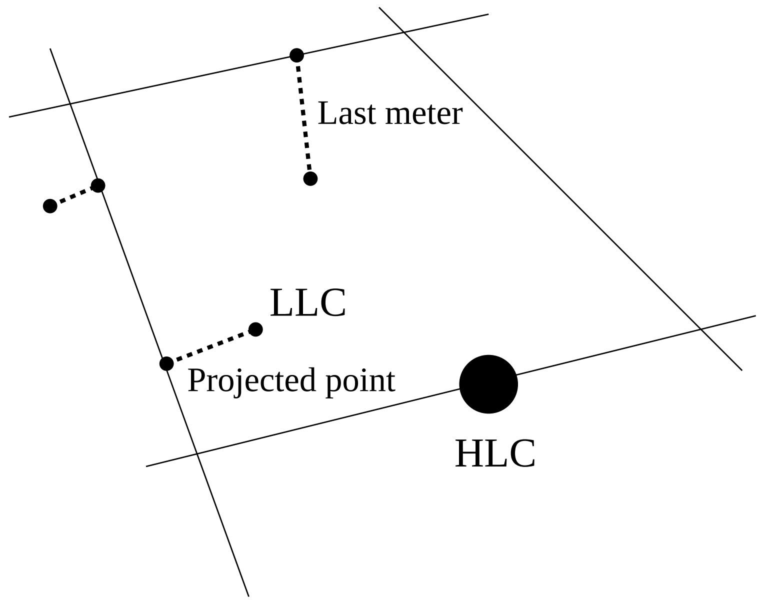
Partitioning of $L(\Xi^*)$ into segments



Mean shortest path length for S

$$\int_S c(P(u,o))\,du = \frac{1}{4}(\nu_1(S))^2 + \frac{1}{2}(c_A(S)+c_B(S))\nu_1(S) + \frac{1}{4}(c_B(S)-c_A(S))^2$$

Mean subscriber line lengths
Placement of lower level components



Spatial placement and projection to nearest line

- Stationary Poisson point process $\{X'_n\}_{n\geq 1}$ (int. λ_L)
- Stationary marked point process $X'_L = \{[X'_n, c(P(X'_n, N(X'_n)))]\}_{n\geq 1}$
  - $N(X'_n)$ location of nearest HLC of X'_n
  - Project X'_n onto X''_n
  - $c(P(X'_n, N(X'_n))) = c'(X'_n, X''_n) + c(P(X''_n, N(X'_n)))$
  - $c'(X'_n, X''_n)$ cost value of last meter

Mean subscriber line lengths
Simulation methods

- **Mean subscriber line length**

$$d_{LH}(W) = \frac{1}{\#\{n : X'_n \in W\}} \sum_{n \geq 1} \mathbb{1}_W(X'_n)\, c(P(X'_n, N(X'_n)))$$

- Ergodicity of X'_L

$$\lim_{i \to \infty} d_{LH}(W_i) = d^*_{LH} = \mathbb{E}_{X'_L} c(P(o, N(o)))$$

Mean subscriber line lengths
Simulation methods

- **Mean subscriber line length**

$$d_{LH}(W) = \frac{1}{\#\{n : X'_n \in W\}} \sum_{n \geq 1} \mathbb{1}_W(X'_n)\, c(P(X'_n, N(X'_n)))$$

- Ergodicity of X'_L

$$\lim_{i \to \infty} d_{LH}(W_i) = d^*_{LH} = \mathbb{E}_{X'_L} c(P(o, N(o)))$$

- Application of Neveu

$$\mathbb{E}_{X'_L}\, c(P(o, N(o))) = \frac{1}{\mathbb{E}_{X_H} \nu_2(\Xi^*)}\, \mathbb{E}_{X_H} \int_{\Xi^*} c(P(u, o))\, du$$

Mean subscriber line lengths
Computational algorithm

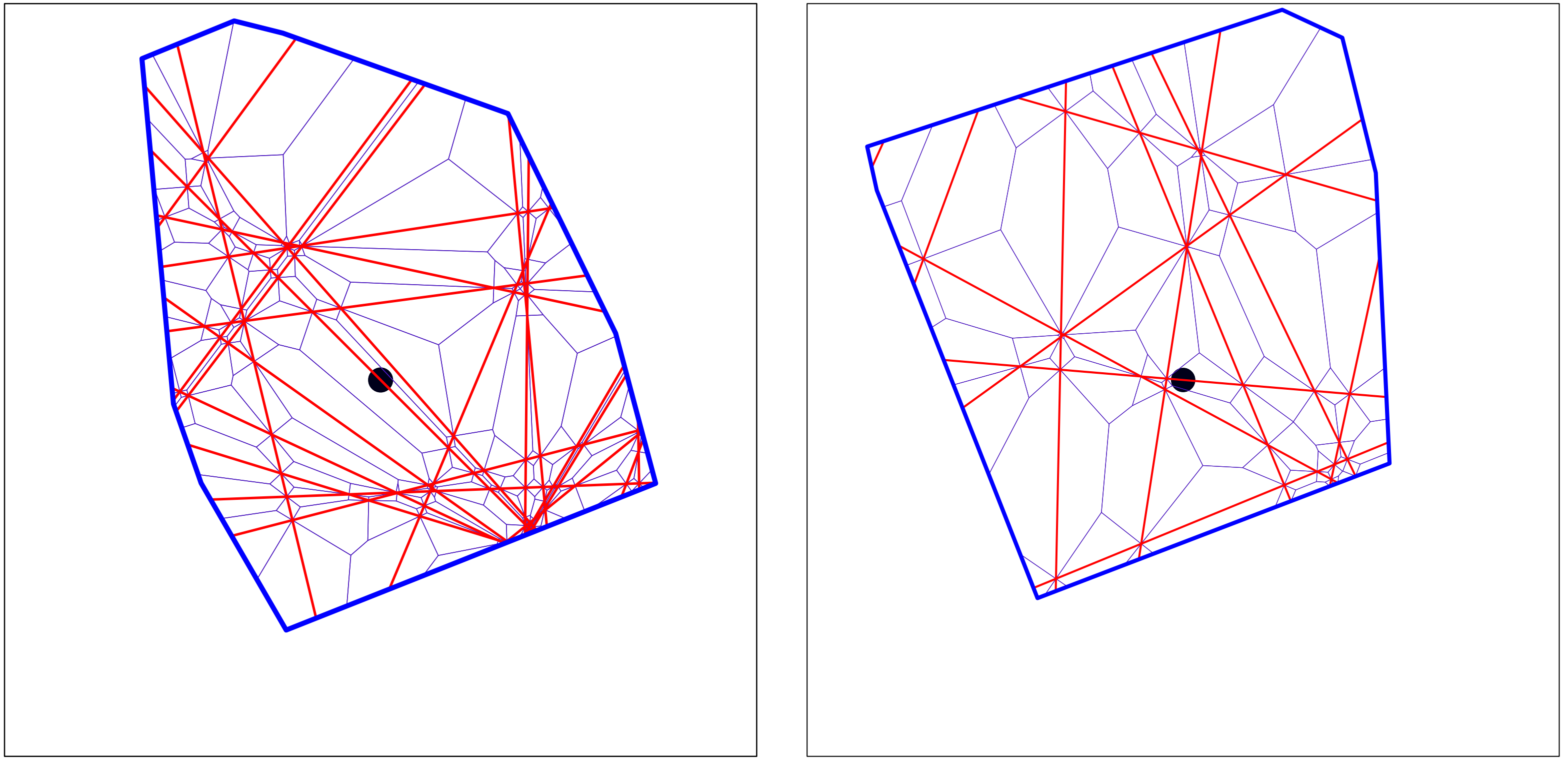
Estimators for d^*_{LC} $(k \geq 1)$,

- $$\widehat{d}_{LH}(k) = \frac{1}{\sum\limits_{i=1}^{k} \nu_2(\Xi^*_i)} \sum_{i=1}^{k} \sum_{j=1}^{K_i} \int\limits_{\Psi_i^{(j)}} c(P(u,o))\, du$$

- $$\check{d}_{LH}(k) = \lambda_1 \gamma \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{K_i} \int\limits_{\Psi_i^{(j)}} c(P(u,o)\, du$$

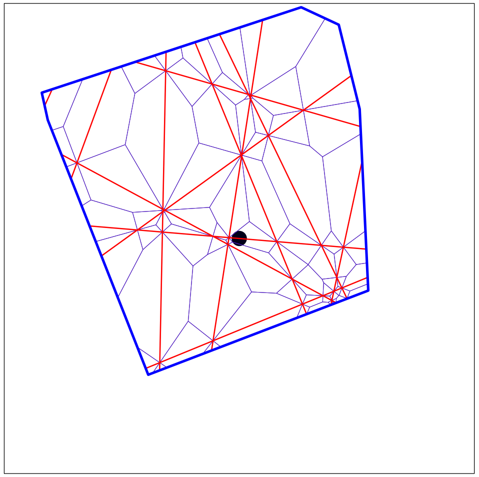
- $\int_{\Psi_i^{(j)}} c(P(u,o))\, du$ can be analytically calculated

Mean subscriber line lengths
Computational algorithm

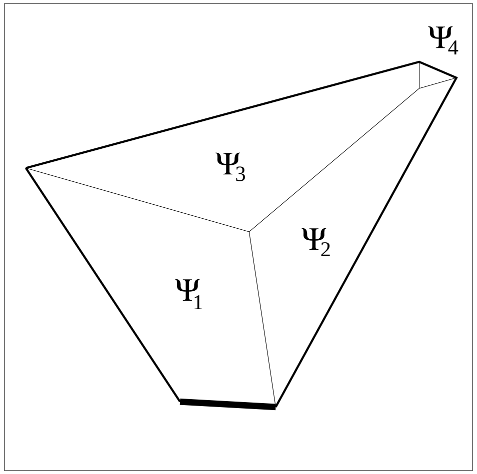


Samples of typical Cox–Voronoi serving zones

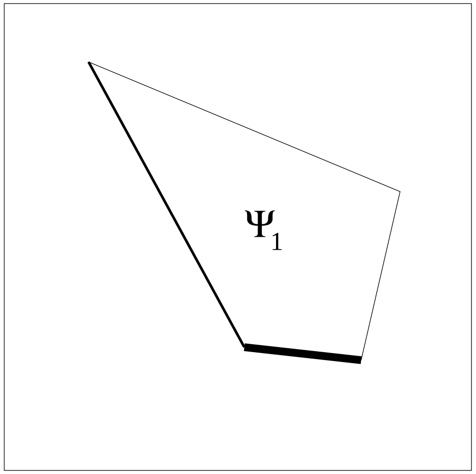
Mean subscriber line lengths
Computational algorithm



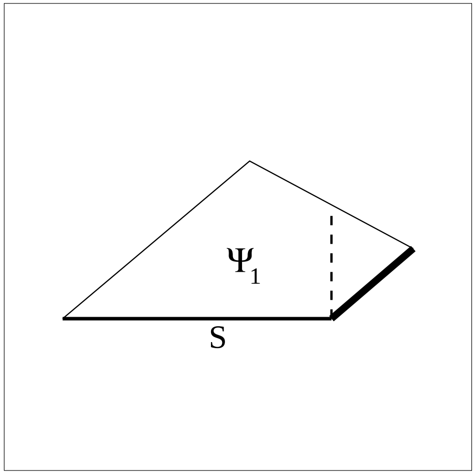
(a) Typical cell



(b) A micro–cell

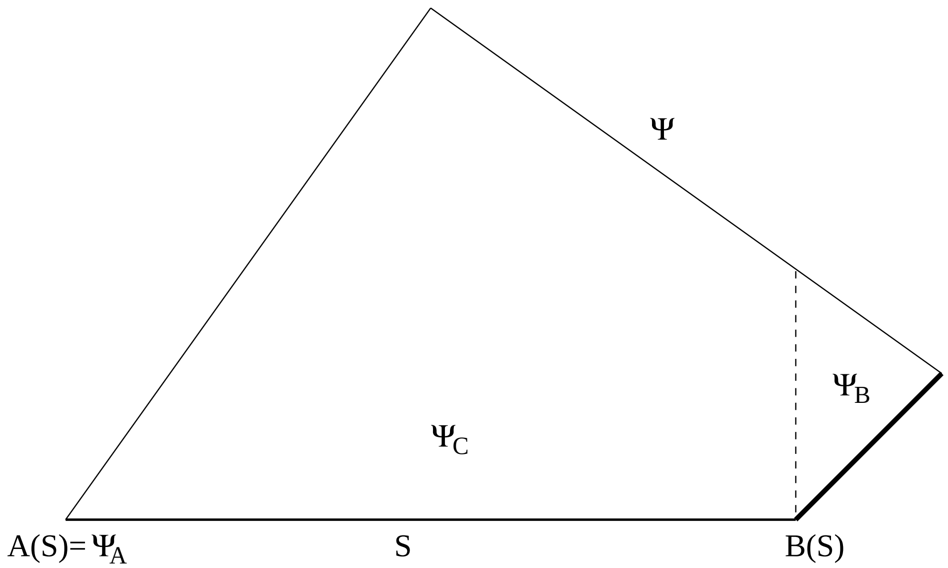


(c) Part of this micro–cell



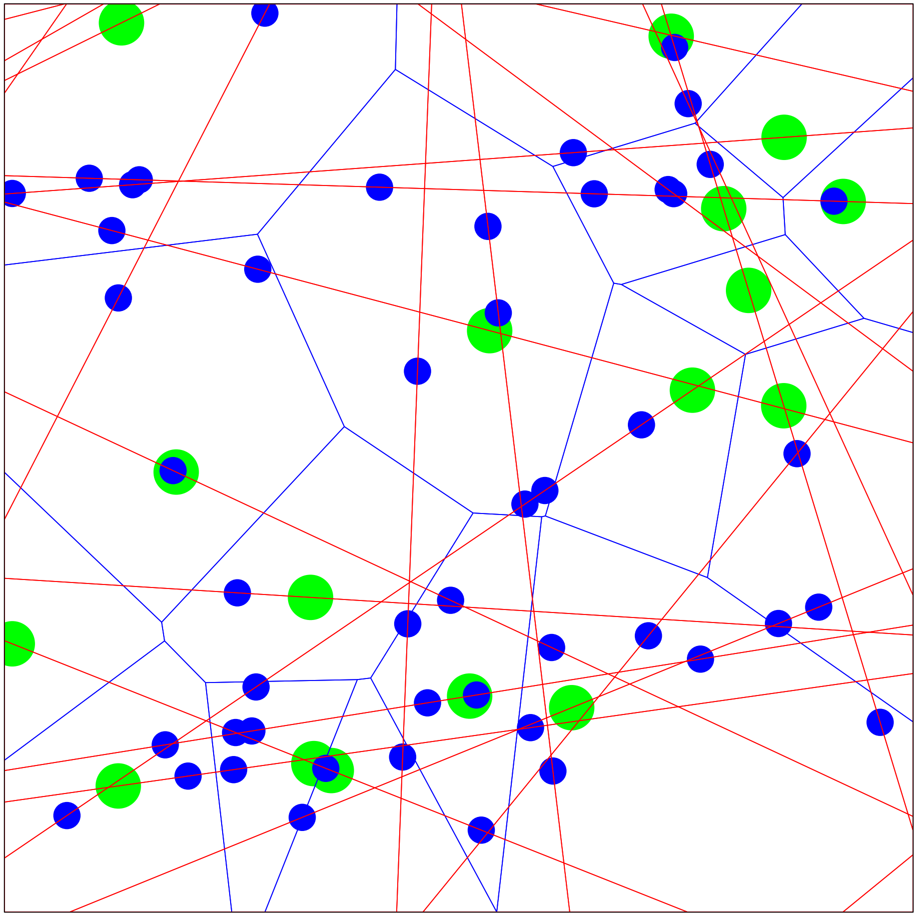
(d) Further decomposition

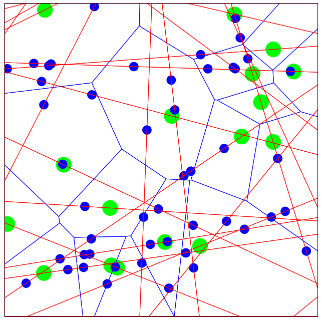
Mean subscriber line lengths
Computational algorithm



$$\int_{\Psi} c(P(u,o))\,du = c_a(S)\,\nu_2(\Psi_A) + c_B(S)\,\nu_2(\Psi_B) + \int_{\Psi_C} c(P(u,o))\,du$$

Numerical results
Scaling property





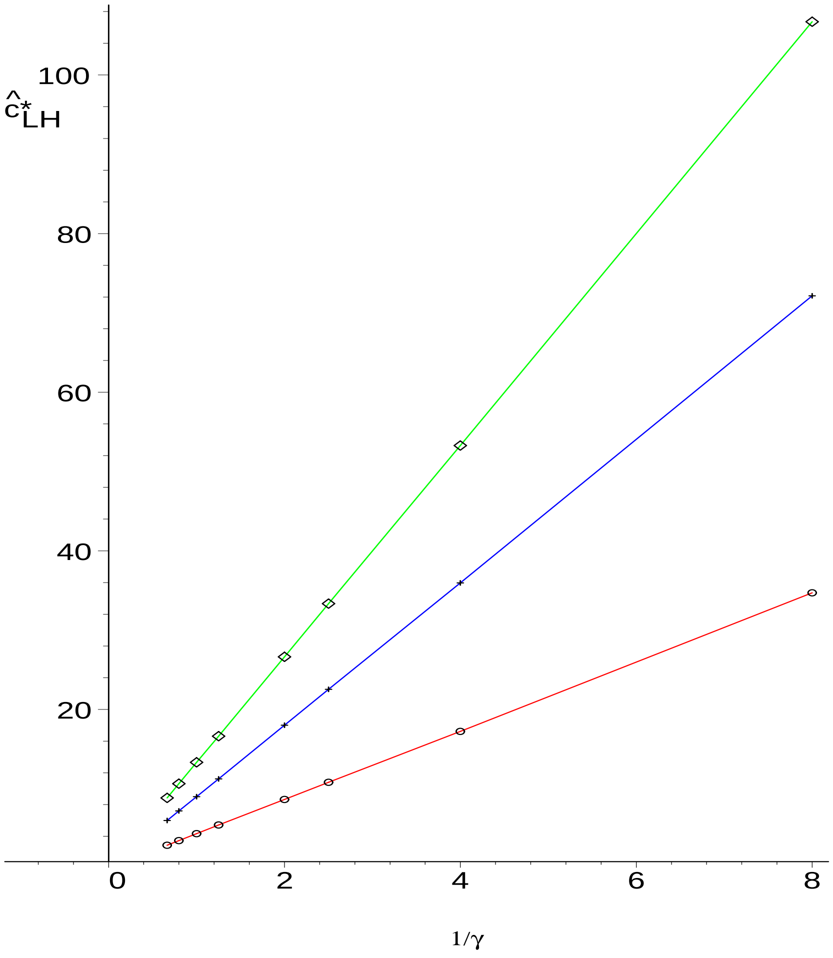
Let $\gamma^{(1)}/\lambda_1^{(1)} = \gamma^{(2)}/\lambda_1^{(2)}$

- $\gamma^{(1)}\, c^*_{LH}(\gamma^{(1)}, \lambda_1^{(1)})$
  $= \gamma^{(2)}\, c^*_{LH}(\gamma^{(2)}, \lambda_1^{(2)})$
- $\gamma^{(1)}\, d^*_{LH}(\gamma^{(1)}, \lambda_1^{(1)})$
  $= \gamma^{(2)}\, d^*_{LH}(\gamma^{(2)}, \lambda_1^{(2)})$

Different intensities but same $\kappa = \gamma/\lambda_1$

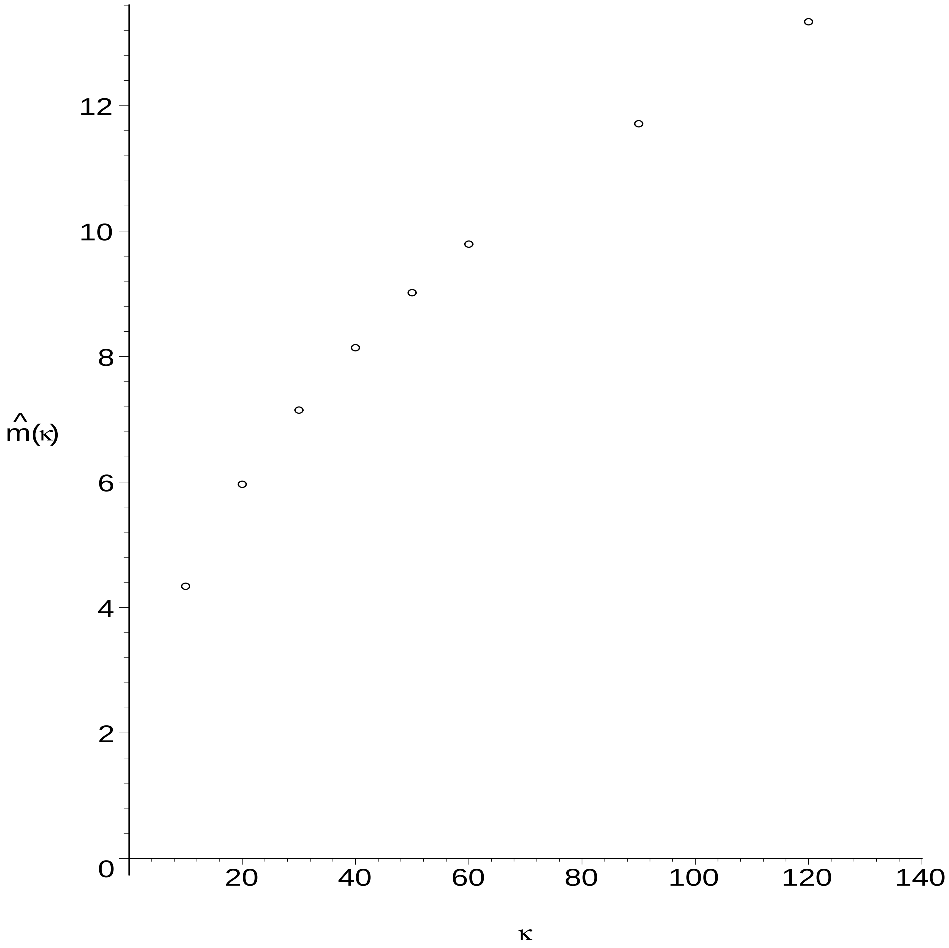
Numerical results

Mean shortest path lengths



- 50000 realizations of Ξ^* (and $L(\Xi^*)$)
- Estimated results of c^*_{LH} for
  - $\kappa = 10$
  - $\kappa = 50$
  - $\kappa = 120$
- $c^*_{LH}(\gamma, \lambda_1) = m(\kappa)/\gamma$

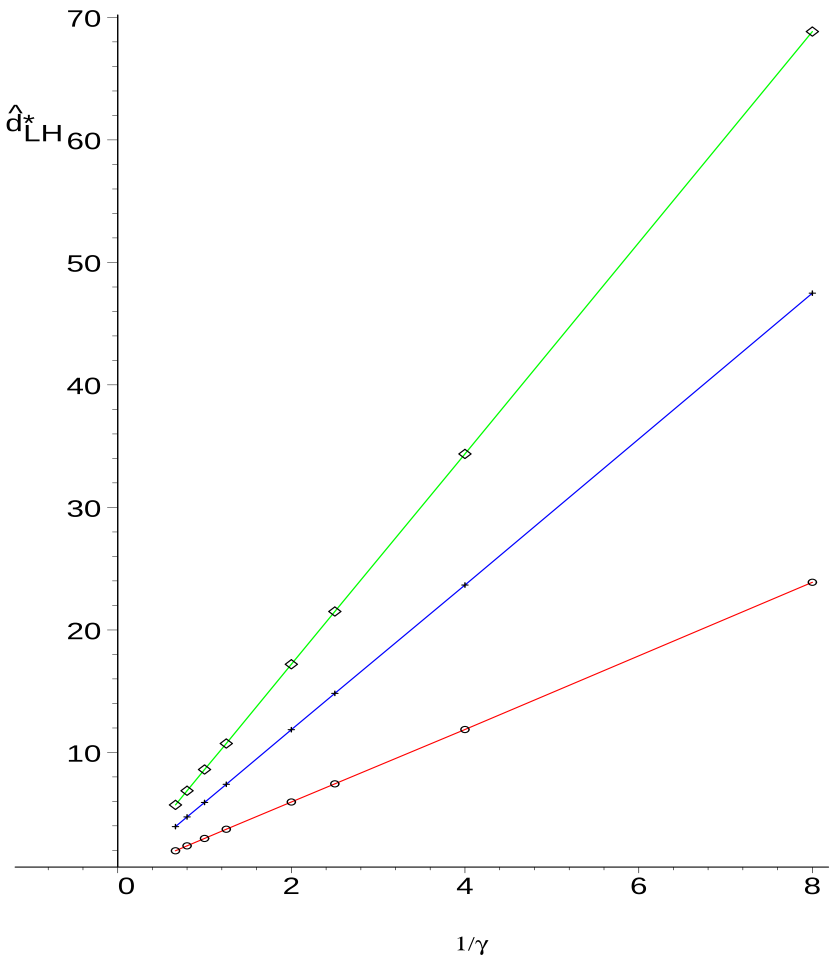
Numerical results
Mean shortest path lengths



$m(\kappa)$ of c^*_{LH} for increasing κ

Numerical results

Mean subscriber line lengths



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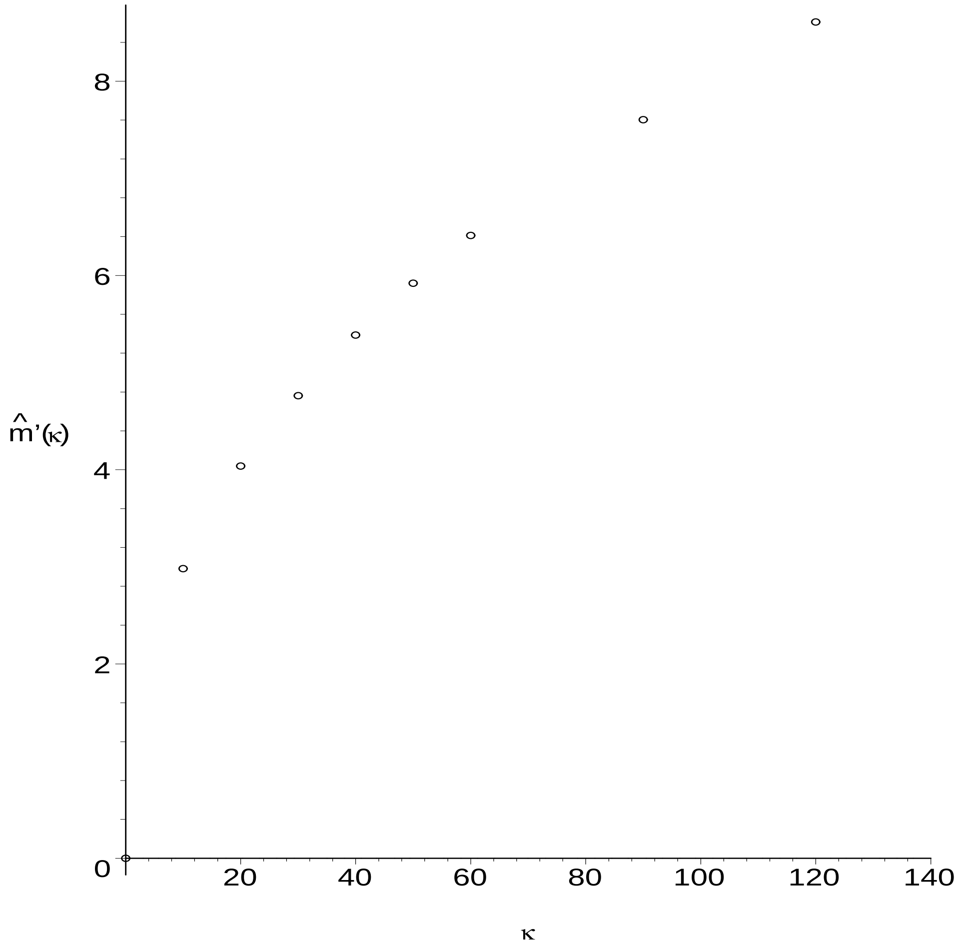
Numerical results
Mean subscriber line lengths



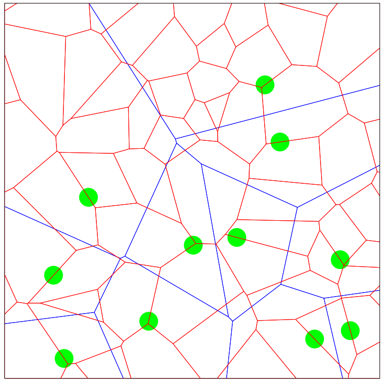
$m'(\kappa)$ of d^*_{LH} for increasing κ

Numerical results

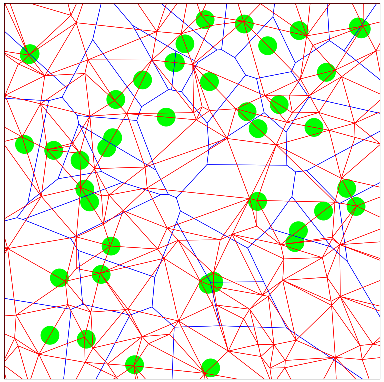
Comparison and Summary

- $c_{LH}^* > d_{LH}^*$ for any parameter pair (λ_1, γ)
- Good approximations $m(\kappa) \approx a\kappa^b$ and $m'(\kappa) \approx a'\kappa^{b'}$
- Thereby estimation of c_{LH}^* and d_{LH}^* without simulation possible
  - $\kappa = \gamma/\lambda_1$
  - $\widehat{m}(\kappa) = a\kappa^b$
  - $\widehat{m}'(\kappa) = a'\kappa^{b'}$
  - $\widehat{c}_{LH}^* = \widehat{m}(\kappa)\gamma^{-1}$
  - $\widehat{d}_{LH}^* = \widehat{m}'(\kappa)\gamma^{-1}$

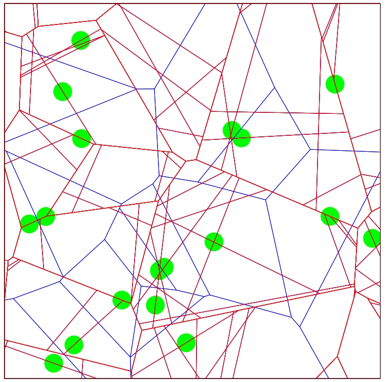
Outlook
Extensions to other geometry models



CVT based on Poisson-Voronoi



CVT based on Poisson-Delaunay



CVT based on iterated tessellations

Literature

- C. Gloaguen, F. Fleischer, H. Schmidt, V. Schmidt. *Simulation of typical Cox-Voronoi cells, with a special regard to implementation tests.* Mathematical Methods of Operations Research **62**(3), 357-373 (2005).
- C. Gloaguen, F. Fleischer, H. Schmidt, V. Schmidt. *Analysis of shortest path and subscriber line lengths in telecommunication access networks.* Preprint, (2005).

- For more information `www.geostoch.de`