# Variance paradoxes in geometric sampling : A generalisation of the 'Jensen-Gundersen Paradox'

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27th March 2006

# Overview

Introduction

Volume estimators based on single probes

Second moment formulas for IUR test probes

Volume estimators based on test systems

Second moment formulas for test systems

Outlook

# Introduction

- Jensen-Gundersen Paradox (1982): Pointcounting more precise than area measurement.

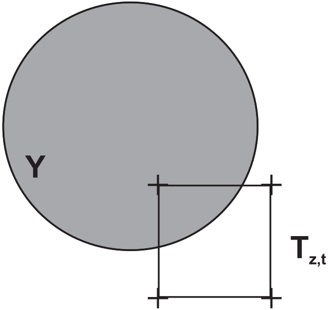


Figure: Original Jensen-Gundersen Paradox (E.B. Jensen, H.J.G. Gundersen (1982))

- K. Schladitz (1999) : Pointcounting optimal in some cases.

# Introduction

- Ohser's Paradox (1990): Counting lines more precise than length measurement.

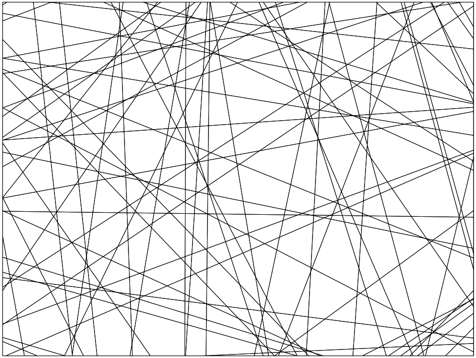

Figure: Ohser's Paradox (J. Ohser (1990))

- K. Schladitz (1996) : Counting optimal.

# Introduction

- What happens in the case of test systems?

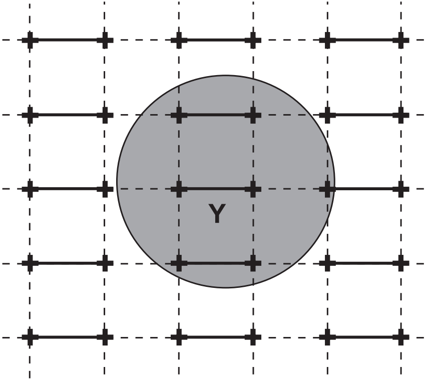


Figure: Test system intersecting a set $Y$.

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# IUR probes hitting a $d$-dimensional set

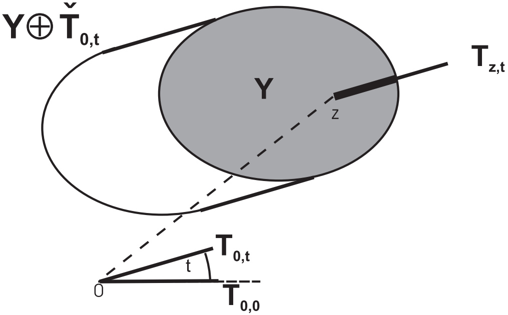


Figure: IUR line segment hitting a set $Y$ in $\mathbb{R}^2$

# Estimation of volume with a single bounded probe

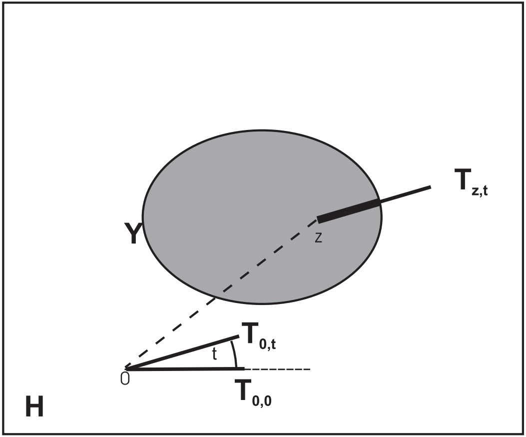


Figure: Estimation with a single test probe

# IUR probes hitting a $d$-dimensional set

Let

- $T_{0,0}$ be a fixed set of dimension $q$
- $T_{z,t}$ be the set after isotropic rotation by $t \in G_{d[0]}$ and uniform random translation by $z \in Y \oplus \check{T}_{0,t}$

Then $T_{z,t}$ is called isotropic uniform random (IUR) hitting $Y$ and

$$\widehat{\nu}_d(Y, T_{z,t}) := \frac{\mathbb{E}_t \nu_d(Y \oplus \check{T}_{0,t})}{\nu_q(T_{0,0})} \nu_q(T_{z,t} \cap Y)$$

is an unbiased estimator for $\nu_d(Y)$.

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# The geometric covariogram

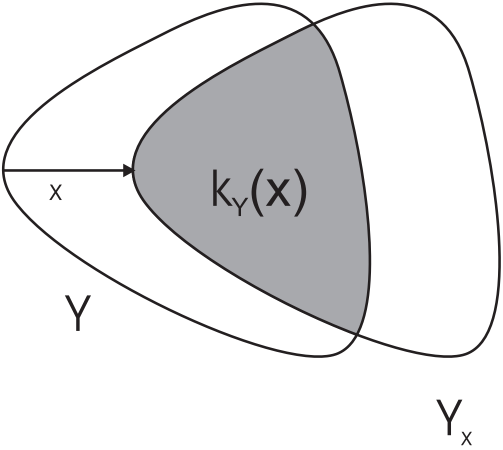


Figure: The geometric covariogram

# Second moment formulas for bounded IUR probes

### Theorem

*Let $Y \subset \mathbb{R}^d$, dim(Y)=d and $T_{z,t}$ be an IUR probe of dimension q hitting Y, then*

$$\mathbb{E}\nu_q^2(Y \cap T_{z,t}) = \frac{1}{\mathbb{E}_t \nu_d(Y \oplus \check{T}_{0,t})} \int_{T_{0,0}} \int_{T_{0,0}} k_Y(\| y - x \|) \nu_q(dx) \nu_q(dy) \quad (1)$$

See [Baddeley and Cruz-Orive, 1995] for similar formulas.

# Second moment formulas for single probes

This formula is in general difficult to use, see the following figure :

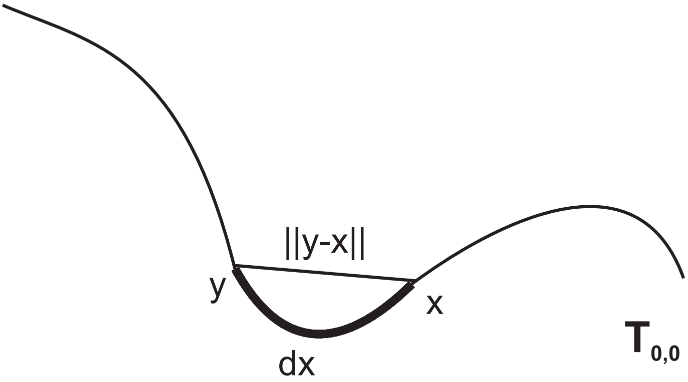


Figure: Values for $\| x - y \|$ for a curve in $\mathbb{R}^2$.

# Second moment formulas for bounded IUR probes

## Corollary

*If $T_{0,0}$ is a subset of a q-dimensional affine-linear subspace, then*

$$\mathbb{E}\nu_q^2(Y \cap T_{z,t}) = \frac{1}{\mathbb{E}_t \nu_d(Y \oplus \check{T}_{0,t})} \int_0^\infty q b_q r^{q-1} k_Y(r) k_T(r) dr \qquad (2)$$

*where $k_T$ is the isotropic geometric covariogram of $T_{0,0}$ regarded as a subset of $\mathbb{R}^q$.*

# Second moment formula for an IUR $q$-flat

**Theorem**

*Let $Y \subset \mathbb{R}^d$, dim($Y$)=$d$ and $L_q(z,t)$ be an IUR $q$-flat hitting $Y$, then*

$$\mathbb{E}\nu_q^2(Y \cap L_q(z,t)) = \frac{1}{\mathbb{E}_t \nu_{d-q}(Y_t')} \int_0^\infty q b_q r^{q-1} k_Y(r) dr \qquad (3)$$

# Example : Disk probed by a line segment and its endpoints

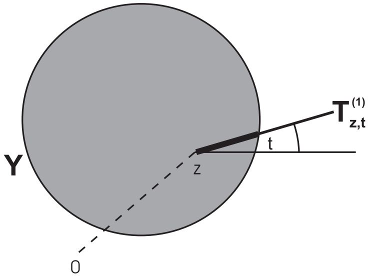


Figure: 2 probes used to estimate $\nu_2(Y)$

# Example : Disk probed by a line segment and its endpoints

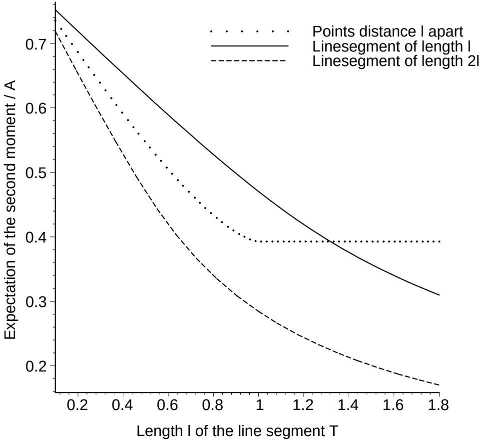


Figure: Second moments of the estimators for a disk $Y$ of diameter 1.

# Example : Disk probed by a line segment and its endpoints

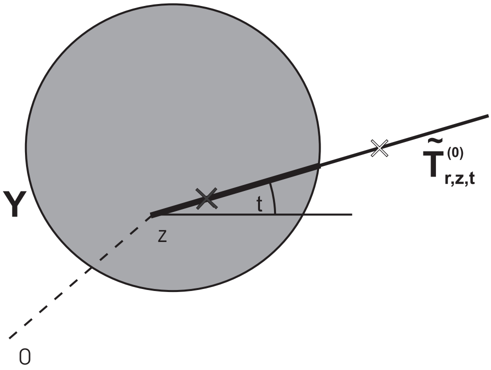


Figure: The 2 connected probes.

# Example : Area of square estimated by different probes

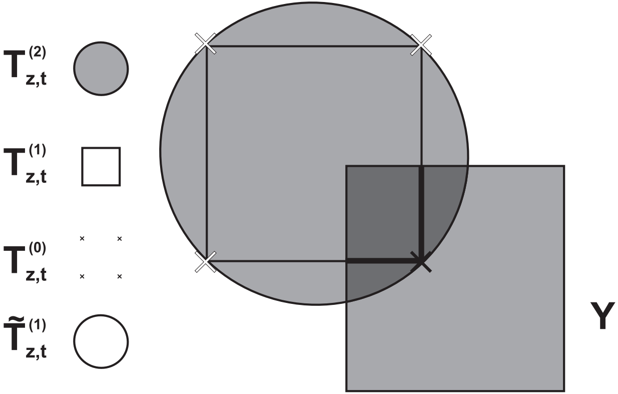


Figure: The 4 probes used to estimate $\nu_2(Y)$.

- $Y \subset \mathbb{R}^2$ square of side 1
- $T^{(0)}_{z,t} \subset T^{(1)}_{z,t} \subset T^{(2)}_{z,t}$ with 3 different probes

# Example : Area of square estimated by different probes

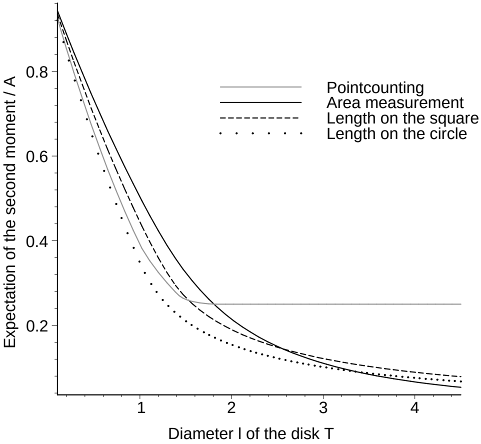


Figure: $\frac{1}{A}\mathbb{E}\widehat{\nu}_2^2(Y, T^{(i)})$ of the 4 estimators with respect to $l$.

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# Estimation of volume with test systems

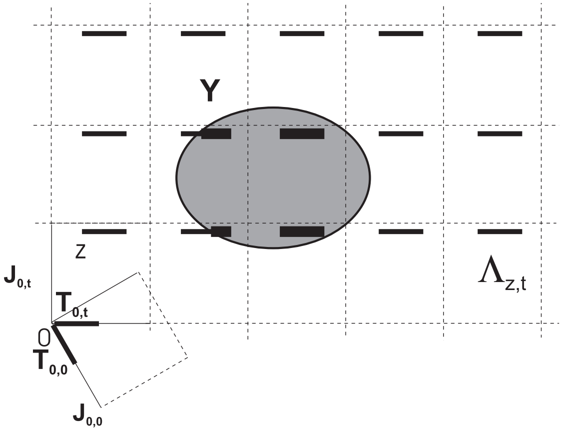


Figure: Estimation with a test system of line segments

# Estimation of volume with test systems

Let

- $T_{0,0}$ be a set of dimension $q$ and $\exists J_{0,0} \subset \mathbb{R}^d$ with
  - $\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}} J_{z_k,0}$ for $z_k \in \mathbb{R}^d \; \forall k \in \mathbb{Z}$
  - $J_{z_k,0} \cap J_{z_j,0} = \emptyset \, \forall k \neq j$
- $\Lambda_{0,0} = \bigcup_{k\in\mathbb{Z}} T_{z_k,0}$
- $\Lambda_{z,t}$ be $\Lambda_{0,0}$ after isotropic rotation by $t \in G_{d[0]}$ and uniform random translation by $z$ in $J_{0,t}$

Then

$$\widehat{\nu}_d(Y, \Lambda_{z,t}) := \frac{\nu_d(J_{0,0})}{\nu_q(T_{0,0})} \nu_q(\Lambda_{z,t} \cap Y)$$

is an unbiased estimator for $\nu_d(Y)$

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# Second moment formulas for test systems

## Theorem

*Let $\Lambda_{0,0}$ be a test system with $q$–dimensional basic probe $T_{0,0}$. Then for a IUR test system $\Lambda_{z,t}$*

$$\mathbb{E}\nu_q^2(Y \cap \Lambda_{z,t}) = \frac{1}{\nu_d(J_{0,0})} \int_{T_{0,0}} \int_{\Lambda_{0,0}} k_Y(\| y - x \|)\, \nu_q(dx)\, \nu_q(dy) \quad (4)$$

See e. g. [Matérn, 1989] for $q = 0$.

# Probes contained in affine-linear subspaces

## Corollary

*If $J_{0,0} = [0, a_1) \times \cdots \times [0, a_d)$ and $T_{0,0}$ is a q-dimensional basic probe with $T_{0,0} \subset [0, a_1) \times \cdots \times [0, a_q)$ and $\Lambda_{z,t}$ is IUR, then*

$$\mathbb{E}\nu_q^2(Y \cap \Lambda_{z,t}) = \frac{1}{\nu_d(J_{0,0})} \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^q} k_Y(\|z + z_j^{(d-q)}\|) k_{T_{0,0}}(z + z_k^{(q)}) \, \nu_q(dz) \tag{5}$$

*where $z_k^{(q)}$ are the translations in $L_q \supset T_{0,0}$, $z_j^{(d-q)}$ are the translations in $L_q^{\perp}$.*

# Test systems of lines, line segments and points in $\mathbb{R}^2$

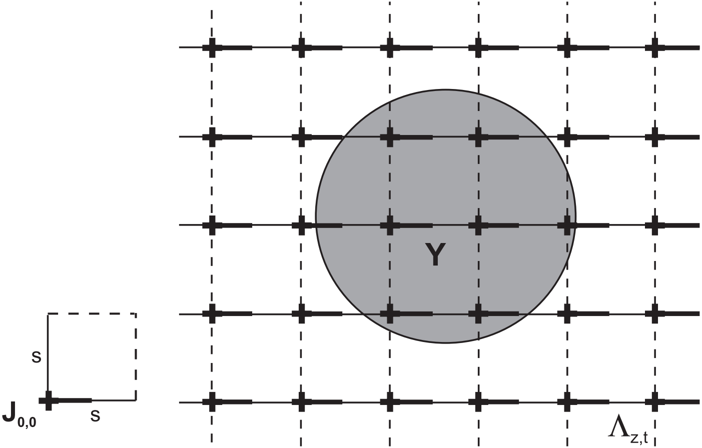


Figure: Different test systems.

# Comparison of lines, line segments and points in $\mathbb{R}^2$

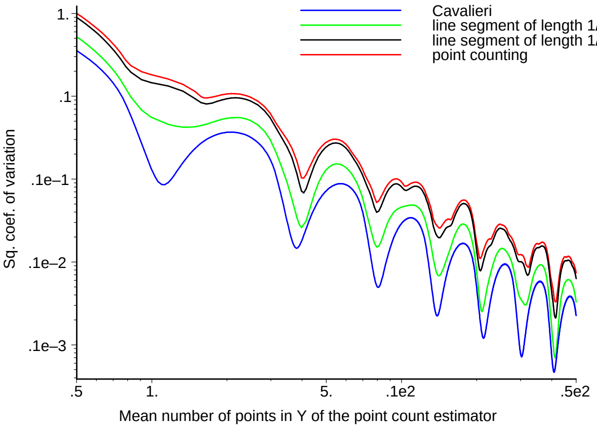


Figure: Square coefficient of variation of different estimators with $J_{0,0} = [0, s) \times [0, s)$ for a disk $Y$ of diameter 1

# Comparison of lines, line segments and points in $\mathbb{R}^2$

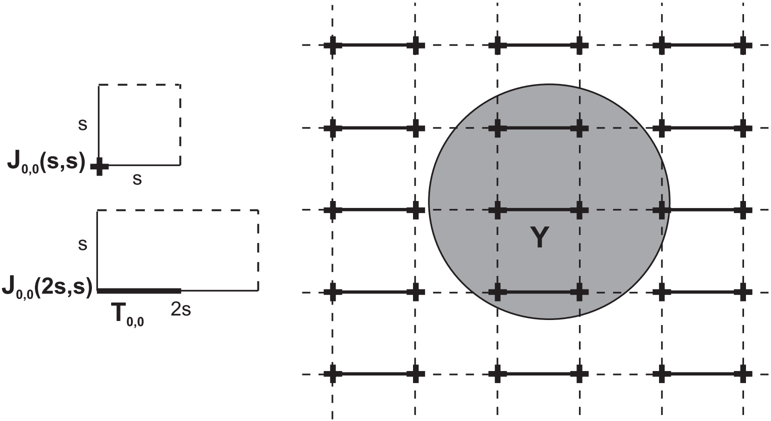


Figure: Test system of points contained in the system of line segments.

# Comparison of lines, line segments and points in $\mathbb{R}^2$

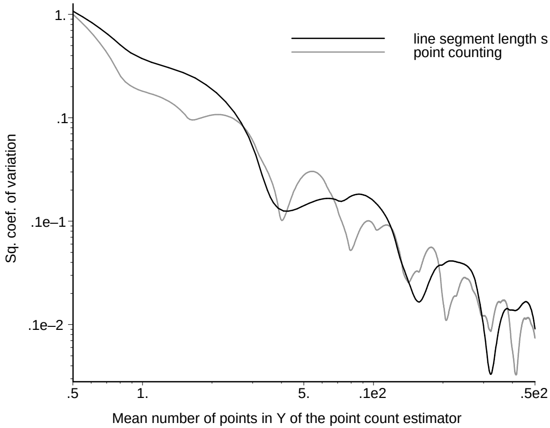


Figure: Square coefficient of variation of the pointcounting and the estimator based on line segments of length $s$.

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# Outlook : Second moment formula for dim($Y$)$< d$

### Theorem

*Let $Y \subset \mathbb{R}^2$, dim($Y$)=1 and $T_{z,t}$ be an IUR line segment. Then*

$$\mathbb{E}\nu_0^2(Y \cap T_{z,t}) \cdot \left(\pi \mathbb{E}_t \nu_2(Y \oplus \check{T}_{0,t})\right)$$
$$= \int_Y \int_Y k_T(\| y - x \|) \frac{\sin\alpha \sin\beta}{\| y - x \|} \nu_1(dy)\nu_1(dx) + 2\nu_1(T)\nu_1(Y)$$

# Outlook : Second moment formula for dim($Y$)$< d$

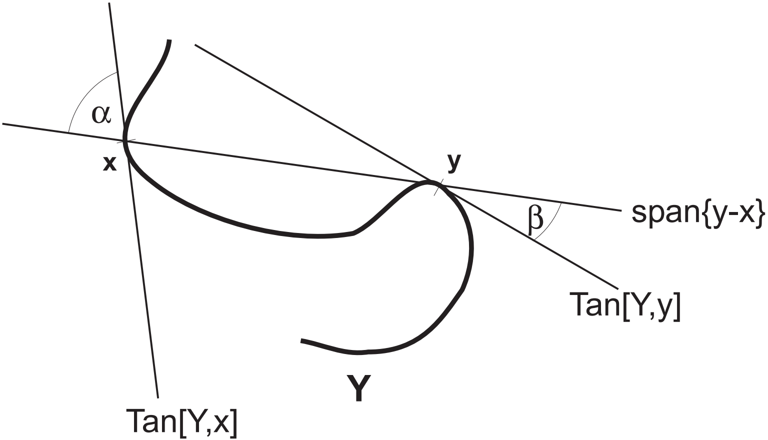


Figure: The angles $\alpha$ and $\beta$ in the theorem.

Baddeley, A. J. and Cruz-Orive, L. M. (1995).
The Rao-Blackwell theorem in stereology and some counterexamples.
*Adv. Appl. Probab.*, 27:2–19.

Jensen, E. B. and Gundersen, H. J. G. (1982).
Stereological ratio estimation based on counts from integral test systems.
*J. Microscopy*, 125:51–66.

Matérn, B. (1989).
Precision of area estimation: a numerical study.
*J. Microscopy*, 153:269–284.

Ohser, J. (1990).
*Grundlagen und praktische Möglichkeiten der Charakterisierung struktureller Inhomogenitäten von Werkstoffen.*
Dr. sc. techn. thesis, Bergakademie Freiberg, Freiberg (Sachsen), Germany.

Schladitz, K. (1996).
*Estimation of the intensity of stationary flat processes.*
PhD thesis, Friedrich–Schiller–Universität Jena.

Schladitz, K. (1999).
Surprising optimal estimators for the area fraction.
*Adv. Appl. Probab.*, 31:995–1001.