TAYLOR-SERIES EXPANSION FOR MULTIVARIATE CHARACTERISTICS OF CLASSICAL RISK PROCESSES

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Abstract: For the continuous-time risk model with compound Poisson input, the (finite-horizont) joint probability $\boldsymbol{P}(\tau \leq t, X \leq x, Y \leq y)$ of ruin time τ, surplus X just before ruin and deficit Y at ruin time τ is considered as a function of the arrival rate λ of claims. It is expanded into a Taylor series at $\lambda = 0$. A certain extension of a corresponding result for infinite-horizont joint probabilities, which previously has been derived in Gerber, Goovaerts and Kaas (1987), is also given. For each $n \geq 1$, the coefficient of λ^n is determined by using a general representation formula for the derivatives of a wide class of functionals of independently marked Poisson processes.

Keywords: risk theory, ruin time, surplus prior to ruin, severity of ruin, joint probabilities, finite time, infinite time, series expansion

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Abbreviated title: Taylor-Series Expansion

1 Introduction

The risk process $\{Z_t\}_{t\geq 0}$ of the classical continuous-time risk model is assumed to be generated from positive interclaim times T_k and claim sizes U_k according to premium 1 per time unit, i.e.

$$Z_t = u + t - \sum_{k=1}^{N_t} U_k \tag{1}$$

where $N_t = \sup\{k : \sigma_k \leq t\}$ and $\sigma_k = T_1 + \ldots + T_k$ is the kth arrival epoch; $u \geq 0$. The sequences $\{T_k\}_{k\geq 1}$ and $\{U_k\}_{k\geq 1}$ are independent; each of them consists of i.i.d. random variables, where the T_k are exponentially distributed with parameter $\lambda > 0$. Let B denote the distribution function of claim sizes U_k; $\mu^{-1} \stackrel{def}{=} \int_0^\infty x \; dB(x)$. We assume that $\rho \stackrel{def}{=} \lambda\mu^{-1} < 1$, i.e. the premium loading factor

$$\theta \stackrel{def}{=} \frac{1-\rho}{\rho} \tag{2}$$

is positive.

The ruin time τ is defined by $\tau = \inf\{t > 0 : Z_t < 0\}$ or, equivalently, by $\tau = \sigma_\kappa$ where $\kappa = \inf\{k : Z_{\sigma_k} < 0\}$. The quantity $Y \stackrel{def}{=} -Z_\tau$ can be interpreted as the deficit at ruin time τ (or as the severity of ruin). The surplus just before the ruin time is $X \stackrel{def}{=} Z_{\tau-}$.

Recently, besides the ruin time τ which is a quite common characteristic of risk processes, also the surplus X and the deficit Y have been considered in risk theory. However, for a finite time horizont $t < \infty$ and for a general claim size distribution it is rather difficult (or even impossible) to get a closed formula for the probabilities $\boldsymbol{P}(\tau \leq t, X \leq x, Y \leq y)$. A certain exception is the case when $t = \infty$ and $u = 0$. In this case, the probabilities $\boldsymbol{P}(\tau < \infty, X \leq x, Y \leq y)$ can be expressed by λ and B, by means of a closed formula, which even remains true for risk models with a general stationary input and, therefore, reaches far beyond the classical risk model with compound Poisson input (see e.g. Bowers et al. (1987) for the compound Poisson case, and Miyazawa and Schmidt (1993), Miyazawa (1994), Asmussen and Schmidt (1995) for the general stationary case). This situation motivated several authors to develop approximative methods in order to determine probabilities of the form $\boldsymbol{P}(\tau \leq t, X \leq x, Y \leq y)$, or their marginals, for arbitrary $t \leq \infty$ and $u \geq 0$. We mention here the method of Panjer (1986) to calculate the ruin probabilities $\boldsymbol{P}(\tau < \infty)$ recursively. To apply this method, one has to use the fact that the probability $\boldsymbol{P}(\tau < \infty)$ can be represented as solution of a (defective) renewal equation:

$$\boldsymbol{P}(\tau < \infty) = \sum_{n=1}^{\infty} \lambda^n F_{\infty,\infty} * G^{*(n-1)}(u), \tag{3}$$

where $F_{\infty,\infty}(v) = \int\limits_{v}^{\infty} (1 - B(s))\, ds$, $G(u) = \int\limits_{0}^{u} (1 - B(s))\, ds$, and $*$ denotes convolution. In Dickson (1989), an algorithm has been given to approximate the probabilities $\boldsymbol{P}(\tau < \infty, Y \leq y)$ which consists of two steps. First, $\boldsymbol{P}(\tau < \infty, Y \leq y)$ is approximately expressed by the survival probabilities $1 - \boldsymbol{P}(\tau < \infty)$. Next, Panjer's method (1986) is used to approximate $1 - \boldsymbol{P}(\tau < \infty)$. In Gerber, Goovaerts and Kaas (1987) it has been observed that also $\boldsymbol{P}(\tau < \infty, Y \leq y)$ can be represented as solution of a (defective) renewal equation:

$$\boldsymbol{P}(\tau < \infty, Y \leq y) = \sum_{n=1}^{\infty} \lambda^n F_{\infty,y} * G^{*(n-1)}(u), \tag{4}$$

where $F_{\infty,y}(v) = \int\limits_{v}^{v+y} (1 - B(s))\, ds$. And this representation formula has been used to find an explicit series expansion of the density $\frac{d}{dy} \boldsymbol{P}(\tau < \infty, Y \leq y)$ in the case in which all claims are of constant size. For further related results concerning the calculation of infinite-horizont probabilities of the form $\boldsymbol{P}(\tau < \infty, X \leq x, Y \leq y)$, partially under specific assumptions on the distribution of claim sizes, we refer to Dufresne and Gerber (1988a,b), Dickson (1992,1993).

In order to determine finite-horizont ruin probabilities $\boldsymbol{P}(\tau \leq t)$ for $t < \infty$, a different algorithm has been proposed in De Vylder and Goovaerts (1988). In that paper the original problem to determine the probability that the continuous-time risk process $\{Z_s\}_{s \geq 0}$ undercrosses the level zero before time t , has been replaced by finding the probability of non-ruin at the end of n consecutive periods of any given length. This idea, combined with using Panjer's recursion (1981), gives approximations of the ruin probabilities $\boldsymbol{P}(\tau \leq t)$. Later on, the work of De Vylder and Goovaerts (1988) was extended by Dickson and Waters (1991,1992) to approximate survival probabilities $1 - \boldsymbol{P}(\tau < \infty)$ in infinite time and probabilities $\boldsymbol{P}(\tau \leq t, Y \leq y)$ of ruin and its severity in finite time.

In the present paper, we first show that similar arguments which have been used in Gerber, Goovaerts and Kaas (1987) to obtain (4), can be used to prove an analogous result for $\boldsymbol{P}(\tau < \infty, X \leq x, Y \leq y)$, see Theorem 1. Next we derive a Taylor-series expansion for the finite-horizont probabilities $\boldsymbol{P}(\tau \leq t, X \leq x, Y \leq y)$ considering these probabilities as a function of the arrival intensity λ . This representation formula given in Theorem 2 suggests that, in the finite-horizont case, a variant of the algorithm proposed in Dickson (1989) could be used to approximate the probabilities $\boldsymbol{P}(\tau \leq t, X \leq x, Y \leq y)$. Using a result of Zazanis (1992) it is not difficult to show that, for each $t < \infty$, the function

$$f_{t,u,x,y}(\lambda) = \boldsymbol{P}(\tau \leq t, X \leq x, Y \leq y) \tag{5}$$

is analytic at $\lambda = 0$. Moreover, we can determine the (right) derivatives $f^{(n)}_{t,u,x,y}(\lambda)$ of the function $f_{t,u,x,y}(\lambda)$ at $\lambda = 0$ for an arbitrary order n using a general representation formula for a wide class of functionals of independently marked Poisson processes. First, for better readability, we additionally assume that the claim size distribution has an exponential tail. In this case, we can use the representation formula derived in Reiman and Simon (1989). Next we drop the additional assumption on the claim size distribution using a modified version of the representation formula of Reiman and Simon (1989). This modified representation formula was considered also in Kroese and Schmidt (1995) and goes back to an idea of Błaszczyszyn (1995). In this way, by the Taylor-series expansion

$$f_{t,u,x,y}(\lambda) = \sum_{n=0}^{\infty} \frac{f^{(n)}_{t,u,x,y}(0)}{n!}\,\lambda^n \quad , \lambda \geq 0, \tag{6}$$

of $f_{t,u,x,y}(\lambda)$ at $\lambda = 0$, we can approximate the probabilities $\boldsymbol{P}(\tau \leq t,\, X \leq x,\, Y \leq y)$ (see Theorem 2 for the precise formulation).

Clearly, the method to approximate the probabilities $\boldsymbol{P}(\tau \leq t,\, X \leq x,\, Y \leq y)$ via the Taylor-series expansion given in Theorem 2 works particularly well for relatively small ρ, i.e. in the cases of moderate and large premium loading factors θ. Then, for getting a high accuracy of the approximation, it suffices to consider in (6) a few summands only. On the other hand, diffusion approximations for the finite-time ruin probabilities $\boldsymbol{P}(\tau \leq t)$ have been discussed e.g. in Iglehart (1969), Grandell (1977), Asmussen (1984), Schmidli (1992). Most of these approximations are based on a limit theorem and give reasonable accuracy when $u \to \infty$, $t \to \infty$ or $\rho \to 1$. Further papers, where power-series expansions with respect to the arrival intensity λ have been investigated in order to approximate probability characteristics of stochastic models driven by a compound Poisson process, are e.g. Baccelli and Schmidt (1995), Hooghiemstra et al. (1988), Reiman and Simon (1988), Blanc (1990,1991), Simon (1993), Kroese and Schmidt (1995).

The present paper is organized as follows. In Section 2 the Taylor-series expansions for the probabilities $\boldsymbol{P}(\tau < \infty,\, X \leq x,\, Y \leq y)$ and $\boldsymbol{P}(\tau \leq t,\, X \leq x,\, Y \leq y)$ are given. Section 3 contains the proof of the latter result. Some examples are discussed in Section 4. They show impressively how much the improvement of approximation can be when higher-order terms of (6) are taken into consideration.

2 Main Results

First, for $t < \infty$, we represent the function $f_{t,u,x,y}(\lambda)$, defined in (5), in such a form that a result of Zazanis (1992) about analyticity of Poisson-driven systems can be used. Let $\mathcal{M}$ be the space of all (counting) measures on $I\!R_+ = [0,\infty)$ that take their values in the set $\overline{I\!N} = I\!N \cup \{\infty\}$ and are finite on bounded sets; $I\!N = \{0,1,\ldots\}$. By $\mathcal{B}(\mathcal{M})$ we denote the smallest σ-algebra of subsets of $\mathcal{M}$ such that the mappings $\varphi_B : \mathcal{M} \to \overline{I\!N}$ with $\varphi_B(\eta) \stackrel{def}{=} \eta(B)$ are measurable for all bounded $B \in \mathcal{B}(I\!R)$. Then, the realizations of the sequence $\{T_k\}$ can be identified with elements of $\mathcal{M}$. Let P^1_λ be the distribution of $\{T_k\}$ on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$. Furthermore, let $I\!R_+^\infty$ be the set of all sequences $u_1, u_2, \ldots$ of non-negative real numbers with the (product) σ-algebra $\mathcal{B}(I\!R_+^\infty)$. By P^2 we denote the distribution of $\{U_k\}$ on $(I\!R_+^\infty, \mathcal{B}(I\!R_+^\infty))$.

Our basic probability space will be $(\Omega, \mathcal{F}, \boldsymbol{P})$ with $\Omega = \mathcal{M} \times I\!R_+^\infty$, $\mathcal{F} = \mathcal{B}(\mathcal{M}) \otimes \mathcal{B}(I\!R_+^\infty)$ and $\boldsymbol{P} = P_\lambda^1 \times P^2$. For each $s \in I\!R_+$, let $\mathcal{B}(\mathcal{M}_s)$ denote the sub-σ-algebra of $\mathcal{B}(\mathcal{M})$ generated by the mappings $\varphi_B : \mathcal{M} \to \overline{I\!N}$ for all $B \in \mathcal{B}([0, s))$. Then, clearly, $\{N_t\}$ is a $(P_\lambda^1, \mathcal{B}(\mathcal{M}_s))$-Poisson process with intensity λ for all $\lambda \geq 0$. Furthermore, consider the filtration $\{\mathcal{F}_s\}_{s \geq 0}$ with $\mathcal{F}_s = \mathcal{B}(\mathcal{M}_s) \otimes \mathcal{B}(I\!R_+^\infty)$, and define the random variable

$$T \stackrel{def}{=} \min\{\tau, t\}. \tag{7}$$

It is easy to see that T is an $\{\mathcal{F}_t\}$-stopping time and that the mapping $\psi : \Omega \to \{0, 1\}$ with $\psi \stackrel{def}{=} \mathbf{1}_{\{\tau \leq t, X \leq x, Y \leq y\}}$ is measurable with respect to the σ-algebra $\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq s\} \in \mathcal{F}_s$ for every $s \geq 0\}$. Moreover, for each $t < \infty$, the stopping time T defined in (7) is bounded for all $\lambda \geq 0$. Thus, from the representation

$$f_{t,u,x,y}(\lambda) = \boldsymbol{E}\, \psi, \tag{8}$$

using Theorem 3 and Corollary 1 of Zazanis (1992), it follows that the function $f_{t,u,x,y}(\lambda)$ is analytic at $\lambda = 0$ and the Taylor-series expansion (6) has an infinite radius of convergence.

However, first we determine the (right) derivatives at $\lambda = 0$ of the infinite-horizont counterpart of $f_{t,u,x,y}(\lambda)$, i.e. we consider the function

$$f_{u,x,y}(\lambda) \stackrel{def}{=} \lim_{t \to \infty} f_{t,u,x,y}(\lambda) = \boldsymbol{P}(\tau < \infty, X \leq x, Y \leq y). \tag{9}$$

In order to show that $f_{u,x,y}(\lambda)$ is analytic at $\lambda = 0$ we can proceed in the following way. Using a similar argument as in Gerber, Goovaerts and Kaas (1987) we get that $f_{u,x,y}(\lambda)$ satisfies the following defective renewal equation:

$$f_{u,x,y}(\lambda) = \lambda F_{x,y}(u) + \lambda \int\limits_0^u f_{u-v,x,y}(\lambda)\, dG(v), \tag{10}$$

where $F_{x,y}(v) = \int\limits_v^{\max\{x,v\}} (B(s+y) - B(s))\, ds$ for $v \geq 0$ and $F_{x,y}(v) = 0$ for $v < 0$, $G(u) = \int\limits_0^u (1 - B(s))\, ds$.

Moreover, analogous to (3) and (4) we get

$$f_{u,x,y}(\lambda) = \sum_{n=1}^{\infty} \lambda^n F_{x,y} * G^{*(n-1)}(u), \tag{11}$$

where the law of total probability is used together with the fact that, for $0 < v < u$,

$$\lambda^n F_{x,y}(u-v)\, dG^{*(n-1)}(v)$$

is the infinitesimal probability that the nth record low of the risk process $\{Z_s\}_{s\geq 0}$ is between $u-v$ and $u-v+dv$ and that ruin occurs with the following record low such that the surplus just before the ruin and the deficit do not exceed the values x and y, respectively (see e.g. Miyazawa (1994), Asmussen and Schmidt (1995)). Thus, we arrived at the following result.

Theorem 1 *The function $f_{u,x,y}(\lambda)$ is analytic at $\lambda = 0$ and, for every $n \geq 1$, the nth (right) derivative $f^{(n)}_{u,x,y}(\lambda)$ at $\lambda = 0$ is given by*

$$\frac{f^{(n)}_{u,x,y}(0)}{n!} = F_{x,y} * G^{*(n-1)}(u). \tag{12}$$

Note that, for $x = \infty$, formula (12) is in accordance with (4), i.e. with formula (30) in Gerber et al. (1987). Moreover, for $x = y = \infty$, from (12) we get (3) which is equivalent to the well-known Pollaczek-Khinchin formula for the stationary waiting-time distribution in the M/G/1 queue (see e.g. Chapter 4 in Franken et al. (1982)).

Next, we give a recursion formula which shows how the derivatives $f^{(n)}_{t,u,x,y}(0)$ of $f_{t,u,x,y}(\lambda)$ can be obtained from those of the functions $f_{v,x,y}(\lambda)$.

Theorem 2 *For each $n \geq 1$, $0 \leq u, t < \infty$, $0 < x, y \leq \infty$, it holds*

$$\frac{f^{(n)}_{t,u,x,y}(0)}{n!} = \frac{f^{(n)}_{u,x,y}(0)}{n!} - \sum_{k=1}^{n} q^{(n-k,k)}_{t,u,x,y} \tag{13}$$

where the quantities $q^{(n,k)}_{t,u,x,y}$ are given recursively by

$$q^{(n,k)}_{t,u,x,y} = \int_0^t \left(\int_0^{u+s} q^{(n-1,k)}_{t-s,u+s-z,x,y}\, dB(z) - q^{(n-1,k)}_{t-s,u+s,x,y} \right) ds \tag{14}$$

and

$$q^{(0,k)}_{t,u,x,y} = \frac{f^{(k)}_{u+t,x,y}(0)}{k!}. \tag{15}$$

3 Proof of Theorem 2

In order to prove Theorem 2 we use a representation formula given in Reiman and Simon (1989) for the nth (right) derivatives $f^{(n)}_{t,u,x,y}(0)$. For better readability, we first use this formula to calculate the n th derivatives of $f_{u,x,y}(\lambda)$ at $\lambda = 0$, although they are already given by (12). For this purpose, we have to assume additionally that the distribution of claim sizes has an exponential tail, i.e. there are constants $0 < c < \infty$ and $\gamma > 0$ such that $c = \lim_{x\to\infty} e^{\gamma x}(1 - B(x))$.

The general representation formula which we will use, was derived in Reiman and Simon (1989) under a certain admissibility condition and it follows from this formula that the derivatives of $f_{t,u,x,y}(\lambda)$ at $\lambda = 0$ converge to those of $f_{u,x,y}(\lambda)$ at $\lambda = 0$ when $t \to \infty$. Thus, we have to show that the functional ψ defined in Section 2, is admissible in the sense of Reiman and Simon (1989). For any random variable $\psi : \Omega \to \mathbb{R}$ and for any $s > 0$ we consider the mapping $\psi_s : \Omega \to \mathbb{R}$ with $\psi_s(\omega) = \psi(\omega_s)$ where $\omega_s = (\eta(\cdot \cap [0, s)), \{u_k\})$ and $\omega = (\eta, \{u_k\})$. Thus, the outcome of ψ_s is equal to that of ψ when the arrival epochs σ_k in $[s, \infty)$ are not taken into consideration. Moreover, let $A_{(s_1,s_2]}(l) = \{\omega : N_{s_2}(\omega) - N_{s_1}(\omega) = l\}$ for any $0 < s_1 < s_2$, $l \in \mathbb{N}$. Following Reiman and Simon (1989), the random variable $\psi : \Omega \to \mathbb{R}$ is said to be *admissible* if there exist constants $0 < \alpha, \beta < \infty$, $1 < a < \infty$, and $\gamma^\star > 0$ such that for any $0 < s_1 < s_2$, $0 \le j, l < \infty$

$$\boldsymbol{E}\left(|\psi_{s_1} - \psi_{s_2}| \,\Big|\, A_{(0,s_1]}(j) \cap A_{(s_1,s_2]}(l)\right) \le \alpha(j+l)^\beta a^{j+l} e^{-\gamma^\star s_1}. \tag{16}$$

Lemma 1 *Assume that the claim size distribution has an exponential tail. Then, the functional $\psi = \psi_{t,u,x,y}$ introduced in Section 2, i.e. $\psi_{t,u,x,y} = \mathbf{1}_{\{\tau\le t,\, X\le x,\, Y\le y\}}$ is admissible for any $0 < t, x, y \le \infty$ and $0 \le u < \infty$.*

Proof: For $0 < s_1 < s_2$, consider the event $D_{s_1,s_2} \stackrel{def}{=} \left\{\omega : \sum_{i=1}^{N_{s_2}(\omega)} U_i(\omega) < s_1 + u\right\}$. Analogously to (a1) of Reiman and Simon (1989) we have

$$\begin{aligned}
&\boldsymbol{E}\left(|(\psi_{t,u,x,y})_{s_1} - (\psi_{t,u,x,y})_{s_2}| \,\Big|\, A_{(0,s_1]}(j) \cap A_{(s_1,s_2]}(l)\right) = \\
&\quad \boldsymbol{E}\left(|(\psi_{t,u,x,y})_{s_1} - (\psi_{t,u,x,y})_{s_2}| \,\Big|\, A_{(0,s_1]}(j) \cap A_{(s_1,s_2]}(l) \cap D_{s_1,s_2}\right) \\
&\quad \times \boldsymbol{P}\left(D_{s_1,s_2} \,\Big|\, A_{(0,s_1]}(j) \cap A_{(s_1,s_2]}(l)\right) \\
&\quad + \boldsymbol{E}\left(|(\psi_{t,u,x,y})_{s_1} - (\psi_{t,u,x,y})_{s_2}| \,\Big|\, A_{(0,s_1]}(j) \cap A_{(s_1,s_2]}(l) \cap D^C_{s_1,s_2}\right) \\
&\quad \times \boldsymbol{P}\left(D^C_{s_1,s_2} \,\Big|\, A_{(0,s_1]}(j) \cap A_{(s_1,s_2]}(l)\right).
\end{aligned}$$

From the definition of D_{s_1,s_2} we get that $(\psi_{t,u,x,y})_{s_1}(\omega) = (\psi_{t,u,x,y})_{s_2}(\omega) = 0$ for $\omega \in D_{s_1,s_2}$. On the other hand, $|(\psi_{t,u,x,y})_{s_1} - (\psi_{t,u,x,y})_{s_2}| \le 1$. Thus, we get

$$\begin{aligned}
&\boldsymbol{E}\left(|(\psi_{t,u,x,y})_{s_1} - (\psi_{t,u,x,y})_{s_2}| \,\Big|\, A_{(0,s_1]}(j) \cap A_{(s_1,s_2]}(l)\right) \\
&\le \quad \boldsymbol{P}\left(D^C_{s_1,s_2} \,\Big|\, A_{(0,s_1]}(j) \cap A_{(s_1,s_2]}(l)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{P}\left(\sum_{i=1}^{j+l} U_i > s_1 + u\right) \\
&\leq \boldsymbol{P}\left(\exp\left[\sum_{i=1}^{j+1} \gamma^\star U_i\right] \geq \mathrm{e}^{\gamma^\star s_1}\right) \\
&\leq \left(\int_0^\infty \mathrm{e}^{\gamma^\star v}\, dB(v)\right)^{j+l} \mathrm{e}^{-\gamma^\star s_1}.
\end{aligned}
$$

This finishes the proof because $\int_0^\infty \mathrm{e}^{\gamma^\star v}\, dB(v) < \infty$ for $0 < \gamma^\star < \gamma$. □

For the moment, we return to a general random variable $\psi : \Omega \to I\!R$. However, throughout this paper we assume that the value of $\psi(\omega)$ depends on the finite sequence $u_1, \ldots, u_n$ only provided that the counting measure η in $\omega = (\eta, \{u_k\})$ has only finitely many atoms $t_1, \ldots, t_n$ with $t_1 \leq \ldots \leq t_n$, i.e. $\eta = \sum_{i=1}^n \delta_{t_i}$. In this case, we therefore write $\psi(\omega) \equiv \psi(\{(t_1, u_1), \ldots, (t_n, u_n)\})$. Moreover, for each $s < \infty$ and $\omega = (\eta, \{u_k\}) \in \Omega$ we identify the restriction ω_s with the sequence $\{(t_1, u_1), \ldots, (t_{n(s)}, u_{n(s)})\}$ where $\eta(\cdot \cap [0, s)) = \sum_{i=1}^{n(s)} \delta_{t_i}$ and $t_1 \leq t_2 \leq \ldots \leq t_{n(s)}$.

Let $\left\{\binom{n}{j}\right\}$ be the set of j-tuples chosen from $\{1, 2, \ldots, n\}$, and for $\pi = \{i_1, i_2, \ldots, i_k\} \in \left\{\binom{n}{k}\right\}$, let $\boldsymbol{t}_\pi = \{(t_{i_1}, u_{i_1}), \ldots, (t_{i_k}, u_{i_k})\}$. Furthermore, we define the function

$$
\psi^*(\{(t_1, u_1), \ldots, (t_n, u_n)\}) = \sum_{j=1}^n (-1)^{n-j} \sum_{\pi \in \left\{\binom{n}{j}\right\}} \psi(\boldsymbol{t}_\pi). \tag{17}
$$

For any $\omega \in \Omega$, $(t, u) \in (I\!R_+)^2$, let

$$
\psi_{(t,u)}(\omega) \stackrel{def}{=} \psi(\omega_t \cup \{(t, u)\}) - \psi(\omega_t) \tag{18}
$$

and, for any $t_1, \ldots, t_n \in I\!R_+$, $u_1, \ldots, u_n \in I\!R_+$, let

$$
\psi_{(t_1,u_1),\ldots,(t_n,u_n)}(\omega) \stackrel{def}{=} \Big(\ldots (\psi_{(t_n,u_n)})_{(t_{n-1},u_{n-1})} \ldots\Big)_{(t_1,u_1)} (\omega). \tag{19}
$$

Then, by induction with respect to n, it is easy to see that

$$
\psi_{(t_1,u_1),\ldots,(t_n,u_n)}(\emptyset) = \begin{cases} \psi^*(\{(t_1, u_1), \ldots, (t_n, u_n)\}) & \text{for } t_1 < \ldots < t_n \\ 0 & \text{otherwise} \end{cases} \tag{20}
$$

Moreover, for any $\omega \in \Omega$ it holds

$$
\psi_{(t_1,u_1),\ldots,(t_n,u_n)}(\omega) = \begin{cases} \sum_{j=0}^n (-1)^{n-j} \sum_{\pi \in \left\{\binom{n}{j}\right\}} \psi(\omega_{t_1} \cup \boldsymbol{t}_\pi) & \text{for } t_1 < \ldots < t_n \\ 0 & \text{otherwise} \end{cases} \tag{21}
$$

In Reiman and Simon (1989) the following result has been proved concerning the (right) derivatives $f^{(n)}(0)$ of the function $f(\lambda) = \boldsymbol{E}\,\psi$.

Lemma 2 *If ψ is admissible, then $f(0)=\psi(\emptyset)$ and for $n \geq 1$,*

$$f^{(n)}(0) = \int\limits_{(\mathbb{R}_+)^{2n}} \psi^*\left(\{(t_1,u_1),\ldots,(t_n,u_n)\}\right)\, dt_n dB(u_n)\cdots dt_1 dB(u_1), \tag{22}$$

and, equivalently,

$$\frac{f^{(n)}(0)}{n!} = \int\limits_0^\infty \int\limits_{t_1}^\infty \cdots \int\limits_{t_{n-1}}^\infty \int\limits_{(\mathbb{R}_+)^n} \psi^*(\{(t_1,u_1),\ldots,(t_n,u_n)\})\, dB(u_n)\cdots dB(u_1)\, dt_n \cdots dt_2\, dt_1 \tag{23}$$

Applying the representation formulas (22),(23) to the functional $\psi = \psi_{t,u,x,y}$ introduced in Section 2, the following auxilliary result is useful.

Lemma 3 *For $n \geq 2$ and $t_1 < \ldots < t_n$ it holds*

$$\begin{aligned}&\psi^*_{t,u,x,y}(\{(t_1,u_1),\ldots,(t_n,u_n)\}) \\ &= \begin{cases} \psi^*_{t,u-u_1,x,y}(\{(t_2,u_2),\ldots,(t_n,u_n)\}) \\ \quad -\psi^*_{t,u,x,y}(\{(t_2,u_2),\ldots,(t_n,u_n)\}) & \text{for } u_1 \leq u+t_1 \\ \\ -\psi^*_{t,u,x,y}(\{(t_2,u_2),\ldots,(t_n,u_n)\}) & \text{for } u_1 > u+t_1 \end{cases}\end{aligned} \tag{24}$$

Proof: From (18) - (20) we get that

$$\begin{aligned}&\psi^*_{t,u,x,y}(\{(t_1,u_1),\ldots,(t_n,u_n)\}) \\ &= (\psi_{t,u,x,y})_{(t_2,u_2),\ldots,(t_n,u_n)}(\{(t_1,u_1)\}) - \psi^*_{t,u,x,y}(\{(t_2,u_2),\ldots,(t_n,u_n)\}).\end{aligned}$$

Now, from (17) and (21) we get that

$$(\psi_{t,u,x,y})_{(t_2,u_2),\ldots,(t_n,u_n)}(\{(t_1,u_1)\}) = \psi^*_{t,u-u_1,x,y}(\{(t_2,u_2),\ldots,(t_n,u_n)\})$$

for $u_1 \leq u+t_1$. On the other hand, if $u_1 > u+t_1$, we have either $\psi_{t,u,x,y}(\{(t_1,u_1)\} \cup \boldsymbol{t}_\pi) = 0$ for all $\pi \in \left\{\binom{n-1}{j}\right\}$ or $\psi_{t,u,x,y}(\{(t_1,u_1)\} \cup \boldsymbol{t}_\pi) = 1$ for all $\pi \in \left\{\binom{n-1}{j}\right\}$. By (21), this gives $(\psi_{t,u,x,y})_{(t_2,u_2),\ldots,(t_n,u_n)}(\{(t_1,u_1)\}) = 0$ for $u_1 > u+t_1$, and (24) follows. □

Now, we are ready to prove Theorem 2, still under the additional assumption that the claim size distribution has an exponential tail. For simplicity, we first consider the infinite-horizont probabilities $f_{u,x,y}(\lambda)$. From Lemma 1 it follows that we can use (23) in order to determine the derivatives $f^{(n)}_{u,x,y}(0)$ of the function $f_{u,x,y}(\lambda)$ defined in (9). These derivatives were already given by (12) without any assumption on the claim size distribution B, but to make the reader more confident with (23) we will calculate them again. Together with (17), for $n = 1$ this gives

$$\begin{aligned} f^{(1)}_{u,x,y}(0) &= \int_0^\infty \int_0^\infty \mathbf{1}_{[0,x-u]}(t_1)\, \mathbf{1}_{(u+t_1,u+t_1+y]}(u_1)\; dt_1\; dB(u_1) \\ &= \int_u^{\max\{x,u\}} (B(s+y) - B(s))\; ds \\ &= F_{x,y}(u). \end{aligned}$$

Thus, using (23) and (24), by induction with respect to n, we have

$$\begin{aligned} &\frac{f^{(n)}_{u,x,y}(0)}{n!} \\ &= \int_0^\infty \left(\int_0^{u+t_1} \frac{f^{(n-1)}_{u+t_1-u_1,x,y}(0)}{(n-1)!}\; dB(u_1) \;-\; \frac{f^{(n-1)}_{u+t_1,x,y}(0)}{(n-1)!} \right) dt_1 \\ &= \int_0^\infty \left(\int_0^{u+t_1} F_{x,y} * G^{*(n-2)}(u+t_1-u_1)\; dB(u_1) \;-\; F_{x,y} * G^{*(n-2)}(u+t_1) \right) dt_1 \\ &= \int_u^\infty \left(\int_0^{z} F_{x,y} * G^{*(n-2)}(z-s)\; dB(s) \;-\; F_{x,y} * G^{*(n-2)}(z) \right) dz \\ &= \int_u^\infty \int_0^\infty \Big(F_{x,y} * G^{*(n-2)}(z-s) \;-\; F_{x,y} * G^{*(n-2)}(z) \Big)\; dB(s)\; dz \\ &= \int_u^\infty \Big(F_{x,y} * G^{*(n-2)} * B(z) \;-\; F_{x,y} * G^{*(n-2)}(z) \Big)\; dz \\ &= F_{x,y} * G^{*(n-1)}(u). \end{aligned}$$

Now, let us consider the finite-horizont probabilities $f_{t,u,x,y}(\lambda)$. Observe that, similar as above, from (23) and (24) we have

$$\frac{f^{(n)}_{t,u,x,y}(0)}{n!} = \int_0^t \left(\int_0^{u+t_1} \frac{f^{(n-1)}_{t-t_1,u+t_1-u_1,x,y}(0)}{(n-1)!}\; dB(u_1) \;-\; \frac{f^{(n-1)}_{t-t_1,u+t_1,x,y}(0)}{(n-1)!} \right) dt_1 \qquad (25)$$

for every $n \geq 2$. Besides this, from (17) and (22) it follows that

$$f^{(1)}_{t,u,x,y}(0) \;=\; \int_0^t \int_0^\infty \mathbf{1}_{[0,x-u]}(t_1)\, \mathbf{1}_{(u+t_1,u+t_1+y]}(u_1)\; dB(u_1)\; dt_1$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty \mathbf{1}_{[0,x-u]}(t_1)\, \mathbf{1}_{(u+t_1,u+t_1+y]}(u_1)\; dB(u_1)\; dt_1 \\
&\quad - \int_0^\infty \int_0^\infty \mathbf{1}_{[0,x-u]}(t_1+t)\, \mathbf{1}_{(u+t+t_1,u+t+t_1+y]}(u_1)\; dB(u_1)\; dt_1 \\
&= f^{(1)}_{u,x,y}(0) - f^{(1)}_{u+t,x,y}(0) \\
&= f^{(1)}_{u,x,y}(0) - q^{(0,1)}_{t,u,x,y}.
\end{aligned}
$$

Thus, by (25) and by induction with respect to n, we have

$$
\begin{aligned}
\frac{f^{(n)}_{t,u,x,y}(0)}{n!} &= \int_0^t \left(\int_0^{u+t_1} \frac{f^{(n-1)}_{t-t_1,u+t_1-u_1,x,y}(0)}{(n-1)!}\, dB(u_1) \;-\; \frac{f^{(n-1)}_{t-t_1,u+t_1,x,y}(0)}{(n-1)!} \right) dt_1 \\
&= \int_0^t \left(\int_0^{u+t_1} \frac{f^{(n-1)}_{u+t_1-u_1,x,y}(0)}{(n-1)!}\, dB(u_1) \;-\; \frac{f^{(n-1)}_{u+t_1,x,y}(0)}{(n-1)!} \right) dt_1 \\
&\quad - \sum_{k=1}^{n-1} \int_0^t \left(\int_0^{u+t_1} q^{(n-1-k,k)}_{t-t_1,u+t_1-u_1,x,y}\, dB(u_1) \;-\; q^{(n-1-k,k)}_{t-t_1,u+t_1,x,y} \right) dt_1 \\
&= \int_0^t \left(\int_0^{u+t_1} \frac{f^{(n-1)}_{u+t_1-u_1,x,y}(0)}{(n-1)!}\, dB(u_1) \;-\; \frac{f^{(n-1)}_{u+t_1,x,y}(0)}{(n-1)!} \right) dt_1 \;-\; \sum_{k=1}^{n-1} q^{(n-k,k)}_{t,u,x,y} \\
&= \int_0^\infty \left\{ \int_0^{u+t_1} \frac{f^{(n-1)}_{u+t_1-u_1,x,y}(0)}{(n-1)!}\, dB(u_1) \;-\; \frac{f^{(n-1)}_{u+t_1,x,y}(0)}{(n-1)!} \right\} dt_1 \\
&\quad - \int_0^\infty \left\{ \int_0^{u+t+t_1} \frac{f^{(n-1)}_{u+t+t_1-u_1,x,y}(0)}{(n-1)!}\, dB(u_1) \;-\; \frac{f^{(n-1)}_{u+t+t_1,x,y}(0)}{(n-1)!} \right\} dt_1 \;-\; \sum_{k=1}^{n-1} q^{(n-k,k)}_{t,u,x,y} \\
&= \frac{f^{(n)}_{u,x,y}(0)}{n!} - \frac{f^{(n)}_{u+t,x,y}(0)}{n!} - \sum_{k=1}^{n-1} q^{(n-k,k)}_{t,u,x,y} \\
&= \frac{f^{(n)}_{u,x,y}(0)}{n!} - \sum_{k=1}^{n} q^{(n-k,k)}_{t,u,x,y}
\end{aligned}
$$

This completes the proof of Theorem 2 in the case of an exponentially tailed claim size distribution. □

Finally, we drop this additional assumption. Then, in order to determine the derivatives $f^{(n)}_{t,u,x,y}(0)$, we will use the following Lemma which is a simple extension of Corollary 4.3 of Błaszczyszyn (1995) to the case of an independently marked Poisson process (see also Kroese and Schmidt (1995)).

Lemma 4 *Let ψ be a continuous at infinities functional of a stationary independently marked Poisson process (i.e. $\lim_{t\to-\infty}\psi(\omega_t \cup \omega') = \psi(\omega')$, $\lim_{t\to\infty}\psi(\omega_t) = \psi(\omega)$ $\forall \omega, \omega' \in \Omega$, such that ω' has only finitely many atoms). If*

$$\int\limits_{\mathbb{R}^{2i}} \int\limits_{\Omega} |\psi_{(t_1,u_1),\ldots,(t_i,u_i)}(\omega)| \; \boldsymbol{P}(d\omega)\, dt_1\, dB(u_1) \cdots dt_i\, dB(u_i) < \infty \tag{26}$$

for all $i = 1, \ldots, n+1$ then

$$\begin{aligned} \boldsymbol{E}\psi &= \psi(\emptyset) + \sum_{i=1}^{n} \lambda^i \int\limits_{\mathbb{R}^{2i}} \psi_{(t_1,u_1),\ldots,(t_i,u_i)}(\emptyset)\, dt_1\, dB(u_1) \cdots dt_i\, dB(u_i) \\ &\quad + \lambda^{n+1} \int\limits_{\mathbb{R}^{2(n+1)}} \int\limits_{\Omega} |\psi_{(t_1,u_1),\ldots,(t_{n+1},u_{n+1})}(\omega)|\, \boldsymbol{P}(d\omega)\, dt_1\, dB(u_1) \cdots dt_{n+1}\, dB(u_{n+1}). \end{aligned}$$

In our case, i.e. for $\psi = \psi_{t,u,x,y}$, the integrability condition (26) holds, since

$$\begin{aligned} \left|\psi_{(t_1,u_1),\ldots,(t_i,u_i)}(\omega)\right| &= \left|\sum_{j=1}^{i} (-1)^{i-j} \sum_{\pi \in \{\binom{\{1,\ldots,i-1\}}{j-1}\}} \psi(\omega_{t_1} \cup \mathbf{t}_\pi \cup \{(t_i,u_i)\}) - \psi(\omega_{t_1} \cup \mathbf{t}_\pi)\right| \\ &\le 2^i\, \mathbf{1}_{\{t_i \le t\}} \end{aligned}$$

for $t_1 < \ldots < t_i$ and $\psi_{(t_1,u_1),\ldots,(t_i,u_i)}(\omega) = 0$ otherwise. Moreover, note that the remainder term is bounded by $(2\lambda t)^{n+1}$. Now, using the same arguments as above one gets that $\int\limits_{\mathbb{R}^{2i}} \psi_{(t_1 u_1),\ldots,(t_i,u_i)}(\emptyset)\, dt_1\, dB(u_1) \cdots dt_i\, dB(u_i) = \frac{f^{(i)}_{t,u,x,y}(0)}{i!}$ given by Theorem 2.
□

4 Examples

Using the formulas developed in the preceding sections of this paper, we calculate explicite expressions for $f^{(n)}_{t,u,\infty,y}(0)$ $(n = 1,2,3)$ and for $f^{(n)}_{t,u,x,y}(0)$ $(n = 1,2)$ in the case where the claim sizes U_k are exponentially distributed. Next, we consider the case where the U_k are Erlang distributed and illustrate by some numerical examples how the algorithm works which has been given in Theorem 2.

Example 1 Assume that the claim sizes are exponentially distributed with parameter μ, i.e. $1 - B(x) = \mathrm{e}^{-\mu x}$. Using Theorem 2 one can get closed formulas for the derivatives $f^{(n)}_{t,u,\infty,y}(\lambda)$ at $\lambda = 0$, e.g.

$$f^{(1)}_{t,u,\infty,y}(0) = \frac{\mathrm{e}^{-\mu u}}{\mu}(1 - \mathrm{e}^{-\mu y})(1 - \mathrm{e}^{-\mu t})$$

$$
\begin{aligned}
\frac{f^{(2)}_{t,u,\infty,y}(0)}{2!} &= \frac{\mathrm{e}^{-\mu u}}{\mu}\,(1-\mathrm{e}^{-\mu y})\Big(u-\mathrm{e}^{-\mu t}(u+\mu t u+\frac{\mu t^2}{2})\Big) \\
\frac{f^{(3)}_{t,u,\infty,y}(0)}{3!} &= \frac{\mathrm{e}^{-\mu u}}{12\mu}\,(1-\mathrm{e}^{-\mu y})\ \times \\
&\quad \Big(6u^2-\mathrm{e}^{-\mu t}\Big(6u^2-2\mu t^3+6\mu t u^2+4\mu^2 t^3 u+\mu^2 t^4+3\mu^2 t^2 u^2\Big)\Big)
\end{aligned}
$$

More general, for $f^{(n)}_{t,u,x,y}(0)$ one gets

$$
\begin{aligned}
f^{(1)}_{t,u,x,y}(0) &= \frac{\mathrm{e}^{-\mu u}}{\mu}\,(1-\mathrm{e}^{-\mu y})\Big[1-\mathrm{e}^{-\mu(x-u)^+}-\mathrm{e}^{-\mu t}(1-\mathrm{e}^{-\mu(x-u-t)^+})\Big] \\
\frac{f^{(2)}_{t,u,x,y}(0)}{2!} &= \frac{\mathrm{e}^{-\mu u}}{\mu}\,(1-\mathrm{e}^{-\mu y})\Big\{\min\{u,x\}+\frac{1}{\mu}(\mathrm{e}^{-\mu x}-\mathrm{e}^{-\mu(x-u)^+})\ - \\
&\quad -\mathrm{e}^{-\mu t}\Big[\min\{u+t,x\}+\frac{1}{\mu}(\mathrm{e}^{-\mu x}-\mathrm{e}^{-\mu(x-u-t)^+})\Big]\ - \\
&\quad -\mathrm{e}^{-\mu t}\Big[\frac{\mu\Big(t^2-((t-x)^+)^2\Big)}{2}+\mu\min\{t,x\}\min\{u,x-t\}\ - \\
&\quad -\ \min\{t,x\}\mathrm{e}^{-\mu(x-u-t)^+}+\frac{1}{\mu}\mathrm{e}^{-\mu(x-t)^+}-\frac{\mathrm{e}^{-\mu x}}{\mu}-t(1-\mathrm{e}^{-\mu(x-u-t)^+})\Big]\Big\},
\end{aligned}
$$

where x^+ denotes the maximum of x and 0.

Example 2 Consider $f_{t,u,x,y}(\lambda)$ and let the claim sizes be 2-order Erlang distributed with expectation 1. Figure 1 shows the simulated and approximated values of $f_{1,0,1,1}(\lambda)$ for different values of λ, where $s(\lambda)$ are the simulated values and $a_1(\lambda), a_2(\lambda), a_3(\lambda)$ are approximations of order 1, 2 and 3 respectively, i.e.

$$
a_k(\lambda)=\sum_{i=1}^{k}\frac{f^{(i)}_{t,u,x,y}(0)}{i!}\lambda^i, \qquad k=1,2,3.
$$

It can be seen that in the case $k=3$ the differences between the simulated and approximated values are rather small, even for large values of λ.

However, one has to be careful. Expansion up to the third-order term does not always give satisfactory approximations, at least not uniformly in λ. Figure 2 shows the simulated and approximated values of $f_{2,2,3,2}(\lambda)$ for different values of λ. In this case it looks like the approximation of order 2 is better than the one of order 3. But, after changing the scale, Figure 3 shows that for small values of λ approximation $a_3(\lambda)$ is better than $a_2(\lambda)$. For small and moderate values of λ both approximations are rather good.

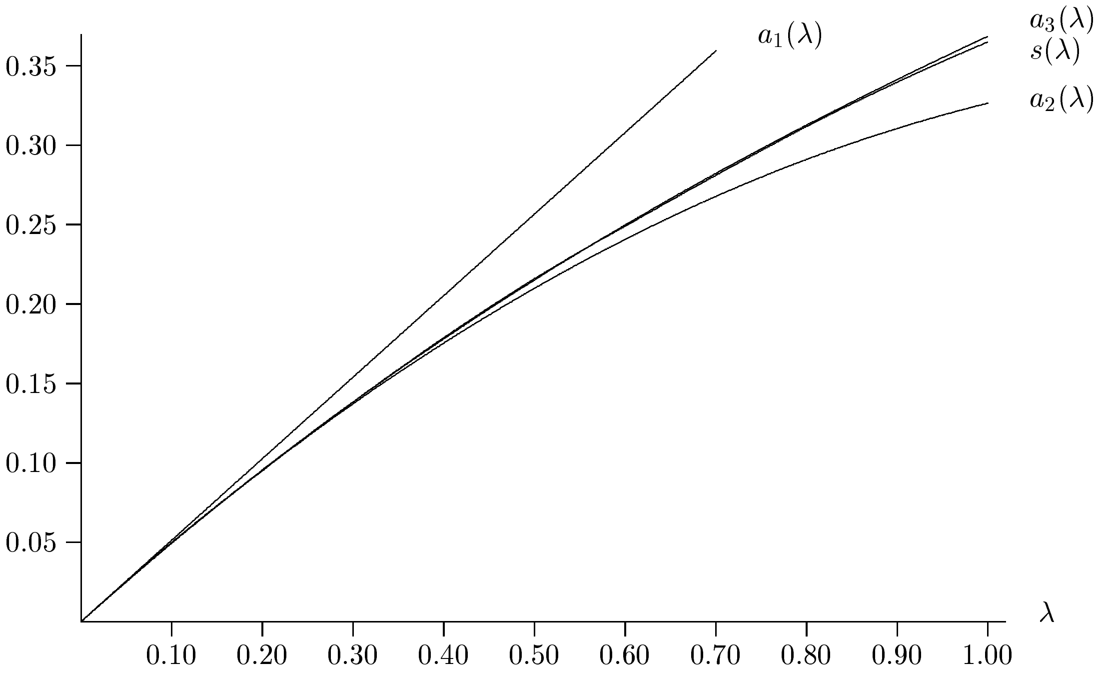

Figure 1: $f_{1,0,1,1}(\lambda)$ *where* $U_k \sim \mathrm{Erl}(2, 0.5)$.

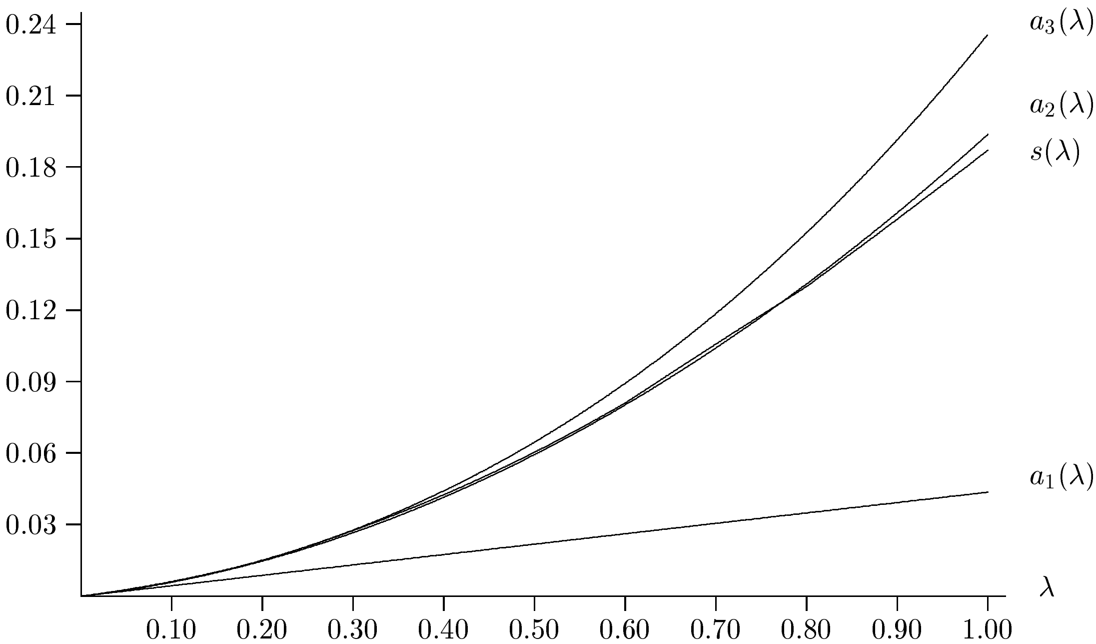

Figure 2: $f_{2,2,3,2}(\lambda)$ *where* $U_k \sim \mathrm{Erl}(2, 0.5)$.

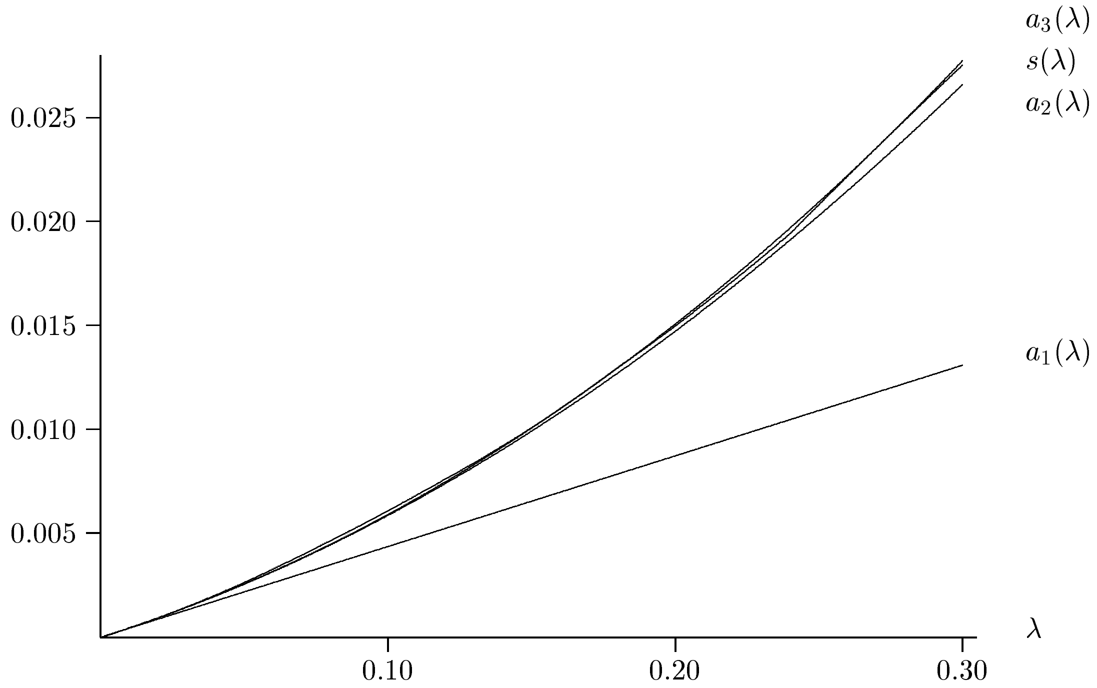

Figure 3: $f_{2,2,3,2}(\lambda)$ *where* $U_k \sim \mathrm{Erl}(2, 0.5)$.

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