A Note on an M/GI/1/N Queue with Vacation Time and Exhaustive Service Discipline

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Abstract

We consider an M/GI/1/N queue with vacation time and exhaustive service discipline, the same queue as in Lee [9]. Focusing only on the service completion epochs (as oposed to both service completion epochs and vacation completion epochs by Lee), we present a simple analysis for the queue length distribution at an arbitrary time as well as for the waiting time distribution.
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1 Introduction

Communication engineers frequently encounter a queueing situation in the time division multiple access (TDMA) environment (see Stuck and Arthurs [12]). Modelling the situation leads to a single-server finite capacity queue with *vacation time* and *exhaustive* service discipline. By *vacation time*, we mean that the server becomes unavailable for occasional intervals of time, and by *exhaustive* we mean that customers are served continuously until there is no customer in the system (see Doshi [5], and Takagi [13]). The vacation time corresponds to a constant slotted time period in the TDMA system.

Under the exhaustive vacation policy, the well known stochastic decomposition formula (see Doshi [5], Fuhrmann and Cooper [6], Kroese and Schmidt [8] and Miyazawa [10]) will be useful for an infinite capacity queue. However, if we would like to evaluate the loss probability we have to treat a finite capacity system rather than infinte capacity systems.

Assuming Poisson input and a finite capacity queue Lee [9] already provided a numerical algorithm for this system via the standard embedded Markov chain technique. As the embedded points, he took the service completion epochs and server vacation completion epochs. To obtain the queue length distribution at an arbitrary time, he applied the supplementary variable technique and the sample biasing technique.

Here, we treat the same queueing system as in Lee [9] but present a simpler analysis than Lee's. We show that service completion epochs are enough for the queue length to form an embedded Markov chain (server vacation completion epochs are redundant). Combining a couple of simple qualitative results, we straightforwardly obtain the queue length distribution at an arbitrary time from which engineers can evaluate their most interesting performance measures. From the queue length distribution we obtain the Laplace-Stieltjes transform (LST) of the waiting time distribution.

2 The model

We consider an M/GI/1/N queue, where the input process is Poissonian with rate λ, the service times form a sequence of i.i.d. random variables with distribution function $S(x)$ and N equals the number of waiting places in the queue, including the space for the customer that may be in service. We assume that accepted customers by the system are served by a single server exhaustively, i.e. the server serves the queue continuously until the queue is empty, where a customer is accepted by the system if the number of customers in the system is less than N. Whenever the queue becomes empty the server starts a vacation with distribution function $V(x)$. If the queue is still empty upon his return, he takes another independent vacation with distribution function $V(x)$. We assume furtheron that the

service discipline is non-preemptive and the service order is FIFO.
By g_j and h_j we denote the probabilities that j customers arrive during a service time S and a vacation time V, respectively. Hence

$$g_j = \int_0^\infty \frac{(\lambda x)^j}{j!} e^{-\lambda x} \, dS(x) \tag{1}$$

$$j = 0, 1, \ldots$$

$$h_j = \int_0^\infty \frac{(\lambda x)^j}{j!} e^{-\lambda x} \, dV(x). \tag{2}$$

The probability that j customers arrive (and are accepted) during an idle period (the server is on vacation) will be denoted by φ_j and is given by

$$\begin{aligned} \varphi_j &= \sum_{l=0}^{\infty} (h_0)^l h_j = \frac{h_j}{1 - h_0}, \qquad j = 1, \ldots, N-1 \\ \varphi_N &= \sum_{l=0}^{\infty} (h_0)^l h_N^c = \frac{h_N^c}{1 - h_0}, \end{aligned} \tag{3}$$

where

$$h_N^c = \sum_{j=N}^{\infty} h_j.$$

3 The queue length distribution at a departure epoch

By π_j, $j = 0, \ldots, N-1$, we denote the steady state probability that j customers are left in the system at a departure epoch of a customer, i.e. if $t_0, t_1, \ldots$ are the departure epochs of the customers and L_t is the number of customers in the system at time instant t, then

$$\pi_j = \lim_{m \to \infty} \mathbf{P}(L_{t_m} = j), \qquad j = 0, \ldots, N-1.$$

If we define

$$g_j^c = \sum_{i=j}^{\infty} g_i,$$

it is easy to see, that π_j satisfy the following equations

$$\begin{aligned} \pi_j &= \pi_0 \sum_{i=1}^{j+1} \varphi_i g_{j-i+1} + \sum_{i=1}^{j+1} \pi_i g_{j-i+1}, \qquad j = 0, \ldots, N-2 \\ \pi_{N-1} &= \pi_0 \sum_{i=1}^{N} \varphi_i g_{N-i}^c + \sum_{i=1}^{N-1} \pi_i g_{N-i}^c \end{aligned} \tag{4}$$

and the normalization condition

$$\sum_{j=0}^{N-1} \pi_j = 1. \tag{5}$$

From equations (4) and (5) we can now evaluate $\{\pi_j; j = 0, \ldots, N-1\}$ recursively.

Remark 3.1 *The above described departure-epoch embedded Markov chain was considered for infinite capacity queues in Cooper [3].*

Remark 3.2 *The probabilities π_j are different from the probabilities p_j given in Lee [9] where also vacation completion epochs are considered. The relationship is given by*

$$\pi_j = \frac{p_j}{\sum_{i=0}^{N-1} p_i}, \qquad j = 0, \ldots, N-1. \tag{6}$$

4 The queue length distribution at an arbitrary time in steady state

In this section we will derive the probabilities π_j^*, that there are j customers in the system at an arbitrary time in steady state ($j = 0, \ldots, N$).
Let ρ' be the probability that the server is busy, then the following lemma holds.

Lemma 4.1 *It holds that*

$$\rho' = \lambda(1 - \pi_N^*)\mathbf{E}(S), \tag{7}$$

where π_N^* is the probability that N customers are in the system, and $\mathbf{E}(S)$ is the expected service time.

Proof: We restrict ourselves to only the service facility (excluding the waiting room). By applying the Poisson Arrivals See Time Averages (PASTA) property (see Wolff [14]), we see that π_N^* is also the probability that N customers are in the system just before an arrival epoch. Hence the rate $\lambda(1 - \pi_N^*)$ is the arrival rate of customers accepted by the system, and it is also the throughput of the service facility. The mean number of customers in the service facility is equal to ρ'. Applying Little's law, we then obtain (7).

□

The following theorem will link π_N^* with π_0 which is the probability that no customer is left in the system at a departure epoch.

Theorem 4.1 *The probability that the server is busy (not on vacation) is given by*

$$\rho' = \frac{\mathbf{E}(S)(1 - h_0)}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1 - h_0)}, \tag{8}$$

where $\mathbf{E}(V)$ is the expected vacation time.

Proof: Let us consider two point processes, where the first one is formed by the beginning epochs of busy periods and the second one by the end epochs of busy periods. By λ_{be} and λ_{ee} we denote the rates (expected number of points in a unit time interval) of these two point processes, respectively. Obviously, $\lambda_{be} = \lambda_{ee}$ and it holds that

$$\lambda_{be} = \frac{(1-\rho')(1-h_0)}{\mathbf{E}(V)}, \tag{9}$$

$$\lambda_{ee} = \frac{\rho'\pi_0}{\mathbf{E}(S)}. \tag{10}$$

The first equation can be seen by noting that $(1-\rho')/\mathbf{E}(V)$ is the rate of the point process formed by the time instants where the server returns from a vacation and $1-h_0$ is the probability that the system is not empty anymore, which means that a busy period starts. The second equation can be seen by noting that $\rho'/\mathbf{E}(S)$ is the rate of the point process formed by the departure epochs of customers (i.e. service completion epochs) and π_0 is the probability that no customer is left in the system, which means that a busy period ends. By setting $\lambda_{be} = \lambda_{ee}$ the assertion follows.

□

By using Lemma 4.1 and Theorem 4.1 we can now calculate $(1-\pi_N^*)$, the probability that a customer is being accepted by the system,

$$1 - \pi_N^* = \frac{(1-h_0)\lambda^{-1}}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1-h_0)}. \tag{11}$$

Because of the PASTA property, we see that π_j^* is also the probability that there are j customers in the system just before an arrival. Thus, the generalized version of Burke's theorem (see Burke [1], Gebhardt [7], Cohen [2] and Cooper [4]) is applied to get

$$\pi_j = \frac{\pi_j^*}{1-\pi_N^*}, \qquad j = 0, \ldots, N-1. \tag{12}$$

Substituting (11) into (12) we obtain the following theorem.

Theorem 4.2 *The queue length distribution $\{\pi_j^*; j = 0, \ldots, N\}$ at an arbitrary time in steady state is obtained as*

$$\pi_j^* = \frac{\pi_j(1-h_0)\lambda^{-1}}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1-h_0)}, \qquad j = 0, \ldots, N-1 \tag{13}$$

$$\pi_N^* = 1 - \frac{(1-h_0)\lambda^{-1}}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1-h_0)}. \tag{14}$$

5 The waiting time distribution

In this section we will show how the above obtained result can be applied to obtain a simple formula for the Laplace-Stieltjes transform (LST) of the waiting time distribution.

The waiting time of an arriving customer depends on the number of customers in the system and on the remaining service (resp. vacation) time. Hence we will need the joint distribution of the forward recurrence service (vacation) time and the number of arrivals during the backward recurrence service (vacation) time. If we define by $\hat{S}$ ($\hat{V}$) the forward recurrence service (vacation) time and by $\mathcal{X}(\tilde{S})$ ($\mathcal{X}(\tilde{V})$) the number of arrivals during the backward recurrence service (vacation) time, the quantities

$$\begin{aligned} \tilde{g}_n(\theta) &= \mathbf{P}(\mathcal{X}(\tilde{S}) = n)\, \mathbf{E}(\mathrm{e}^{-\theta \hat{S}} \mid \mathcal{X}(\tilde{S}) = n) \\ \tilde{h}_n(\theta) &= \mathbf{P}(\mathcal{X}(\tilde{V}) = n)\, \mathbf{E}(\mathrm{e}^{-\theta \hat{V}} \mid \mathcal{X}(\tilde{V}) = n) \end{aligned} \tag{15}$$

were derived in the Appendix of Lee [9] as

$$\begin{aligned} \tilde{g}_n(\theta) &= \frac{1}{(\lambda-\theta)\mathbf{E}(S)} \left\{ S^*(\theta) \left(\frac{\lambda}{\lambda-\theta}\right)^n - \sum_{j=0}^{n} g_j \left(\frac{\lambda}{\lambda-\theta}\right)^{n-j} \right\} \\ \tilde{h}_n(\theta) &= \frac{1}{(\lambda-\theta)\mathbf{E}(V)} \left\{ V^*(\theta) \left(\frac{\lambda}{\lambda-\theta}\right)^n - \sum_{j=0}^{n} h_j \left(\frac{\lambda}{\lambda-\theta}\right)^{n-j} \right\} \end{aligned} \tag{16}$$

where $S^*(\theta)$ ($V^*(\theta)$) defines the LST of the service (vacation) time distribution.

Theorem 5.1 *The LST of the waiting time distribution $W^*(\theta)$ is given by*

$$\begin{aligned} W^*(\theta) &= S^*(\theta)^{N-1} \sum_{j=0}^{N-1} \pi_j \left(\frac{\lambda}{\lambda-\theta}\right)^{N-j} \\ &+ \lambda \frac{\left(1 - \left(\frac{\lambda}{\lambda-\theta}\right)^N (S^*(\theta))^N\right) \frac{\pi_0}{1-h_0} (V^*(\theta)-1)}{\lambda - \theta - \lambda S^*(\theta)}. \end{aligned} \tag{17}$$

Proof: An arriving customer sees the server either serving (with probability ρ') or on vacation (with probability $1-\rho'$). If the server is serving, the probability that the actual service epoch started with k customers is $\pi_k + \pi_0 \varphi_k$. Hence the LST of the waiting time distribution $W^*(\theta)$ is given by

$$W^*(\theta) = \frac{1}{1-\pi_N^*} \left\{ \sum_{j=1}^{N-1} \sum_{k=1}^{j} \rho'(\pi_k + \pi_0\varphi_k) \tilde{g}_{j-k}(\theta) \left(S^*(\theta)\right)^{j-1} \right.$$

$$+ \quad \sum_{j=0}^{N-1} (1-\rho')\tilde{h}_j(\theta) \left(S^*(\theta)\right)^j \Bigg\}. \tag{18}$$

Substituting (16) into (18) yields (17) after some algebraic manipulations.

□

Remark 5.1 *Via transform-free method, Niu and Cooper [11] derived the waiting time distribution in terms of* σ_k*. Here,* σ_k *is the stationary probability that there are* k *customers waiting in the queue immediately after a service-start epoch. Note that* σ_k *is given by*

$$\sigma_k = \pi_{k+1} + \pi_0 \varphi_{k+1}, \qquad k = 0, \ldots, N-1. \tag{19}$$

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