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PALM REPRESENTATION AND APPROXIMATION OF THE COVARIANCE OF RANDOM CLOSED SETS

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Abstract

The covariance $C(r)$, $r \geq 0$, of a stationary isotropic random closed set Ξ is typically complicated to evaluate. This is the reason that an exponential approximation formula for $C(r)$ has been widely used in the literature, which matches $C(0)$, $C^{(1)}(0)$, and in many cases also $\lim_{r\to\infty} C(r)$. However, for $0 < r < \infty$, the accuracy of this approximation is not very high in general. In the present paper, we derive representation formulas for the covariance $C(r)$ and its derivative $C^{(1)}(r)$ using Palm calculus, where $r \geq 0$ is arbitrary. As a consequence, an explicit expression is obtained for the second derivative $C^{(2)}(0)$. These results are then used to get a refined exponential approximation for $C(r)$, which additionally matches the second derivative $C^{(2)}(0)$.

Keywords: Stochastic geometry; Random closed set; Covariance; Palm calculus, Exponential approximation; Boolean model

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1. Introduction

The covariance of stationary random closed sets is a fundamental characteristic of this class of stochastic models. However, in many cases, the covariance $C(\mathbf{r}) = \mathbb{P}(o \in \Xi, \mathbf{r} \in \Xi)$, $\mathbf{r} \in \mathbb{R}^d$, of such a random set Ξ cannot be expressed by means of simple analytical formulas. This is the reason that for stationary isotropic random closed sets the following exponential approximation

$$C_a(r) = p(1-p)\,\mathrm{e}^{-ar} + p^2 \tag{1.1}$$

for $C(\mathbf{r}) = C(r\mathbf{u}) = C(r)$, $r > 0$, has been widely used in the literature, where $p = \mathbb{P}(o \in \Xi)$ is the volume fraction of Ξ, $\mathbf{u}$ is an arbitrary unit vector, and a is some positive constant; see, for example, the monographs by Molchanov (1997), Ohser and Mücklich (2000), and Stoyan, Kendall and Mecke (1995). Usually, the constant a is chosen in such a way that the (first-order) exponential approximation $C_a(r)$ matches $C^{(1)}(0)$. However, for $0 < r < \infty$, the accuracy of this approximation is not very high in general. In the present paper, we derive representation formulas for the covariance $C(r)$ and its derivative $C^{(1)}(r)$ using Palm calculus, where $r \geq 0$ is arbitrary. As a consequence, an explicit expression is obtained for the second derivative $C^{(2)}(0)$. This complements a well-known formula for the first derivative $C^{(1)}(0)$; see Stoyan,

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Kendall, and Mecke (1995), Section 6.2. These results are then used to get a refined (second-order) exponential approximation

$$C_{a,b}(r) = p(1-p)\,\mathrm{e}^{-(ar+br^2)} + p^2\,, \tag{1.2}$$

for $C(r)$, $r > 0$, which additionally matches the second derivative $C^{(2)}(0)$, where $a > 0$ is the same constant as in (1.1) and b is a constant which can be positive, zero, or negative. Notice that our results also complement formulas for the first and the second derivatives of the spherical contact distribution of stationary random closed sets, derived in [4, 5, 6].

2. Some Preliminaries

In Section 2.1 we recall some basic notions and results from stochastic geometry. More details on this can be found, for example, in [7, 8, 9, 10]. Furthermore, some elementary properties of stationary point processes in $\mathbb{R}^1$, which are used in the following, are summarized in Section 2.2. For more details, we refer e.g. to [1, 2].

2.1. Covariance of Linear Section

Let Ξ be an arbitrary stationary random closed set (RACS) in $\mathbb{R}^d$. Suppose that the *volume fraction*

$$p = \mathbb{P}(o \in \Xi) \tag{2.1}$$

of Ξ satisfies $0 < p < 1$. The *covariance* $C(\mathbf{r})$ of Ξ is defined as

$$C(\mathbf{r}) = \mathbb{P}(o \in \Xi, \mathbf{r} \in \Xi)\,, \tag{2.2}$$

where $o \in \mathbb{R}^d$ denotes the origin and $\mathbf{r} \in \mathbb{R}^d$ is an arbitrary d-dimensional vector. We additionally assume throughout this paper that Ξ is isotropic. Because of this, the covariance $C(\mathbf{r})$ depends only on the length r of the vector $\mathbf{r}$, i.e., $C(\mathbf{r}) = C(r\mathbf{u}) = C(r)$, where $\mathbf{u}$ is an arbitrary unit vector and $r \geq 0$.

In order to get a representation formula of the covariance $C(r)$ of Ξ, the *linear section* $\Xi \cap g$ is considered, where $g \in \mathbb{R}^d$ is an arbitrary oriented line through o. Suppose that the set $\partial\Xi \cap g$ is locally finite with probability 1. The stationarity of Ξ then implies that $\partial\Xi \cap g$ is a stationary point process on g. Suppose that it has finite positive intensity λ^*. By $\{X_n\}$ we denote the stationary point process of the left end-points of the intervals forming the connected components of $\Xi \cap g$, where the left end-points are put according to the orientation of g. Furthermore, two marks L_n and D_n are attached to each X_n by defining L_n to be the length of the corresponding connected component of $\Xi \cap g$ and D_n to be the length of the corresponding spacing, i.e., $D_n = X_{n+1} - (X_n + L_n)$. If Ξ is a germ-grain model, then L_n and D_n can be interpreted as chord lengths. The marked point process $\Phi = \{X_n, L_n, D_n\}$ also inherits stationarity from Ξ and has the intensity

$$\lambda_\Phi = \frac{\lambda^*}{2}\,. \tag{2.3}$$

Without loss of generality, g can (and will) be chosen the real line $g = \mathbb{R}$ and the RACS $\Xi \cap g$ will be viewed as a subset of $\mathbb{R}$. Its covariance $C(r)$, $r \geq 0$, which coincides with the covariance of Ξ, is equal to the length fraction of the eroded set

$$(\Xi \cap g) \ominus \{o, r\} = \{x \in \mathbb{R} : \{x, x+r\} \subset \Xi \cap g\},$$

that is $C(r) = \mathbb{P}(o \in \Xi \cap g \ominus \{o, r\}) = \mathbb{P}(\{x, x+r\} \subset \Xi \cap g)$ for each $x \in \mathbb{R}$. Notice that in the planar case, i.e., $d = 2$, we have

$$\lambda^* = \frac{2\ell}{\pi}\,, \tag{2.4}$$

where ℓ is the specific boundary length of Ξ. This is a classical stereological formula; see [9], Sections 9.5 and 11.2.

2.2. Palm Distribution and Campbell's Theorem

The points X_n of Φ will be numbered in the following way:

$$\ldots < X_{-1} < X_0 \le 0 < X_1 < \ldots\,,$$

i.e., X_0 is the largest point of Φ on the halfline $(-\infty, 0]$. Furthermore, let $\mathbb{P}_0$ denote the *Palm distribution* of Φ and $\mathbb{E}_0$ the expectation taken with respect to $\mathbb{P}_0$, where

$$\mathbb{P}_0(A) = \frac{\mathbb{E} \sum\limits_{i=-\infty}^{\infty} \mathbb{1}(X_i \in B)\, \mathbb{1}(\Phi - X_i \in A)}{\mathbb{E} \sum\limits_{i=-\infty}^{\infty} \mathbb{1}(X_i \in B)} \tag{2.5}$$

for each $A \in \mathcal{N}(\mathbb{R} \times [0,\infty)^2)$ and an arbitrary Borel set $B \in \mathcal{B}(\mathbb{R})$ of positive and finite Lebesgue measure. Here, for each i fixed, $\Phi - X_i$ denotes the shifted marked point process $\Phi - X_i = \{(X_n - X_i, L_n, D_n),\ n = \ldots, -1, 0, 1, \ldots\}$ and $\mathcal{N}(\mathbb{R} \times [0,\infty)^2)$ is the Borel-σ-algebra on the point process space $N(\mathbb{R} \times [0,\infty)^2)$. Then, $\mathbb{P}_0(0 \in \Phi) = 1$, i.e., $\mathbb{P}_0(X_0 = 0) = 1$, which motivates the interpretation of $\mathbb{P}_0$ as conditional distribution of Φ given that there is a point at the origin. Using (2.5), it can be easily seen that

$$\mathbb{E}_0 f(\Phi) = \frac{1}{\lambda_\Phi}\, \mathbb{E} \sum_{X_n \in (0,1)} f(\Phi - X_n) \tag{2.6}$$

for each Borel-measurable function $f : N(\mathbb{R} \times \mathbb{R}^2) \to [0, \infty)$. Equation (2.6) is usually called *Campbell's theorem*. Notice that (2.6) implies in particular that

$$\lambda_\Phi = \frac{1}{\mathbb{E}_0(X_1 - X_0)} = \frac{1}{\mathbb{E}_0(L_0 + D_0)} = \frac{1}{\mathbb{E}_0(X_{n+1} - X_n)}\,, \tag{2.7}$$

where the last equality follows from the well-known fact that the sequence $\{X_{n+1} - X_n\}$ of distances between consecutive points is stationary under the Palm distribution $\mathbb{P}_0$.

3. Representation Formulas

In this section, representation formulas for the covariance $C(r)$ and its derivative $C^{(1)}(r)$ will be derived in terms of the Palm distribution $\mathbb{P}_0$. These formulas lead to expressions for $C^{(1)}(0)$ and $C^{(2)}(0)$.

Theorem 3.1. *For each $r \ge 0$,*

$$C(r) = \frac{\mathbb{E}_0 \max\{0, L_0 - r\} + \sum\limits_{n=1}^{\infty} \mathbb{E}_0 |\{x \in \mathbb{R} : 0 \le x \le L_0,\ X_n \le x + r \le X_n + L_n\}|}{\mathbb{E}_0(L_0 + D_0)}\,. \tag{3.1}$$

Proof. We have

$$
\begin{aligned}
C(r) &= \mathbb{P}(\{o,r\} \subset \Xi \cap \mathbb{R}) \\
&= \int_0^1 \mathbb{P}(\{x, x+r\} \subset \Xi \cap \mathbb{R})\, dx \\
&= \mathbb{E} \int_0^1 \mathbb{1}(\{x, x+r\} \subset \Xi \cap \mathbb{R})\, dx \\
&= \mathbb{E} \sum_{X_n \in (0,1)} |\{x \in \mathbb{R} :\, X_n \le x \le X_n + L_n,\, x + r \in \Xi \cap \mathbb{R}\}|\,.
\end{aligned}
$$

Since $\mathbb{E}\sum_{X_n\in(0,1)}(L_n + D_n) = 1$, we can conclude that

$$
C(r) = \frac{\mathbb{E}\sum_{X_n\in(0,1)} |\{x \in \mathbb{R} :\, X_n \le x \le X_n + L_n,\, x + r \in \Xi \cap \mathbb{R}\}|}{\mathbb{E}\sum_{X_n\in(0,1)}(L_n + D_n)}\,.
$$

Using Campbell's theorem (2.6), this gives

$$
C(r) = \frac{\sum\limits_{n=0}^{\infty} \mathbb{E}_0|\{x \in \mathbb{R} : 0 \le x \le L_0,\, X_n \le x + r \le X_n + L_n\}|}{\mathbb{E}_0(L_0 + D_0)}\,,
$$

which immediately implies (3.1).

We now use the result of Theorem 3.1 in order to derive a representation formula for the derivative $C^{(1)}(r)$ of $C(r)$. For this purpose, we additionally assume that the intensity measure of $\{X_n\}$ under the Palm distribution $\mathbb{P}_0$ is *locally finite*, i.e.,

$$
\sum_{n=1}^{\infty} \mathbb{P}_0(X_n < r) < \infty \tag{3.2}
$$

for each $r \ge 0$. Notice that (3.2) holds, for example, for the Boolean model with isotropic compact and convex grains, because in this case the random variables $X_1 - X_0, X_2 - X_1, \ldots$ are independent and identically distributed under $\mathbb{P}_0$.

Theorem 3.2. *The derivative $C^{(1)}(r)$ exists for almost all $r \ge 0$, where*

$$
C^{(1)}(r) = \frac{-\,\mathbb{P}_0(L_0 > r) + \sum\limits_{n=1}^{\infty} \mathbb{P}_0(0 < r - (X_n - L_0) < \min\{L_0, L_n\})}{\mathbb{E}_0(L_0 + D_0)}\,. \tag{3.3}
$$

Proof. The right derivative of the first summand in the numerator of (3.1) gives

$$
\begin{aligned}
\frac{d^+}{dr}\mathbb{E}_0 \max\{0, L_0 - r\} &= \lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}_0 \max\{0, L_0 - (r+\varepsilon)\} - \mathbb{E}_0 \max\{0, L_0 - r\}}{\varepsilon} \\
&= \lim_{\varepsilon\downarrow 0} \frac{-\varepsilon\, \mathbb{P}_0(L_0 - r \ge \varepsilon) - \mathbb{E}_0\big((L_0 - r)\,\mathbb{1}(0 < L_0 - r \le \varepsilon)\big)}{\varepsilon} \\
&= -\,\mathbb{P}_0(L_0 > r)\,.
\end{aligned}
$$

Furthermore, for each $n \geq 1$, we have

$$\mathbb{E}_0|\{x \in \mathbb{R} : 0 \leq x \leq L_0,\ X_n \leq x+r \leq X_n+L_n\}| = \mathbb{E}_0 \min\bigl\{L_0, L_n, \max\{0, r-(X_n-L_0)\}\bigr\}$$

and, for each $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{E}_0 \min\bigl\{L_0, L_n, \max\{0, r+\varepsilon-(X_n-L_0)\}\bigr\} - \mathbb{E}_0 \min\bigl\{L_0, L_n, \max\{0, r-(X_n-L_0)\}\bigr\} \\
&\quad = \varepsilon\, \mathbb{P}_0\bigl(0 < r-(X_n-L_0),\ r-(X_n-L_0)+\varepsilon \leq \min\{L_0, L_n\}\bigr) \\
&\qquad + \mathbb{E}_0\bigl(\min\{L_0, L_n\} - \max\{0, r-(X_n-L_0)\}\bigr\} \\
&\qquad\qquad \mathbb{1}\bigl(r-(X_n-L_0) \leq \min\{L_0, L_n\} < r-(X_n-L_0)+\varepsilon\bigr)\bigr) \\
&\quad \leq 2\varepsilon\, \mathbb{P}_0(X_n - L_0 < r+\varepsilon)\,.
\end{aligned}$$

Condition (3.2) implies that for arbitrary fixed $r \geq 0$ and $\varepsilon_0 > 0$, the infinite sum $\sum_{n=1}^{\infty} \mathbb{P}_0(X_n - L_0 < r+\varepsilon)$ is uniformly bounded for all $\varepsilon \in [0, \varepsilon_0)$, because

$$\sum_{n=1}^{\infty} \mathbb{P}_0(X_n - L_0 < r+\varepsilon) \leq \sum_{n=1}^{\infty} \mathbb{P}_0(X_n - X_1 < r+\varepsilon_0) = 1 + \sum_{n=1}^{\infty} \mathbb{P}_0(X_n < r+\varepsilon_0) < \infty\,.$$

Thus, we can differentiate each summand of the infinite sum in the numerator of (3.1) separately, where the above computations imply that

$$\frac{d^+}{dr}\mathbb{E}_0 \min\bigl\{L_0, L_n, \max\{0, r-(X_n-L_0)\}\bigr\} = \mathbb{P}_0\bigl(0 < r-(X_n-L_0) < \min\{L_0, L_n\}\bigr)\,.$$

This and an analogous consideration for the left derivative gives (3.3).

Notice that, by the definition of the random variables L_0 and D_0, we have

$$\mathbb{P}_0(L_0 > 0) = \mathbb{P}_0(D_0 > 0) = 1\,, \tag{3.4}$$

which means in particular that the point process $\{X_n\}$ of the left end-points of coverage intervals has no multiple points under $\mathbb{P}_0$. Moreover, suppose that the random variables L_0 and D_0 have continuous densities $f_{L_0}(s)$ and $f_{D_0}(s)$ on $[0, \infty)$ under the Palm distribution $\mathbb{P}_0$, i.e., $\mathbb{P}_0(L_0 > r) = \int_r^\infty f_{L_0}(s)\, ds$ and $\mathbb{P}_0(D_0 > r) = \int_r^\infty f_{D_0}(s)\, ds$ for each $r \geq 0$. We further assume that

$$\lim_{r \downarrow 0} \frac{1}{r}\bigl(\mathbb{P}_0(L_0 + D_0 \leq r) + \mathbb{P}_0(D_0 + L_1 \leq r)\bigr) = 0\,. \tag{3.5}$$

Notice that these assumptions are mild technical conditions which are fulfilled in many applications. In particular, condition (3.5) holds for the Boolean model with isotropic compact convex grains, because in this case the random variables L_0, D_0, and L_1 are independent under $\mathbb{P}_0$.

Corollary 3.1. *Under the above conditions,*

$$C^{(1)}(0) = \frac{-1}{\mathbb{E}_0(L_0 + D_0)} \tag{3.6}$$

and

$$C^{(2)}(0) = \frac{f_{L_0}(0) + f_{D_0}(0)}{\mathbb{E}_0(L_0 + D_0)}\,. \tag{3.7}$$

Proof. Statement (3.6) immediately follows from (3.3) and (3.4), because condition (3.2) implies that

$$\lim_{r\downarrow 0}\sum_{n=1}^{\infty}\mathbb{P}_0\big(0<r-(X_n-L_0)<\min\{L_0,L_n\}\big)=0\,.$$

In order to show (3.7), notice that

$$\lim_{r\downarrow 0}\frac{-\,\mathbb{P}_0(L_0>r)+1}{r}=f_{L_0}(0)\,.$$

Furthermore,

$$\begin{aligned}
&\lim_{r\downarrow 0}\frac{\mathbb{P}_0(0<r-(X_1-L_0)<\min\{L_0,L_1\})}{r}\\
&=\ \lim_{r\downarrow 0}\frac{\mathbb{P}_0(D_0<r<D_0+\min\{L_0,L_1\})}{r}\\
&=\ \lim_{r\downarrow 0}\frac{\mathbb{P}_0(D_0<r)}{r}-\lim_{r\downarrow 0}\frac{\mathbb{P}_0(D_0<r,D_0+\min\{L_0,L_1\}\le r)}{r}\\
&=\ \lim_{r\downarrow 0}\frac{\mathbb{P}_0(D_0<r)}{r}=f_{D_0}(0)\,,
\end{aligned}$$

because (3.5) implies that

$$\lim_{r\downarrow 0}\frac{\mathbb{P}_0(D_0<r,D_0+\min\{L_0,L_1\}\le r)}{r}=0\,.$$

Since

$$\begin{aligned}
\sum_{n=2}^{\infty}\mathbb{P}_0(0<r-(X_n-L_0)<\min\{L_0,L_n\}) &\le \sum_{n=2}^{\infty}\mathbb{P}_0(X_n-L_0<r<X_n+L_n-L_0)\\
&\le \sum_{n=2}^{\infty}\mathbb{P}_0(X_n-L_0<r<X_{n+1}-L_0)\\
&= \mathbb{P}_0\Big(\bigcup_{n=2}^{\infty}\{X_n-L_0<r<X_{n+1}-L_0\}\Big)\\
&\le \mathbb{P}_0(X_2-L_0<r)\\
&\le \mathbb{P}_0(D_0+L_1<r)\,,
\end{aligned}$$

condition (3.5) also implies that

$$\lim_{r\downarrow 0}\frac{\sum\limits_{n=2}^{\infty}\mathbb{P}_0(0<r-(X_n-L_0)<\min\{L_0,L_n\})}{r}=0\,.$$

Thus, statement (3.7) follows.

We remark that formula (3.6) is well known and has a geometric interpretation in terms of the specific surface content of Ξ (see Section 6.2 of [9]), whereas formula (3.7) seems to be new. We also remark that the above considerations show a way how the third derivative $C^{(3)}(0)$ can be determined. Indeed, by analyzing the application of conditions (3.2) and (3.5) in the proofs of Theorem 3.2 and Corollary 3.1, sufficient conditions can be formulated for the formula

$$C^{(3)}(0) = \frac{f^{(1)}_{L_0}(0) + f^{(1)}_{D_0}(0) - f^{(1)}_{D_0+\min\{L_0,L_1\}}(0)}{\mathbb{E}_0(L_0 + D_0)} \tag{3.8}$$

to be valid. However, these conditions are rather technical. They are therefore omitted.

4. Refined Exponential Approximation

The function $C_a : [0,\infty) \to [0,1]$ with

$$C_a(r) = p(1-p)\,\mathrm{e}^{-ar} + p^2\,, \qquad r \geq 0\,, \tag{4.1}$$

is called an *exponential covariance* of Ξ, where $p = \mathbb{P}(o \in \Xi)$ and

$$a = \frac{1}{p(1-p)\mathbb{E}_0(L_0 + D_0)}\,. \tag{4.2}$$

This approximation for $C(r)$ has been widely used in the literature; see, for example, [7, 8, 9]. By (3.6), it is clear that the exponential covariance given in (4.1) matches $C(0)$ and $C^{(1)}(0)$. Since in many cases we have

$$\lim_{r\to\infty} C(r) = p^2\,, \tag{4.3}$$

$C_a(r)$ then matches also the asymptotics of $C(r)$ at infinity. However, for $0 < r < \infty$, the accuracy of the exponential approximation $C_a(r)$ is not very high in general. Using Corollary 3.1, a refined exponential approximation for $C(r)$ can be given. Indeed, consider the function $C_{a,b} : [0,\infty) \to [0,\infty)$ with

$$C_{a,b}(r) = p(1-p)\,\mathrm{e}^{-(ar+br^2)} + p^2\,, \qquad r \geq 0\,, \tag{4.4}$$

where a is given in (4.2) and

$$b = \frac{a}{2}\big(a - f_{L_0}(0) - f_{D_0}(0)\big)\,. \tag{4.5}$$

By (3.6) and (3.7), it is easily seen that $C_{a,b}(r)$ matches not only $C(0)$ and $C^{(1)}(0)$, but also $C^{(2)}(0)$. Notice that the advantage of the exponential covariance $C_a(r)$ of Ξ is that a can be estimated by a very stable stereological estimator of the specific surface content of Ξ; see Section 6.2 of [9]. In contrast, the parameter b, which additionally goes into the refined exponential approximation $C_{a,b}(r)$, is based on probability density estimates, which can be instable. To avoid this, large samples are needed.

Furthermore, to compare the covariance $C(r)$ with the exponential approximations $C_a(r)$ and $C_{a,b}(r)$ given in (4.1) and (4.4), we considered the planar Boolean model for different classes of compact and convex grains. Using, for example, discs or uniformly oriented rectangles as grains, it has been shown that $C_{a,b}(r)$ fits the covariance $C(r)$ much better than $C_a(r)$ does. However, it can not be guaranteed that the refined exponential approximation always improves the approximation. This was oberved, for example, for the planar Boolean model with Poisson polygons as grains, where the refined exponential approximation $C_{a,b}(r)$ is better than $C_a(r)$ for small and moderate r only, but not for large r.

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