Kernel Estimation of the Spectral Density of Stationary Random Closed Sets

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Abstract

A nonparametric kernel estimator of the spectral density of stationary random closed sets is studied. Conditions are derived under which this estimator is asymptotically unbiased and mean-square consistent. For the planar Boolean model with isotropic compact and convex grains, an averaged version of the kernel estimator is compared with the theoretical spectral density.

Keywords Stationary random set; spectral density; nonparametric kernel estimator; asymptotic unbiasdness; mean-square consistency; Boolean model

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1 Introduction

The problem of nonparametric estimation of the spectral density $f(x)$ of stationary random closed sets Ξ in $\mathbb{R}^d$ is studied, where Ξ is observed in a sampling window $W \subset \mathbb{R}^d$ with positive and finite volume $|W|$. Motivated by an approximation formula, which can be concluded from Lemma 2.2, that is

$$f(x) \approx \frac{1}{|W|}\, \mathbb{E} \left| \int_{\mathbb{R}^d} g(z)\, \mathrm{e}^{i\langle x,z\rangle}\, dz \right|^2 , \quad x \in \mathbb{R}^d ,$$

where $g(z) = \mathbb{1}_W(z)(\mathbb{1}_\Xi(z) - p)$ and $p = \mathbb{P}(o \in \Xi)$, the kernel estimator

$$\widehat{f}_n(x) = \frac{1}{|W_n|\, {b_n}^d} \int_{\mathbb{R}^d} k\left(\frac{x-y}{b_n}\right) \left| \int_{W_n} (\mathbb{1}_\Xi(z) - p)\, \mathrm{e}^{i\langle y,z\rangle}\, dz \right|^2 dy\,, \quad x \in \mathbb{R}^d ,$$

is considered for an unboundedly increasing sequence of observation windows W_n, for a certain kernel function k and bandwidths b_n.

Under some mild conditions, among them the absolute integrability of the covariance function $Cov(h) = \mathbb{P}(o \in \Xi, h \in \Xi) - p^2$, we show that $\widehat{f}_n(x)$ is an asymptotically unbiased and mean-square consistent estimator of $f(x)$; see Theorems 3.1 and 3.2.

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Notice that in the above definition of $\widehat{f}_n(x)$ we suppose that the volume fraction p is given. It can be shown that the statements of Theorems 3.1 and 3.2 remain true if in the definition of $\widehat{f}_n(x)$ the (hypothetical) volume fraction p is replaced by its unbiased estimator $\widehat{p}_n = |\Xi \cap W_n|/|W_n|$. However, a rigorous proof of this fact requires lengthy and tedious calculations, which will be not carried out here.

On the other hand, Theorems 3.1 and 3.2 imply that $\widehat{f}_n(o)$ converges in probability to $f(o)$ as $n \to \infty$. This and the central limit theorem for $\widehat{p}_n$ give that

$$\sqrt{\frac{|W_n|}{\widehat{f}_n(o)}}\Big(\widehat{p}_n - p\Big) \Longrightarrow N(0,1)\,, \tag{1.1}$$

as $n \to \infty$, which suggests an asymptotic test to check the hypothetical volume fraction p; see also the comments at the end of Section 3.3.

For the planar Boolean model with isotropic compact and convex grains, we compare the theoretical spectral density $f(x)$ with an averaged version $\overline{f_n}(x)$ of the kernel estimator $\widehat{f}_n(x)$. In particular, the estimation of the spectral density $f(x)$ is done from binary images sampled from the Boolean model for three different classes of grain distributions: deterministic discs, uniformly oriented rectangles, and Poisson polygons; see Figure 1. It turns out that the estimator $\overline{f_n}(x)$ fits the theoretical spectral density $f(x)$ quite good for all examples considered in Section 4; see Figure 3.

Notice that $f(x)$ could also be estimated using formula (2.4), that is

$$f(x) = \int_{\mathbb{R}^d} Cov(h)\, \mathrm{e}^{i\langle x,h\rangle}\, dh\,, \quad x \in \mathbb{R}^d\,.$$

However, this would require the estimation of the covariance function $Cov(h)$, which makes it necessary to compute two Fast Fourier transforms, one into the spectral domain and one inverse transformation. Finally, a third Fast Fourier transformation has to be done to calculate an estimator of $f(x)$. On the other hand, using the kernel estimator $\widehat{f}_n(x)$, the complexity can be reduced to one Fast Fourier transformation into the spectral domain.

2 Preliminaries

In the following we recall some basic notions and results from stochastic geometry. More details on this can be found, for example, in [13, 14, 18, 19]. Let Ξ be an arbitrary stationary random closed set (RACS) in $\mathbb{R}^d$. Suppose that the *volume fraction* $p = \mathbb{P}(o \in \Xi)$ of Ξ is (hypothetically) given, where $0 < p < 1$. Furthermore, the *covariance* $C(h)$ of Ξ is defined as $C(h) = \mathbb{P}(o \in \Xi, h \in \Xi)$, where $o \in \mathbb{R}^d$ denotes the origin and $h \in \mathbb{R}^d$ is an arbitrary d-dimensional vector. The (*centered*) *covariance function* $Cov(h)\,, h \in \mathbb{R}^d$, of Ξ is defined as $Cov(h) = C(h) - p^2$. Notice that $Cov(h)$, $h \in \mathbb{R}^d$, coincides with the covariance function of the stationary random field $\{\,\mathbb{1}_\Xi(z),\ z \in \mathbb{R}^d\}$, where $\mathbb{1}_\Xi$ denotes the indicator of the random set Ξ, i.e.,

$$\mathbb{1}_\Xi(z) \;=\; \begin{cases} 1 & \text{if} \quad z \in \Xi\,, \\ 0 & \text{if} \quad z \notin \Xi\,. \end{cases}$$

Lemma 2.1 *The (centered) covariance function* $Cov(h) = C(h) - p^2$ *of any stationary* RACS Ξ *is uniformly continuous on* $\mathbb{R}^d$ *.*

Proof For arbitrary events A, B we have

$$| \mathbb{P}(A) - \mathbb{P}(B) | \;\leq\; \max\{ \mathbb{P}(A \setminus B), \mathbb{P}(B \setminus A)\} \,. \tag{2.1}$$

Furthermore, if $A = \{o \in \Xi\} \cap \{x \in \Xi\}$ and $B = \{o \in \Xi\} \cap \{x + h \in \Xi\}$, then

$$A \setminus B \subseteq \{x \in \Xi\} \cap \{x + h \in \Xi^c\}\,, \qquad B \setminus A \subseteq \{x + h \in \Xi\} \cap \{x \in \Xi^c\}\,.$$

Thus, by the stationarity of Ξ, inequality (2.1) yields

$$\begin{aligned} | C(x) - C(x+h) | \;&\leq\; \max\{ \mathbb{P}(-h \in \Xi, o \in \Xi^c), \mathbb{P}(h \in \Xi, o \in \Xi^c)\} \\ &\leq\; \mathbb{P}(b(o, \|h\|) \cap \Xi \neq \emptyset, o \in \Xi^c) \end{aligned}$$

for any $x, h \in \mathbb{R}^d$, where $\|h\|$ denotes the length of the vector h. Since Ξ is closed, we have $\{b(o, r) \cap \Xi \neq \emptyset, o \in \Xi^c\} \downarrow \emptyset$ as $r \downarrow 0$. Hence, by the continuity of the probability measure $\mathbb{P}$,

$$\lim_{h \to o} \sup_{x \in \mathbb{R}^d} |C(x+h) - C(x)| = 0 \,.$$

□

Suppose that

$$\int_{\mathbb{R}^d} |Cov(h)| \, dh < \infty \,. \tag{2.2}$$

By the result of Lemma 2.1, it then follows from *Bochner's theorem* of the general theory of stationary random fields that there exists a nonnegative Borel-measurable function $f : \mathbb{R}^d \to [0, \infty)$ such that

$$Cov(h) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x) \, \mathrm{e}^{-i\langle x, h\rangle} \, dx \tag{2.3}$$

for each $h \in \mathbb{R}^d$, where $\langle x, h\rangle = \sum_{i=1}^d x_i h_i$ denotes the scalar product of the vectors $x = (x_1, \ldots, x_d)$ and $h = (h_1, \ldots, h_d)$.

The function $f(x)$ appearing in (2.3) is called the *spectral density* of Ξ, where formula (2.3) itself is called the *spectral representation* of the covariance function $Cov(h)$. Furthermore, it is well-known that (2.3) can be inverted in the following way:

$$f(x) = \int_{\mathbb{R}^d} Cov(h) \, \mathrm{e}^{i\langle x, h\rangle} \, dh \,, \quad x \in \mathbb{R}^d \,. \tag{2.4}$$

Using the stationarity of Ξ we obtain for a given *sampling window* $W \subset \mathbb{R}^d$ that for each $h \in \mathbb{R}^d$

$$C(h) = \frac{\mathbb{E}|\Xi \cap (\Xi + h) \cap W \cap (W + h)|}{|W \cap (W + h)|} \,, \tag{2.5}$$

where we assume that W is a Borel set with positive and finite Lebesgue measure $|W|$. Notice that (2.5) can be rewritten as

$$
\begin{aligned}
Cov(h)\,|W\cap(W+h)| &= \mathbb{E}|\Xi\cap(\Xi+h)\cap W\cap(W+h)| - p^2\,|W\cap(W+h)| \\
&= \mathbb{E}\int_{W\cap(W+h)} (\,\mathbb{1}_\Xi(z)\,\mathbb{1}_{\Xi+h}(z) - p^2)\,dz \\
&= \mathbb{E}\int_{W\cap(W+h)} (\,\mathbb{1}_\Xi(z)\,\mathbb{1}_{\Xi}(z-h) - p^2)\,dz \\
&= \mathbb{E}\int_{W\cap(W+h)} (\,\mathbb{1}_\Xi(z) - p)\,(\,\mathbb{1}_{\Xi}(z-h) - p)\,dz\,.
\end{aligned}
$$

Thus, with the notation $g(z) = \mathbb{1}_W(z)(\,\mathbb{1}_\Xi(z) - p)$ we obtain that

$$
Cov(h)\,|W\cap(W+h)| \;=\; \mathbb{E}\int_{\mathbb{R}^d} g(z)\,g(z-h)\,dz\,. \tag{2.6}
$$

An immediate consequence of formula (2.6) is the following result, which is similar to the *Wiener-Khinchin theorem* in the general theory of stationary random fields; see [16], for example.

Lemma 2.2 *For each $x \in \mathbb{R}^d$*

$$
\int_{\mathbb{R}^d} Cov(h)\,|W\cap(W+h)|\;\mathrm{e}^{i\langle x,h\rangle}\,dh = \mathbb{E}\;|G(x)|^2\;, \tag{2.7}
$$

where $G(x) = \int_{\mathbb{R}^d} g(z)\,\mathrm{e}^{i\langle x,z\rangle}\,dz$.

Proof From (2.6) we get that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} Cov(h)\,|W\cap(W+h)|\;\mathrm{e}^{i\langle x,h\rangle}\,dh \\
&\qquad= \mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(z)\,g(z-h)\,dz\;\mathrm{e}^{i\langle x,h\rangle}\,dh \\
&\qquad= \mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(z-h)\,\mathrm{e}^{-i\langle x,z-h\rangle}\,dh\;\; g(z)\,\mathrm{e}^{-i\langle x,-z\rangle}\,dz \\
&\qquad= \mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(h)\,\mathrm{e}^{-i\langle x,h\rangle}\,dh\;\; g(z)\,\mathrm{e}^{-i\langle x,-z\rangle}\,dz \\
&\qquad= \mathbb{E}\int_{\mathbb{R}^d}\overline{\int_{\mathbb{R}^d} g(h)\,\mathrm{e}^{i\langle x,h\rangle}\,dh}\;\; g(z)\,\mathrm{e}^{i\langle x,z\rangle}\,dz \\
&\qquad= \mathbb{E}\left(\overline{G(x)}\,G(x)\right) \\
&\qquad= \mathbb{E}\left(|G(x)|^2\right)\,.
\end{aligned}
$$

$\square$

3 Kernel estimation

3.1 Description of the model

In view of (2.4) and (2.7), a first idea would be to consider the random variable

$$\widehat{f}(x) = \frac{1}{|W|} \left| \int_W (\mathbb{1}_\Xi(z) - p)\, \mathrm{e}^{i\langle x,z\rangle}\, dz \right|^2 \tag{3.1}$$

in order to estimate the value $f(x)$ of the spectral density at point $x \in \mathbb{R}^d$. Although this estimator of the spectral density is asymptotically unbiased, it seems to be difficult (or even impossible) to show that $\widehat{f}(x)$ is a consistent estimator of $f(x)$ as $|W| \to \infty$.

Therefore, in the following, we will consider a kernel estimator of $f(x)$, which turns out to be asymptotically unbiased and mean-square consistent under some integrability conditions on the covariance function $Cov(h)$. For simplicity, let W_n be a sequence of *sampling windows* with $W_n = [0, a_1^{(n)}] \times \cdots \times [0, a_d^{(n)}]$, where

$$\min_{1 \le j \le d} a_j^{(n)} \to \infty\,, \qquad n \to \infty\,. \tag{3.2}$$

Let $k : \mathbb{R}^d \to \mathbb{R}$ be a *kernel function*, which has the following properties. There exists $R > 0$ such that

$$k(x) = 0 \quad \text{for all } x \in \mathbb{R}^d \text{ with } ||x|| > R\,. \tag{3.3}$$

Furthermore, suppose that

$$K = \sup_{x:\, ||x|| \le R} |k(x)| < \infty\,, \qquad \text{and} \qquad \int_{\mathbb{R}^d} k(x)\, dx = 1\,. \tag{3.4}$$

Assume that this kernel function and the above defined sequence of sampling windows W_n are connected with a sequence of *bandwidths* b_n, which has the properties

$$\lim_{n\to\infty} b_n = 0\,, \qquad \text{and} \qquad \lim_{n\to\infty} |W_n|\, {b_n}^d = \infty\,. \tag{3.5}$$

For the above defined kernel function k, for the sampling window W_n and bandwidth b_n, we consider the random variable

$$\widehat{f}_n(x) = \frac{1}{|W_n|\, {b_n}^d} \int_{\mathbb{R}^d} k\left(\frac{x-y}{b_n}\right) \left| \int_{W_n} (\mathbb{1}_\Xi(z) - p)\, \mathrm{e}^{i\langle y,z\rangle}\, dz \right|^2 dy\,, \tag{3.6}$$

which is called a *kernel estimator* of the spectral density $f(x)$ of Ξ at point $x \in \mathbb{R}^d$.

3.2 Asymptotic unbiasedness

Theorem 3.1 *Let the stationary* RACS Ξ *satisfy condition* (2.2)*. Furthermore, assume that conditions* (3.2) – (3.5) *are fulfilled. Then the kernel estimator* $\widehat{f}_n(x)$ *given in* (3.6) *is pointwise asymptotically unbiased, that is*

$$\lim_{n\to\infty} \mathbb{E}\widehat{f}_n(x) = f(x)\,, \quad x \in \mathbb{R}^d\,. \tag{3.7}$$

Proof By Lemma 2.2 we have

$$\mathbb{E}\left|\int_{W_n} (\,1\!\!1_\Xi(z)-p)\,e^{i\langle y,z\rangle}\,dz\right|^2 \;=\; \int_{\mathbb{R}^d} Cov(h)\,|W_n\cap(W_n+h)|\,e^{i\langle y,h\rangle}\,dh\,.$$

Using (3.6), this gives

$$\begin{aligned}\mathbb{E}\widehat{f}_n(x) &= \frac{1}{|W_n|\,b_n^d}\int_{\mathbb{R}^d} k(\frac{x-y}{b_n})\,\mathbb{E}\left|\int_{W_n}(\,1\!\!1_\Xi(z)-p)\,\mathrm{e}^{i\langle y,z\rangle}\,dz\right|^2 dy\\ &= \frac{1}{b_n^d}\int_{\mathbb{R}^d} k(\frac{x-y}{b_n})\int_{\mathbb{R}^d} Cov(h)\,\frac{|W_n\cap(W_n+h)|}{|W_n|}\,e^{i\langle y,h\rangle}\,dh\,dy\\ &= \int_{\mathbb{R}^d} k(w)\int_{\mathbb{R}^d} Cov(h)\,\frac{|W_n\cap(W_n+h)|}{|W_n|}\,e^{i\langle x,h\rangle}\,e^{-ib_n\langle w,h\rangle}\,dh\,dw\,,\end{aligned}$$

where the substitution $w=(x-y)/b_n$ has been used in the last equality. By (2.2), (3.3) and (3.4), the integrand of the last expression is bounded by an absolutely integrable function. We thus pass with limit $n\to\infty$ under the integrals. Using (3.5), this gives

$$\lim_{n\to\infty}\mathbb{E}\widehat{f}_n(x) \;=\; \int_{\mathbb{R}^d} k(w)\,dw\int_{\mathbb{R}^d} Cov(h)\,e^{i\langle x,h\rangle}\,dh = f(x)\,. \qquad \square$$

Notice that we get the following rate of convergence in (3.7) if (2.2) is replaced by the slightly stronger condition

$$\int_{\mathbb{R}^d} |Cov(h)|\,\|h\|\,dh<\infty\,. \tag{3.8}$$

Indeed, for each n the inequality

$$|\,\mathbb{E}\widehat{f}_n(x)-f(x)\,| \le \int_{\mathbb{R}^d}|k(w)|\,dw\int_{\mathbb{R}^d}|Cov(h)|\,\|h\|\,dh\left(R\,b_n+\frac{\sqrt{d}}{\min_{1\le j\le d}a_j^{(n)}}\right) \tag{3.9}$$

holds uniformly in $x\in\mathbb{R}^d$, which is an immediate consequence of the inequalities $|\,e^{-ib_n\langle w,h\rangle}-1\,|\le b_n\|w\|\|h\|$ and

$$1-\frac{|W_n\cap(W_n+h)|}{|W_n|} \;\le\; \sum_{j=1}^d\frac{|h_j|}{a_j^{(n)}} \;\le\; \frac{\sqrt{d}\,\|h\|}{\min_{1\le j\le d}a_j^{(n)}}\,.$$

3.3 Mean-square consistency

The main reason for employing kernel-smoothing techniques in spectral density estimation is that they produce consistent estimators in the sense that the mean square error $\mathbb{E}(\,\widehat{f}_n(x)-f(x)\,)^2$ disappears as $n\to\infty$, which is, in view of (3.7), equivalent to $\mathrm{Var}\widehat{f}_n(x)\to 0$. For stationary random fields on d-dimensional lattices the corresponding asymptotic theory is derived in [3] and [17]. However, similar results for stationary (0-1-valued) random fields on $\mathbb{R}^d$ seem to be unknown so far.

To state our result on the consistency of $\widehat{f}_n(x)$ we consider the fourth-order cumulant density $c^{(4)}(o,u,v,w)$ of the stationary random field $\{\,\mathbb{1}_\Xi(x), x\in\mathbb{R}^d\}$, which is given by

$$\begin{aligned} c^{(4)}(x_1,x_2,x_3,x_4) &= \mathbb{E}\prod_{i=1}^{4}(\,\mathbb{1}_\Xi(x_i)-p) - Cov(x_2-x_1)\,Cov(x_4-x_3) \\ &- \; Cov(x_3-x_1)\,Cov(x_4-x_2) - Cov(x_4-x_1)\,Cov(x_3-x_2) \end{aligned}$$

for $x_1,...,x_4\in\mathbb{R}^d$. We assume that $c^{(4)}(o,u,v,w)$ is absolutely integrable, i.e.,

$$C_4 \;=\; \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\,c^{(4)}(o,u,v,w)\,|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \;<\; \infty\,. \tag{3.10}$$

Theorem 3.2 *Let the stationary* RACS Ξ *satisfy conditions* (2.2) *and* (3.10). *Furthermore, assume that* (3.2) – (3.5) *hold. Then $\widehat{f}_n(x)$ is a pointwise mean-square consistent estimator of $f(x)$. More precisely, for each $x\in\mathbb{R}^d$,*

$$\overline{\lim_{n\to\infty}}\; b_n^d|W_n|\,\mathrm{Var}\widehat{f}_n(x) \le C < \infty\,. \tag{3.11}$$

If, in addition, the kernel function $k(.)$ is Lebesgue-almost everywhere continuous and the assumption $b_n^d\,|W_n| \longrightarrow \infty$ is strengthened to $b_n\min_{1\le j\le d} a_j^{(n)} \longrightarrow \infty$ as $n\to\infty$, then the limit

$$l(x,y) = \lim_{n\to\infty}\; b_n^d\,|W_n|\,\Big(\mathbb{E}(\widehat{f}_n(x)\,\widehat{f}_n(y)) - \mathbb{E}\widehat{f}_n(x)\,\mathbb{E}\widehat{f}_n(y)\Big) \tag{3.12}$$

exists, where

$$l(x,y) = \begin{cases} 0 & \textit{if} \quad x\neq \pm y\,, \\ f^2(x)\;(2\pi)^d\,\int_{\mathbb{R}^d} k^2(x)\,dx & \textit{if} \quad x=y\neq o\,, \\ f^2(x)\;(2\pi)^d\,\int_{\mathbb{R}^d} k(x)\,k(-x)\,dx & \textit{if} \quad x=-y\neq o\,, \\ f^2(o)\;(2\pi)^d\,\int_{\mathbb{R}^d} k(x)\,(k(x)+k(-x))\,dx & \textit{if} \quad x=y=o\,. \end{cases}$$

Proof For any two points $x, y \in \mathbb{R}^d$, the covariance

$$\mathrm{Cov}(\widehat{f}_n(x), \widehat{f}_n(y)) = \mathbb{E}(\widehat{f}_n(x)\, \widehat{f}_n(y)) - \mathbb{E}\widehat{f}_n(x)\, \mathbb{E}\widehat{f}_n(y)$$

of the random variables $\widehat{f}_n(x)$, $\widehat{f}_n(y)$ can be written as

$$b_n^d |W_n| \mathrm{Cov}(\widehat{f}_n(x), \widehat{f}_n(y)) = \frac{1}{b_n^d |W_n|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k\Big(\frac{x-u}{b_n}\Big) k\Big(\frac{y-v}{b_n}\Big) \int_{W_n^4} e^{i\langle u, x_1 - x_2\rangle} e^{i\langle v, x_3 - x_4\rangle}$$
$$\times \mathrm{Cov}\Big((\mathbb{1}_\Xi(x_1) - p)(\mathbb{1}_\Xi(x_2) - p), (\mathbb{1}_\Xi(x_3) - p)(\mathbb{1}_\Xi(x_4) - p)\Big)\, d(x_1, x_2, x_3, x_4)\, dv\, du\,.$$

Taking into account the stationarity of the RACS Ξ we see from the above definition of the fourth-order cumulant density $c^{(4)}(x_1, x_2, x_3, x_4)$ that

$$\begin{aligned}\mathrm{Cov}\Big((\mathbb{1}_\Xi(x_1) - p)(\mathbb{1}_\Xi(x_2) - p), (\mathbb{1}_\Xi(x_3) - p)(\mathbb{1}_\Xi(x_4) - p)\Big) = \\ c^{(4)}(o, x_2 - x_1, x_3 - x_1, x_4 - x_1) + Cov(x_3 - x_1)\, Cov(x_4 - x_2) \\ + Cov(x_4 - x_1)\, Cov(x_3 - x_2)\,.\end{aligned}$$

Since, by (3.10),

$$\Big| \int_{W_n^4} e^{i\langle u, x_1 - x_2\rangle} e^{i\langle v, x_3 - x_4\rangle}\, c^{(4)}(o, x_2 - x_1, x_3 - x_1, x_4 - x_1)\, d(x_1, x_2, x_3, x_4) \Big| \le C_4\, |W_n|$$

and

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big| k\Big(\frac{x-u}{b_n}\Big) k\Big(\frac{y-v}{b_n}\Big) \Big|\, dv\, du \le b_n^{2d} \Big(\int_{\mathbb{R}^d} |k(u)| du \Big)^2$$

we find that

$$b_n^d |W_n|\, \mathrm{Cov}(\widehat{f}_n(x), \widehat{f}_n(y)) = \frac{1}{b_n^d |W_n|}\, (\, J_n^{(+)}(x,y) + J_n^{(-)}(x,y)\,) + O(b_n^d) \qquad (3.13)$$

where

$$\begin{aligned} J_n^{(\pm)}(x,y) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k\Big(\frac{x-u}{b_n}\Big)\, k\Big(\frac{y \pm v}{b_n}\Big) \int_{W_n^4} e^{i\langle u, x_1 - x_2\rangle + i\langle v, x_3 - x_4\rangle} \\ &\times \quad Cov(x_3 - x_1)\, Cov(x_4 - x_2)\, d(x_1, x_2, x_3, x_4)\, dv\, du \end{aligned}$$
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k\Big(\frac{x-u}{b_n}\Big) k\Big(\frac{y \pm v}{b_n}\Big) \Big| \int_{\mathbb{R}^d} \int_{W_n \cap (W_n - w)} e^{i\langle u+v, z\rangle} dz\, e^{i\langle v, w\rangle} Cov(w)\, dw \Big|^2 dv\, du\,.$$

The latter equality is easily checked after substituting $z = x_1$, $z' = x_2$ and $w = x_3 - x_1$, $w' = x_4 - x_2$. To prove (3.11) we derive a uniform bound of $J_n^{(\pm)}(x,x)/b_n^d |W_n|$ for all $x \in \mathbb{R}^d$ and $n \ge 1$. To this end we apply Schwarz's inequality to the inner integral over the variable w, which gives

$$\begin{aligned} |J_n^{(\pm)}(x,x)| &\le C_2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big| k\Big(\frac{x-u}{b_n}\Big)\, k\Big(\frac{x \pm v}{b_n}\Big) \Big| \\ &\times \int_{\mathbb{R}^d} \Big| \int_{W_n \cap (W_n - w)} e^{i\langle u+v, z\rangle}\, dz \Big|^2\, |Cov(w)|\, dw\, dv\, du\,, \end{aligned}$$

where $C_2 = \int_{\mathbb{R}^d} |Cov(w)|\, dw$. Next replace $u+v$ by u' and $x \pm v$ by $v' b_n$. This, together with (3.4), implies that

$$|J_n^{(\pm)}(x,x)| \le b_n^d\, C_2\, K \int_{\mathbb{R}^d} |k(v')|\, dv' \int_{\mathbb{R}^d} \left| \int_{W_n \cap (W_n - w)} e^{i\langle u', z\rangle}\, dz \right|^2 |Cov(w)|\, dw\, du' .$$

Using the identity

$$\left| \int_{W_n \cap (W_n - w)} e^{i\langle u', z\rangle}\, dz \right|^2 = \prod_{j=1}^d \left| \frac{e^{iu'_j\, (a_j^{(n)} - |w_j|)} - 1}{u'_j} \right|^2 \mathbb{1}_{[-a_j^{(n)}, a_j^{(n)}]}(w_j) ,$$

which holds for all $u' = (u'_1, ..., u'_d)$ and $w = (w_1, ..., w_d)$ with the convention that $(e^{ixy} - 1)/x = iy$ for $x = 0$, we obtain after substituting $r_j = u'_j\, (a_j^{(n)} - |w_j|)$ for $j = 1, ..., d$ that

$$\frac{|J_n^{(\pm)}(x,x)|}{b_n^d\, |W_n|} \le C_2^2\, K \int_{\mathbb{R}^d} |k(v')|\, dv' \left(\int_{\mathbb{R}^1} \frac{|e^{i\,r} - 1|^2}{r^2}\, dr \right)^d$$

for any $x \in \mathbb{R}^d$. This combined with (3.13) proves (3.11).

Statement (3.12) can be derived directly from (3.13) after rewriting the integrals $J_n^{(\pm)}(x,y)$. Some obvious rearrangements lead to the representation

$$\begin{aligned} \frac{J_n^{(\pm)}(x,y)}{b_n^d\, |W_n|} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k\Big(\frac{x \mp y}{b_n} \pm v - f_n(u) \Big)\, k(v) \\ &\times \left| \int_{\mathbb{R}^d} \int_{W_n \cap (W_n - w)} e^{i\langle u, g_n(z)\rangle}\, dg_n(z)\, e^{\pm b_n i\langle v, w\rangle \mp i\langle y, w\rangle}\, Cov(w)\, dw \right|^2 dv\, du , \end{aligned}$$

where

$$f_n(u) = \Big(\frac{u_1}{b_n a_1^{(n)}}, ..., \frac{u_d}{b_n a_d^{(n)}} \Big) \quad \text{and} \quad g_n(z) = b_n f_n(z) .$$

Using the assumptions made on the sequence of bandwidths b_n we get that

$$f_n(u) \underset{n\to\infty}{\longrightarrow} o \quad \text{and} \quad \int_{W_n \cap (W_n - w)} e^{i\langle u, g_n(z)\rangle}\, dg_n(z) \underset{n\to\infty}{\longrightarrow} \int_{[0,1]^d} e^{i\langle u, z\rangle}\, dz = \prod_{j=1}^d \frac{e^{iu_j} - 1}{iu_j}$$

for any $u = (u_1, ..., u_d) \in \mathbb{R}^d$. Finally, by means of (2.4) and Lebesgue's dominated convergence theorem we find that

$$\frac{J_n^{(\pm)}(x, \pm x)}{b_n^d\, |W_n|} \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} k(\pm v)\, k(v)\, dv \left(\int_{\mathbb{R}^1} \frac{|e^{i\,r} - 1|^2}{r^2}\, dr \right)^d f^2(x)$$

for all $x \in \mathbb{R}^d$, whereas $J_n^{(\pm)}(x,y) = o(b_n^d\, |W_n|)$ as $n \to \infty$ for $y \ne \pm x$. This result combined with relation (3.13) and the well-known integral

$$\int_{\mathbb{R}^1} \frac{|e^{i\,r} - 1|^2}{r^2}\, dr = 4 \int_0^\infty \frac{1 - \cos r}{r^2}\, dr = 2\,\pi$$

completes the proof. $\square$

In case of a stationary Boolean model with random compact (not necessarily isotropic) typical grain M_0 condition (2.2) is equivalent to $\mathbb{E}|M_0|^2 < \infty$ and (3.10) is satisfied whenever $\mathbb{E}|M_0|^4 < \infty$; see [6]. Furthermore, notice that $\mathbb{E}|M_0| \, \|M_0\|^{d+1} < \infty$ implies condition (3.8), where $\|M_0\| = \sup\{\|x\| : x \in M_0\}$.

Notice that, as a consequence of the Theorems 3.1 and 3.2, the random variables $\widehat{f}_n(o)$ converge in probability to $f(o)$ as $n \to \infty$. This is of particular interest since $f(o)$ coincides with the asymptotic variance of the empirical volume fraction $\widehat{p}_n = |\Xi \cap W_n|/|W_n|$, i.e.,

$$f(o) = \int_{\mathbb{R}^d} Cov(h)\, dh = \lim_{n\to\infty} |W_n|\, \mathbb{E}(\widehat{p}_n - p)^2 \,.$$

Thus, employing Slutsky-type arguments, the central limit theorem for $\widehat{p}_n$ (see [6, 12]) gives (1.1). See also the appendix for further related results.

3.4 Further related estimators

Notice that expression (2.4) could also directly be used in order to get another estimator of the spectral density $f(x)$. This would require the estimation of the covariance function $Cov(h)$. For a nonparametric estimator $\widehat{Cov_n}(h)$ of $Cov(h)$, sufficient conditions can be deduced from Theorem 4.2.1 in [7], which imply that

$$\widetilde{f}_n(x) = \int_{\mathbb{R}^d} \widehat{Cov_n}(h)\, \mathrm{e}^{i\langle x,h\rangle}\, dh \,, \quad x \in \mathbb{R}^d \,,$$

is a consistent estimator of $f(x)$ as $n \to \infty$. In the case of random fields on a d-dimensional lattice, this method to estimate the spectral density has been used, for example, in [3, 15, 17]. However, even an effective estimation of the covariance function makes it necessary to compute two Fast Fourier transforms, one into the spectral domain and one inverse transformation; see [8, 16], for example. Finally, a third Fast Fourier transformation has to be done to calculate an estimator of the spectral density since $f(x)$ is the d-dimensional Fourier transform of the covariance function $Cov(h)$, as one can see from (2.4).

On the other hand, the kernel estimator $\widehat{f}_n(x)$ of $f(x)$ given in (3.6) does not depend on the covariance function. Using this estimator the complexity can be reduced to one Fast Fourier transformation into the spectral domain. Thus, in terms of the implementation of this estimator, a considerable decrease of the computation time as well as of the needed memory can be achieved.

Methods for parametric estimation of the spectral density of continuously indexed random fields have been investigated, for example, in [9, 10]. For random fields on a d-dimensional lattice, parametric estimation of the spectral density has been considered, for example, in [3, 9].

4 Numerical evaluation of the spectral density

4.1 Representation formula: isotropic case

Suppose now that Ξ is a stationary RACS in $\mathbb{R}^2$. If we additionally assume that Ξ is isotropic, the following well-known simplifications arise; see [14]. The covariance function depends only on the radial coordinate $r = ||h||$. Correspondingly, the spectral density $f(x)$ depends only on $\varrho = ||x||$, as it will be shown in expression (4.1) below. Particularly, in the case of isotropy the following *representation formula* of the spectral density $f(x)$ is obtained, where (r, φ) are the polar coordinates of the vector $h = (h_1, h_2) \in \mathbb{R}^2$ and (ϱ, ϕ) are those of $x = (x_1, x_2) \in \mathbb{R}^2$. Then,

$$\begin{aligned} f(x_1, x_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Cov(h_1, h_2)\, \mathrm{e}^{i(x_1 h_1 + x_2 h_2)}\, dh_1\, dh_2 \\ &= \int_0^{\infty} r\, Cov(r) \int_0^{2\pi} \mathrm{e}^{ir(x_1 \cos\varphi + x_2 \sin\varphi)}\, d\varphi\, dr\,, \end{aligned}$$

and

$$\begin{aligned} f(\varrho, \phi) &= \int_0^{\infty} r\, Cov(r) \int_0^{2\pi} \mathrm{e}^{ir\varrho\cos(\varphi - \phi)}\, d\varphi\, dr \\ &= \int_0^{\infty} r\, Cov(r) \int_0^{2\pi} \mathrm{e}^{ir\varrho\cos\varphi}\, d\varphi\, dr\,. \end{aligned}$$

Using the *Bessel function* $J_0(s)$ of the first kind of order 0, i.e.,

$$J_0(s) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{is\cos\vartheta}\, d\vartheta\,, \quad s \geq 0\,,$$

the spectral density $f(\varrho) = f(\varrho, \phi)$ can be written as

$$f(\varrho) = 2\pi \int_0^{\infty} r\, Cov(r) J_0(r\varrho)\, dr\,, \quad \varrho \geq 0\,. \tag{4.1}$$

Thus, to be able to plot the theoretical spectral density $f(\varrho)$ we have to know the covariance function $Cov(r) = C(r) - p^2$, $r \geq 0$. This is the case, for example, for the planar Boolean model with three different classes of compact and convex grains.

4.2 Boolean model with known covariance

Consider a stationary Poisson process in $\mathbb{R}^2$ with finite positive intensity λ, and let $\{Y_n,\, n \geq 1\}$ be a radial enumeration of its points. Let $\{M_n,\, n \geq 1\}$ be a sequence of independent copies of a non-empty, compact and isotropic RACS M_0 (called *typical grain*) such that

$$\mathbb{E}(||M_0||^2) < \infty\,, \tag{4.2}$$

where $\|M_0\| = \sup\{\|x\| : x \in M_0\}$. Using (4.2), it can be shown that the union $\Xi = \bigcup_n (M_n + Y_n)$ is a closed set with probability 1. This stationary RACS Ξ is called a *Boolean germ-grain model*, where the Y_n are called *germs*, and the M_n *grains*.

Furthermore, it can be shown that for each $r \geq 0$,

$$C(r) = 2p - 1 + (1-p)^2 \exp(\lambda \gamma_{M_0}(r))\,, \tag{4.3}$$

where $p = 1 - \exp(-\lambda\, \mathbb{E}|M_0|)$ is the area fraction of Ξ and $\gamma_{M_0}(r) = \mathbb{E}|M_0 \cap (M_0 - ru)|$ for any $u \in \mathbb{R}^2$ with $\|u\| = 1$.

Deterministic Discs. Let M_0 be a disc of center o and of (deterministic) radius R. Then, the covariance $C(r)$ can be given in closed form. Namely,

$$C(r) = \begin{cases} 2p - 1 + (1-p)^2 \exp\Big(\lambda\Big(2R^2 \arccos\dfrac{r}{2R} - \dfrac{r}{2}\sqrt{4R^2 - r^2}\Big)\Big) & \text{if } 0 \leq r < 2R, \\ p^2 & \text{if } r \geq 2R, \end{cases}$$

where $p = 1 - \exp(-\pi\lambda R^2)$; see [18], p. 83.

Uniformly Oriented Rectangles. If M_0 is a uniformly oriented rectangle of area α and side length ratio β (≥ 1), where both α and β are deterministic, then, similar to the example of deterministic discs, an analytical expression can be given for the covariance $C(r)$. Namely, for the function $\gamma_{M_0}(r)$ appearing in (4.3), we have (see [19], p. 123) the following expression:

$$\gamma_{M_0}(r) = \frac{\alpha}{\pi} \begin{cases} \pi - 2x + \dfrac{x^2 - 2x}{\beta} & \text{if } 0 \leq x \leq 1, \\ 2\arcsin\Big(\dfrac{1}{x}\Big) - \dfrac{1}{\beta} - 2(x - u) & \text{if } 1 < x \leq \beta, \\ 2\arcsin\Big(\dfrac{\beta - uv}{x^2}\Big) + 2u + \dfrac{2v}{\beta} - \beta - \dfrac{1 + x^2}{\beta} & \text{if } \beta < x < \sqrt{\beta^2 + 1}, \\ 0 & \text{if } x \geq \sqrt{\beta^2 + 1}, \end{cases}$$

where

$$x = \frac{r}{\sqrt{\alpha/\beta}}\,, \qquad u = \sqrt{x^2 - 1}\,, \qquad v = \sqrt{x^2 - \beta^2}\,.$$

Clearly, the area fraction is $p = 1 - \exp(-\lambda\alpha)$.

Poisson Polygons. Assume now that the distribution of M_0 is that of a so-called Poisson polygon, that is the typical cell of a stationary Poisson line process in $\mathbb{R}^2$, with parameter $\rho > 0$. Here, we have

$$p = 1 - \exp\Big(-\frac{4\lambda}{\pi\rho^2}\Big)\,,$$

and

$$C(r) = 2p - 1 + (1-p)^2 \exp\Big(\frac{4\lambda}{\pi\rho^2} \exp(-\rho r)\Big) \tag{4.4}$$

for all $r \geq 0$; see [18], p.83.

4.3 Rotation average and discrete Fourier transform

Notice that the estimation of the covariance function $Cov(h)$ or the spectral density $f(x)$ from a 2-dimensional binary image can only be done for vectors h and x, respectively, which take values on a 2-dimensional lattice. Therefore, in the isotropic case it is convenient to estimate the spectral density using the *rotation average* $\overline{f_n}(\varrho)$, where

$$\overline{f_n}(\varrho) \quad = \quad \frac{1}{\#\{x : ||x|| \approx \varrho\}} \sum_{x:\, ||x|| \approx \varrho} \widehat{f}_n(x)\,, \tag{4.5}$$

and $\widehat{f}_n(x)$ is given in (3.6). Here, the set $\{x : ||x|| \approx \varrho\}$ can be obtained using the so-called *midpoint circle algorithm*, which is a common algorithm in image analysis to detect a circle on a lattice; see, for example, [4]. This averaging method enables us to improve the accuracy of the estimation. Furthermore, we consider the random field $\{\widehat{h}(y),\, y \in W\}$, where

$$\widehat{h}(y) = \left| \int_W (\, 1\!\!1_\Xi(z) - p)\, \mathrm{e}^{i\langle y,z\rangle}\, dz \right|^2, \quad y \in W\,. \tag{4.6}$$

Notice that $\{\widehat{h}(y),\, y \in W\}$ represents the basis for the computation of the kernel estimator $\widehat{f}_n(x)$ of the spectral density $f(x)$; see formula (3.6). The estimation of the spectral density is done from binary images within a sampling window W being a square spanned by a rectangular lattice of 512×512 pixels points. Since for the computation of $\{\widehat{h}(y),\, y \in W\}$ on a lattice the *discrete Fourier transform* of the stationary random field $\{(\, 1\!\!1_\Xi(z) - p),\, z \in W\}$ is used, the result of $\{\widehat{h}(y),\, y \in W\}$ is obtained again on a lattice of 512×512 pixel points, which can be visualized as a grayscale image. Notice that in the spectral domain the physical size of the grayscale image is $2\pi \times 2\pi$ and consequently the stepwidth on the lattice equals $2\pi/512$. Furthermore, the discrete Fourier transformation is modified such that the quantity $\widehat{h}(o)$ is located in the center of the grayscale image.

For the numerical examples considered below, the well-known *Epanechnikov kernel* is used as the kernel function $k(x)$ appearing in expression (3.6), where in this case $x \in \mathbb{R}^2$. That is,

$$k(x) \quad = \quad \begin{cases} 2/\pi\ (1 - \|x\|^2) & \text{if } \ \|x\| \le 1\,, \\ 0 & \text{if } \ \|x\| > 1\,. \end{cases} \tag{4.7}$$

Simulation studies showed that the behavior and the goodness of fit of the kernel estimator $\overline{f_n}(\varrho)$ in (4.5) does not change significantly if different kernel functions such as the *quadratic kernel* are used.

4.4 Examples

For the planar Boolean model with isotropic compact and convex grains, we compare the theoretical spectral density $f(x)$ with the averaged version $\overline{f_n}(x)$ of the kernel estimator $\widehat{f}_n(x)$. In particular, the estimation of the spectral density $f(x)$ is done from the binary images given in Figure 1, which are sampled from the Boolean model for the three classes of grain distributions discussed in Section 4.2. First, the random field $\{\widehat{h}(y),\, y \in W\}$ given in (4.6) is evaluated from the corresponding realisations of the Boolean model shown in Figure 1; see Figure 2. Recall that $\widehat{h}(o)$ is located in the center of the grayscale image, where in Figure 2 bright gray-tones indicate high values of $\widehat{h}(y)$ whereas dark gray-tones indicate low values. Notice that the grayscale images in Figure 2 differ significantly concerning the chosen grain distribution. On the other hand, for the same class of grain distribution the images are similarly structured; see Figure 2 (b) and (c). Therefore, analyzing the grayscale image of $\{\widehat{h}(y),\, y \in W\}$ for a given realisation of a Boolean model with unknown grain distribution, it seems to be possible to draw conclusions concerning the shapes of the underlying grains.

For the different grain distributions, the fit of the estimator $\overline{f_n}(\varrho)$ to the theoretical function $f(\varrho)$ is illustrated in Figure 3. Notice that for small values of ϱ only a small number of vectors x go into the estimation of the spectral density; see formula (4.5). Therefore, the kernel estimator $\overline{f_n}(\varrho)$ underlies some variation for small values of ϱ concerning different realisations of the Boolean model with one and the same grain distribution. Furthermore, it will be investigated for which bandwidth b_n the kernel estimator $\overline{f_n}(\varrho)$ fits the theoretical spectral density $f(\varrho)$ best in the given setup of parameters.

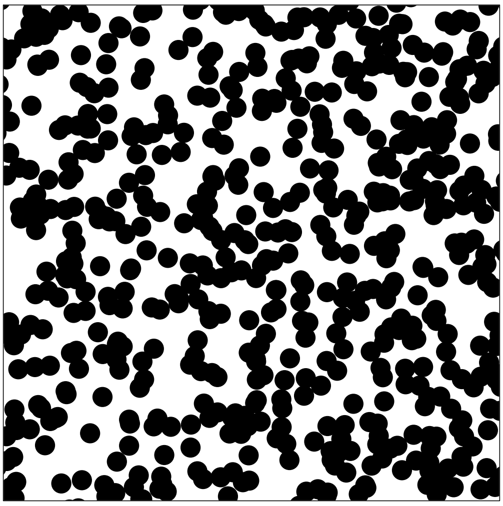
For the numerical examples considered below, the same intensity will be chosen for the Poisson germ process, i.e., $\lambda = 2.2064 \cdot 10^{-3}$. In addition, the parameters of the compact and convex grains will be chosen in such a way that the area fraction p of the Boolean model equals 0.5. Using (4.1), the numerical values of $f(\varrho)$ for $\varrho = 0$ to 0.7 by 0.05 are presented in Table 1 for the different grain distributions.

Deterministic Discs. For $\lambda = 2.2064 \cdot 10^{-3}$ and $R = 10$, the functions $\overline{f_n}(\varrho)$ and $f(\varrho)$ are illustrated in Figure 3 (a), where the bandwidth b_n is chosen equal to 6, 8, and 10 (the upper dotted line corresponds to $b_n = 6$, the medially line to $b_n = 8$, and the lower line to $b_n = 10$). It can be seen that in this setup of parameters the best fit of $\overline{f_n}(\varrho)$ to the theoretical function $f(\varrho)$ is reached for $b_n = 8$. The same is true for the other two grain distributions considered below. Therefore, the bandwidth b_n will be chosen equal to 8 in these models, too.

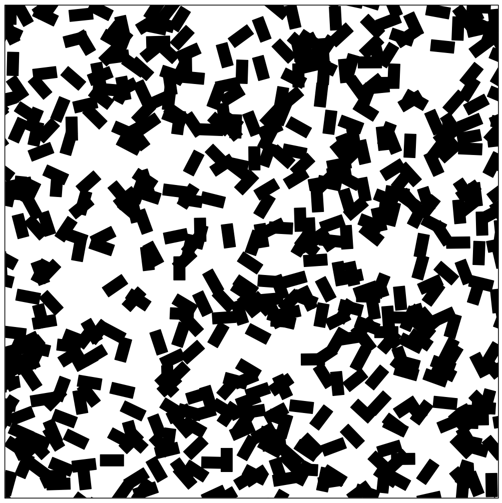
Uniformly Oriented Rectangles. For $\lambda = 2.2064 \cdot 10^{-3}$, $\alpha = 314.16$, and $\beta = 2$ or $\beta = 4$, respectively, a comparison of the functions $\overline{f_n}(\varrho)$ and $f(\varrho)$ is shown in Figure 3 (b) and (c), where the bandwidth b_n is chosen equal to 8.

Poisson Polygons. For $\lambda = 2.2064 \cdot 10^{-3}$ and $\rho = 6.3662 \cdot 10^{-2}$, a comparison of the functions $\overline{f_n}(\varrho)$ and $f(\varrho)$ is shown in Figure 3 (d), where the bandwidth b_n is again chosen equal to 8. Algorithms for sampling the Poisson polygon have been proposed, for example, in [2, 11], where we used the radial simulation procedure discussed in [11].

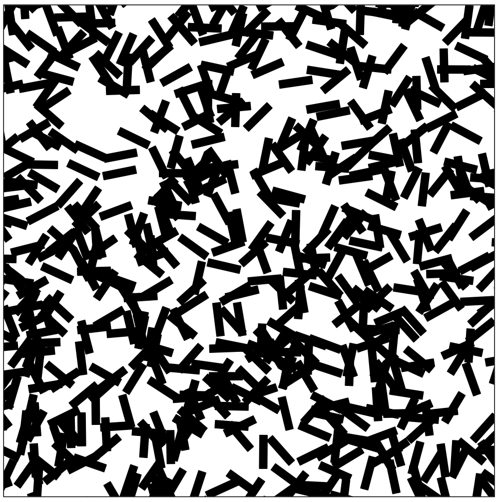
Notice that in [1] similar numerical examples have been used to illustrate the accuracy of different exponential approximations for the covariance $C(r)$, $r \geq 0$.



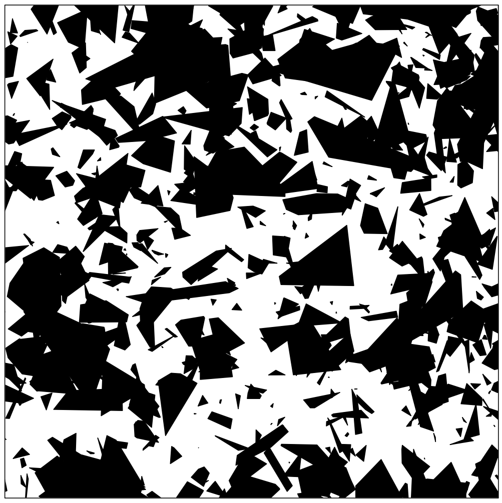
(a)



(b)



(c)



(d)

Figure 1: Samples from the Boolean model with $\lambda = 2.2064 \cdot 10^{-3}$ and $p = 0.5$, where the primary grains are (a) discs with $R = 10$, (b) rectangles with $\alpha = 314.16$ and $\beta = 2$, (c) rectangles with $\alpha = 314.16$ and $\beta = 4$, (d) Poisson polygons with $\rho = 6.3662 \cdot 10^{-2}$.

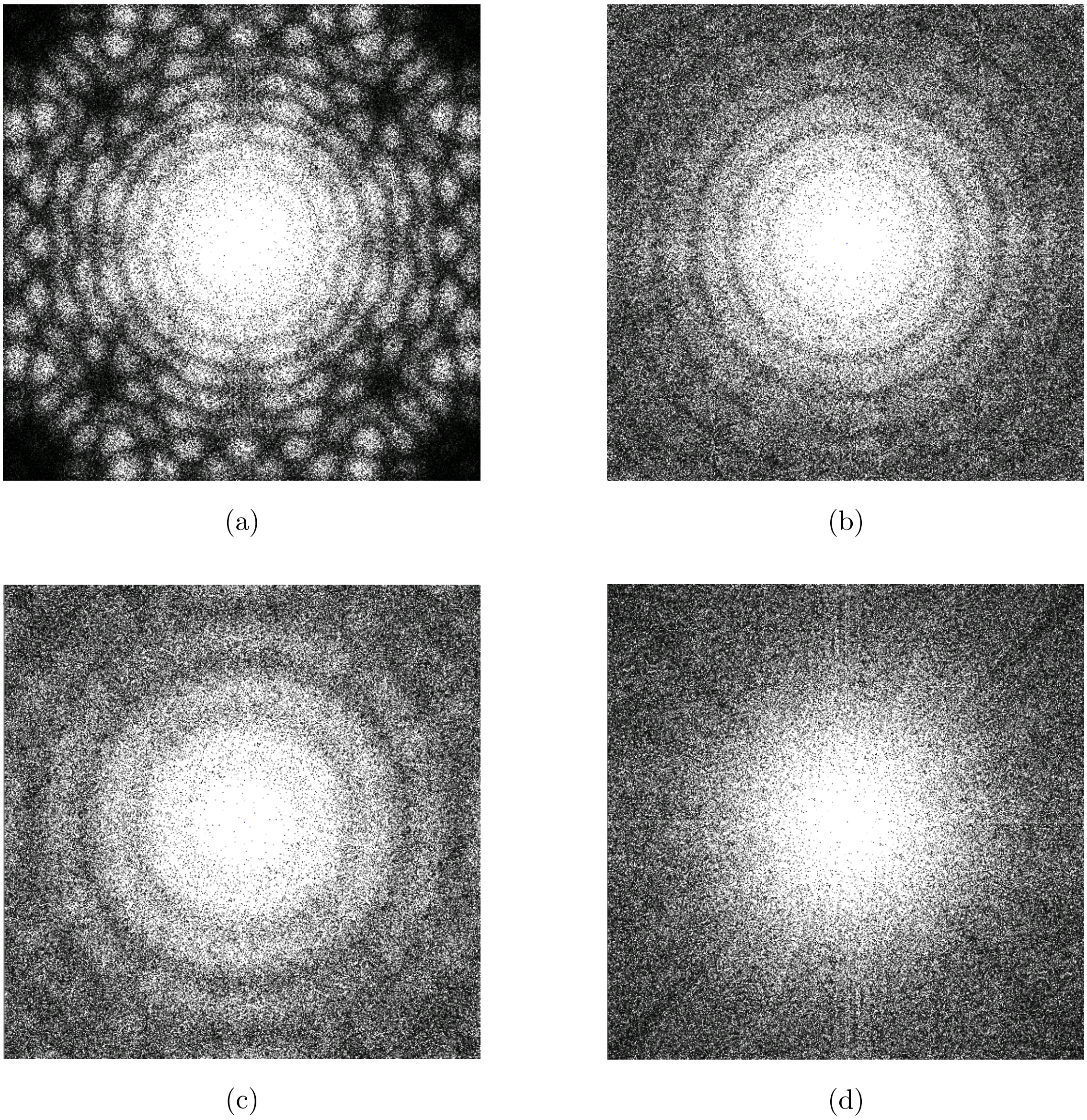


Figure 2: Samples from the random field $\{\widehat{h}(y),\, y \in W\}$, which are evaluated from the corresponding realisations of the Boolean model shown in Figure 1 (a) – (d).

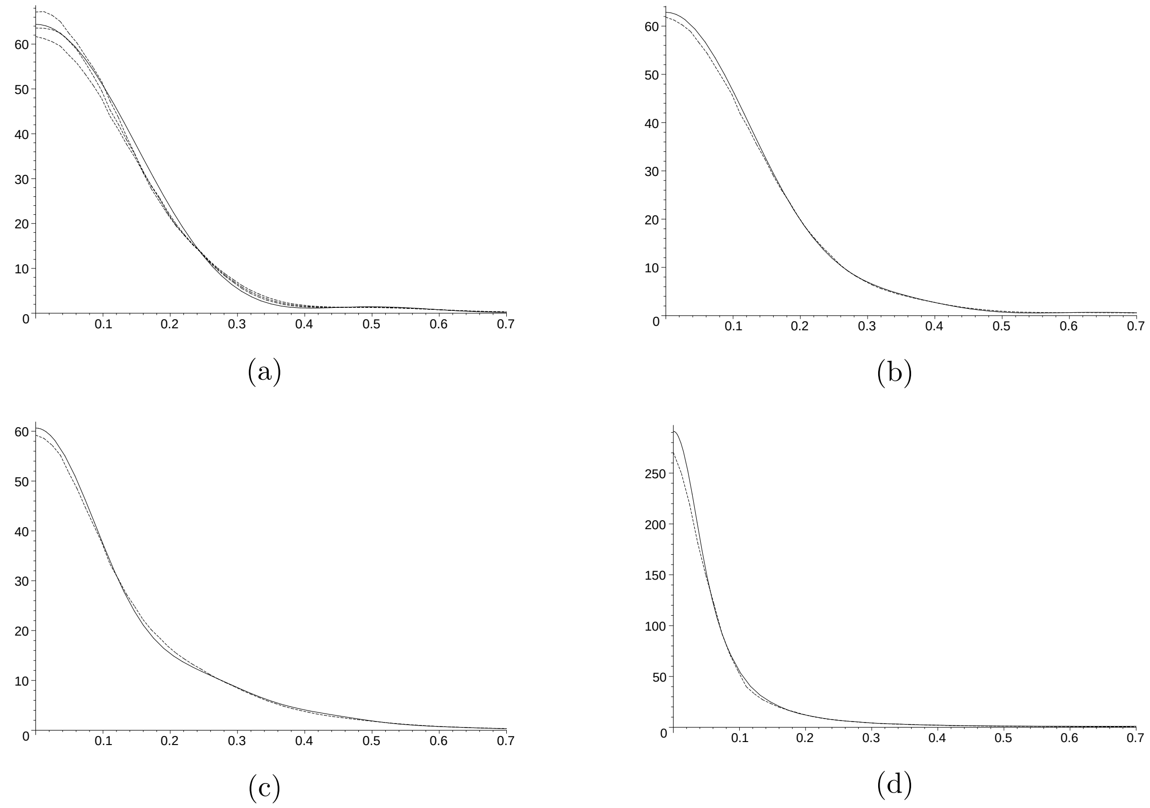


Figure 3: Comparison of $f(\varrho)$ (——) and $\overline{f_n}(\varrho)$ (- - -), where $f(\varrho)$ is calculated for the Boolean model with the primary grains shown in Figure 1 (a) − (d) and $\overline{f_n}(\varrho)$ is evaluated from the corresponding realisations. Here, $b_n = 6, 8, 10$ in Figure (a) and $b_n = 8$ in Figures (b) − (d).

	Determin. discs	Deterministic rectangles		Poisson polygons
	$R = 10$	$\alpha = 314.16,\ \beta = 2$	$\alpha = 314.16,\ \beta = 4$	$\rho = 6.3662 \cdot 10^{-2}$
$\varrho = 0$	64.4134	62.8691	60.6739	291.1698
$\varrho = 0.05$	60.7425	58.2920	53.4301	151.6950
$\varrho = 0.1$	50.7910	46.5243	37.4131	55.0788
$\varrho = 0.15$	37.3113	32.1952	23.1610	23.8848
$\varrho = 0.2$	23.6665	19.8098	15.3853	12.0278
$\varrho = 0.25$	12.6109	11.5342	11.5993	6.7369
$\varrho = 0.3$	5.4771	6.9491	8.5512	4.1729
$\varrho = 0.35$	2.0636	4.4354	5.8520	2.8217
$\varrho = 0.4$	1.1517	2.7233	4.0675	1.9828
$\varrho = 0.45$	1.2929	1.4487	2.8848	1.3878
$\varrho = 0.5$	1.4502	0.7271	1.8655	0.9864
$\varrho = 0.55$	1.2407	0.5503	1.1207	0.7566
$\varrho = 0.6$	0.7875	0.6352	0.7326	0.6242
$\varrho = 0.65$	0.3827	0.6798	0.4890	0.5083
$\varrho = 0.7$	0.2060	0.5966	0.2964	0.3879

Table 1: Theoretical spectral density $f(\varrho)$ computed from (4.1) for the Boolean model with different grain distributions, where $\lambda = 2.2064 \cdot 10^{-3}$ and $p = 0.5$.

5 APPENDIX

To close with we state some results on the asymptotic behaviour of the scaled fluctuation vector $(\Delta_n(x_1),...,\Delta_n(x_N))$ and its length, where the points $x_1,...,x_N \neq o$ are pairwise distinct and $\Delta_n(x) = \sqrt{\widehat{f}_n(x)} - \sqrt{f(x)}$.

Theorem 5.1 *Let the stationary Boolean model* $\Xi = \bigcup_n (M_n + Y_n)$ *with typical grain* M_0 *satisfy* $\mathbb{E}|M_0|^4 < \infty$, $\mathbb{E}\|M_0\|^d < \infty$ *and* $\mathbb{E}|M_0| \, \|M_0\|^{d+1} < \infty$. *Furthermore, let* $k(.)$ *be an almost-everywhere continuous function satisfying* (3.3) *and* (3.4). *If in addition* $b_n^{d+2} \, |W_n| \underset{n\to\infty}{\longrightarrow} 0$ *and* $b_n \min_{1\le j\le d} a_j^{(n)} \underset{n\to\infty}{\longrightarrow} \infty$, *then, for any collection of pairwise distinct points* $x_1,...,x_N \in \mathbb{R}^d \setminus \{o\}$,

$$\sqrt{b_n^d \, |W_n|} \, (\Delta_n(x_1),...,\Delta_n(x_N)) \underset{n\to\infty}{\Longrightarrow} \mathcal{N}(0, \tau^2 \, I_N) \; . \tag{5.1}$$

Here $\underset{n\to\infty}{\Longrightarrow}$ *stands for convergence in distribution (as* $n \to \infty$*) and* $\mathcal{N}(0,\tau^2 \, I_N)$ *denotes an* $N-$ *dimensional Gaussion vector having independent components with mean zero and variance* $\tau^2 = (2\,\pi)^d \int k^2(u)\,du/4$. *Consequently, by the continuous mapping theorem,*

$$\frac{b_n^d \, |W_n|}{\tau^2} \sum_{i=1}^{N} \Delta_n^2(x_i) \underset{n\to\infty}{\Longrightarrow} \chi_N^2 \; , \tag{5.2}$$

where the random variable χ_N^2 *is* χ^2-*distributed with* N *degrees of freedom.*

We only sketch roughly the idea of the proof of Theorem 5.1. In the first step we prove a CLT for $Y_n(x) = \sqrt{b_n^d|W_n|}\,(\widehat{f}_n(x) - \mathbb{E}\widehat{f}_n(x))$ in case of truncated grains $M_n^* = M_n \cap b(o,\rho)$, where $0 < \rho < \infty$ is fixed. This is shown by rewriting the kernel estimator (3.6) as a partial sum of $d-$dimensionally indexed random variables forming an $m-$dependent random field and then by applying a well-known CLT for this type of weakly dependent random fields. Passing to the limit as $\rho \to \infty$ is possible by employing Slutsky-type arguments and the above Theorem 3.2, see [5] for details in a similar situation. Relation (3.9) and both assumptions on the behaviour of the bandwidths allows to replace $\mathbb{E}\widehat{f}_n(x)$ by $f(x)$. Since

$$\frac{Y_n(x)}{\sqrt{\widehat{f}_n(x)} + \sqrt{f(x)}}$$

has the same limit distribution as $\sqrt{b_n^d \, |W_n|} \, \Delta_n(x)$ (which does not depend on $f(x)$), one gets relation (5.1) for $N = 1$. The multivariate version of this CLT is proved quite similarly using the well-known Cramér-Wold device. Notice that relation (5.2) suggests an asymptotic χ^2- goodness-of-fit test to check a hypothetical spectral density $f(x)$.

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