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# Markov chains - Assignment 5

## Exercise 1

Let $p, q \in [0,1]$ be arbitrary fixed numbers. For each of the following matrices, check if it can be considered as the transition matrix of a reversible Markov chain with stationary initial distribution $\boldsymbol{\alpha}$, where $\alpha_i > 0$ for all $i \in E$.

(a) $P = \begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}$ (b) $P = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$

(c) On $E = \{0, 1, 2, ...\}$ let $p_{01} = 1, p_{ii+1} = p, p_{ii-1} = q$ for $i \geq 1$, and $p_{ij} = 0$ else

(d) $p_{ij} = p_{ji}, i, j \in \{1, ..., l\}$

## Exercise 2

Let $\{X_n, n \geq 1\}$ be a Markov chain with transition matrix $\mathbf{P} = (p_{ij})$ and stationary initial distribution $\boldsymbol{\alpha}$, where $\alpha_i > 0 \; \forall \, i \in E$. Define matrix $\mathbf{Q}$ by the property that $\alpha_i \, q_{ij} = \alpha_j \, p_{ji}$ for all $i, j \in E$. Next, consider the sequence $\{X_{-n}, n \geq 1\}$ satisfying:

$$\begin{aligned} & \mathbb{P}(X_{-1} = i_1, X_{-2} = i_2, ..., X_{-k} = i_k \mid X_0 = i, X_2 = j_2, ..., X_n = j_n) \\ = \; & \mathbb{P}(X_{-1} = i_1, X_{-2} = i_2, ..., X_{-k} = i_k \mid X_0 = i) \\ = \; & q_{ii_1} \cdot q_{i_1 i_2} \cdot ... \cdot q_{i_{k-1} i_k} \end{aligned}$$

$\forall \, k \geq 1, n \geq 1, i, i_1, ..., i_k, j_1, ..., j_n \in E$.

(a) Show that $\mathbf{Q}$ is again a stochastic matrix.

(b) Prove that the sequence $\{X_n, n \in \mathbb{Z}\}$ is a homogeneous Markov chain with transition matrix $\mathbf{P}$ and one-dimensional marginal distribution $\boldsymbol{\alpha}$, i.e., show that for all $k \leq n \in \mathbb{Z}$ it holds that

$$\mathbb{P}(X_k = i) = \alpha_i \quad \text{and} \quad \mathbb{P}(X_k = i_k, ..., X_n = i_n) = \alpha_{i_k} \cdot p_{i_k i_{k+1}} \cdot ... \cdot p_{i_{n-1} i_n}.$$

## Exercise 3

Consider the state space

$$E = \{(a, b, c) : a + b + c = 0 \text{ und } a, b, c \in \{-9, -8, ..., 8, 9\}\}.$$

Construct a reversible Markov chain on $E$ such that the limit distribution $\pi$ of this Markov chain is equal to the uniform distribution on $E$.

## Exercise 4

Consider a linear congruential generator (LCG).

(a) Determine the periodicity of the LCG having seed $z_0 = 1$ and the parameters

| | | | |
|---|---|---|---|
| (i) | $m_1 = 512,$ | $a_1 = 51,$ | $c_1 = 0,$ |
| (ii) | $m_2 = 131,$ | $a_2 = 5,$ | $c_2 = 0,$ |
| (iii) | $m_3 = 18,$ | $a_3 = 9,$ | $c_3 = 5$ or |
| (iv) | $m_4 = 12,$ | $a_4 = 2,$ | $c_4 = 1.$ |

(b) Write an implementation of the LCGs given in (a)-(i) and (a)-(ii) and print out the first 10 pseudo-random numbers generated by your LCGs.