Simulation of typical Poisson-Voronoi-Cox-Voronoi cells with applications to telecommunication network modelling

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Motivation

Telecommunication networks

Infrastructure of Paris
Motivation

Agenda

- Motivation
- Introduction of PVCVT
- Palm representation of the typical cell
- Simulation algorithm
- Related cost functionals
- Conclusion and outlook
System of main roads
System of main roads and side streets
Motivation
Modelling

System of roads with network components
Motivation
Modelling

Serving zones
Motivation

Aims

- Modelling by random-geometric networks
  - Description of networks by few parameters
  - Simulation of present and future network design scenarios
  - Cost analysis and risk evaluation
- Models necessary
  - Streets => geometry model
  - Network components => equipment model
Introduction of PVCVT

PLCVT

Geometry model PLT
Introduction of PVCVT

PVCVT

Geometry model PVT
A sequence $\tau = \{C_n\}_{n \in \mathbb{N}}$ of convex compact polygons $C_n \in \mathbb{R}^2$ is called a (deterministic) tessellation of $\mathbb{R}^2$ if:

- $C_n \neq \emptyset \forall n \in \mathbb{N}$, $\text{int} C_n \cap \text{int} C_m = \emptyset \forall n \neq m$
- $\bigcup_{n \in \mathbb{N}} C_n = \mathbb{R}^2$
- $\sum_{n \in \mathbb{N}} 1\{C_n \cap K \neq \emptyset\} < \infty$ for all compact sets $K \subset \mathbb{R}^2$

The sets $C_n$ are called the cells of $\tau$. 

Introduction of PVCVT

(Deterministic) Tessellation
Introduction of PVCVT

Associated point and intensity

- Let $C_n$ be a cell of a tessellation. Then the point $\alpha(C_n)$ is called associated point of $C_n$ if
  - $\alpha(C_n) \in C_n$
  - $\alpha(C_n + x) = \alpha(C_n) + x$

- Let $T$ be a stationary random tessellation. Then
  $$\lambda_T := \mathbb{E}\#\{n : \alpha(\Xi_n) \in [0, 1)^2\}$$

is called the intensity of $T$. 
Introduction of PVCVT

Cells of $T$ are formed with respect to a point process $\{X_n\}_{n \in \mathbb{N}}$

$$\Xi_n = \left\{ x \in \mathbb{R}^2 : \| x - X_n \| \leq \| x - X_m \| \ \forall n \neq m \right\}$$

$$= \bigcap_{m \in \mathbb{N} : m \neq n} H(X_n, X_m)$$

with half spaces

$$H(X_n, X_m) = \left\{ x \in \mathbb{R}^2 : \| x - X_n \| \leq \| x - X_m \| \right\}$$

Possible associated point: $\alpha(\Xi_n) = X_n$

$X_p$ Poisson point process $\Rightarrow \tau_{X_p}$ Poisson-Voronoi tessellation (PVT)
Introduction of PVCVT

Realization of a Poisson process and corresponding Voronoi tessellation
Introduction of PVCVT

**PVCVT**

- Consider Cox process $X_c$ with (random) intensity measure $\Lambda_{X_c}$
  - $\Lambda_{X_c}(B) = \lambda_\ell \nu_1 (B \cap \tau_{X_p}^{(1)})$
  - $\tau_{X_p}^{(1)}$ process of edges of PVT $\tau_{X_p}$
  - Linear Poisson processes on the the edges of $\tau_{X_p}$ with intensity $\lambda_\ell$
- PVT induced by $X_c \Rightarrow$ Poisson-Voronoi-Cox-Voronoi tessellation (PVCVT) $\tau_{X_c}$
- Intensity $\lambda_c = 2\lambda_\ell \sqrt{\lambda_p}$
Introduction of PVCVT

Realization of a PVCVT
Palm representation of the typical cell

Aims

- Typical cell <=> cell chosen randomly out of all cells available
- According to Palm mark distribution
- Important
  - Serving zone (geometric characteristics)
  - Estimation of cost functionals
- Simulation algorithm based on suitable Palm representation
  - For generating Cox process
  - One-to-one correspondence to Voronoi tessellation
Palm representation of the typical cell

**Palm distribution**

Let $\tau_X$ be a random tessellation. The Palm distribution $\mathbb{P}^*_\tau_X$ is defined as

$$
\mathbb{P}^*_\tau_X(A \times G) = \frac{\mathbb{E} \# \{ n : \alpha(\Xi_n) \in [0, 1)^2, T- \alpha(\Xi_n) \in A, \Xi_n-\alpha(\Xi_n) \in G \} }{\lambda_{\tau_X}}
$$

where $A \in \mathcal{T}$, $G \in \mathcal{B}(\mathcal{F}) \cap \mathcal{P}^{(o)}$ with

- $\mathcal{P}^{(o)}$ family of all convex polytopes with associated point at $o$
- $\mathcal{F}$ family of all closed sets in $\mathbb{R}^2$
Palm representation of the typical cell

Palm mark distribution

- The Palm mark distribution $\mathbb{P}^*$ of $\tau_X$ is defined as

$$\mathbb{P}^*(G) = \frac{\mathbb{E}\#\{n : \alpha(\Xi_n) \in [0, 1)^2, \Xi_n - \alpha(\Xi_n) \in G\}}{\lambda_{\tau_X}}$$

- A random polygon with distribution $\mathbb{P}^*$ is called the typical cell of the tessellation $\tau_X$

- Local characterisation

$$\mathbb{P}^*(G) = \lim_{\epsilon \to 0} \mathbb{P}(\Xi^{(o)} - \alpha(\Xi^{(o)}) \in G \mid \alpha(\Xi^{(o)}) \in B(o, \epsilon))$$
Palm representation of the typical cell

- PVT $\tau_{X_p}$ generated by $X_p$
- PVCVT $\tau_{X_c}$ generated by $X_c$
- $h : \mathcal{P}^0 \rightarrow \mathbb{R}_+$ measurable

\[
\int h(\Xi^*) \mathbb{P}^*_{X_c} = \frac{1}{\mathbb{E}_\nu_1(\partial\Xi^*)} \int \int h(\Xi_c(u)) du \mathbb{P}^*_{X_p}
\]

where

- $\mathbb{P}^*_{X_c}, \mathbb{P}^*_{X_p}$ palm mark distributions of $\tau_{X_c}$ and $\tau_{X_p}$
- $\Xi^*, \Xi^*$ typical cells of $\tau_{X_c}$ and $\tau_{X_p}$
- $\Xi_c(u)$ cell of $\tau_{X_c}$ with nucleus $u \in \partial\Xi^*$ given $X_c$ and $\tau_{X_p}$
Palm representation of the typical cell

- \( \mathbb{E} \nu_1(\partial \Xi^*_p) = \frac{4}{\sqrt{\lambda_p}} \)

- \( \mathbb{E}_{X_p} \int_{\partial \Xi^*_p} h(\Xi_c(u)) \, du = \mathbb{E}_{X_p} [\nu_1(\partial \Xi^*_p) \mathbb{E} h(\Xi_c(Z) | X_p)] \)

- where \( Z \in \partial \Xi^*_p \ a.s. \) and \( Z \sim U(\partial \Xi^*_p) \)

- \( \int h(\Xi^*_c) \mathbb{P}^*_X \Xi_c = \frac{\sqrt{\lambda_p}}{4} \mathbb{E}_{X_p} [\nu_1(\partial \Xi^*_p) \mathbb{E} h(\Xi_c(Z) | X_p)] \)

- Representation induces simulation algorithm

- No direct simulation of the typical cell

- Information about characteristics w.r.t
  - Moments
  - Distribution
Simulation algorithm

General considerations

- Averaging over large sampling windows
  - Edge effects
  - Memory and runtime problems
- Direct simulation of typical cell avoids these problems
  - Simulation not clear/very complicated
  - Usage of Palm representation
Simulation algorithm

Concept

Simulation algorithm for distributional properties of the typical cell of PVCVT

- Radial simulation
- Simulate PVT $\tau_{X_p}$ w.r.t. its Palm distribution
- Put points on the edges, one additional (random) point $Z$ on $\partial \Xi^*_p$
- Construct cell $\Xi_c(Z)$ around $Z$
- Weight characteristic $h(\Xi_c(Z))$ by $\nu_1(\partial \Xi^*_p) \sqrt{\lambda_p}/4$
- Estimation by sample means of weighted characteristic
Simulation algorithm

Typical cell of PVT

Simulate $X_p \cup \{o\}$ radially until typical cell $\Xi^*_p$ of $\tau X_p$ can be constructed.
Simulation algorithm

First points on edges

Simulate $N + 1$ points uniform random on $\partial \Xi_p^*$, where

$$N \sim \text{Poi}(\lambda \nu_1(\partial \Xi_p^*))$$
Simulate further cells of $\tau_{X_p}$ and place $N_i$ points on the new edges $s_i$, $N_i \sim \text{Poi}(\lambda \nu_1(s_i))$
Simulation algorithm

Initial cell

Construct initial cell
Simulation algorithm

Further intersections of initial cell

Simulate further points and cells, intersect initial cell with bisectors
Simulation algorithm
Stopping criterion

Stopping criterion
Simulation algorithm

Final cell

Realization of cell $\Xi_c(Z)$
Simulation algorithm

Functionals of interest

- Computation of distributions (and moments) of
  - Number of vertices
  - Perimeter
  - Area
- Computation of related cost functionals
  - Shortest path length
  - Subscriber line length
  - Capacity
- Comparison to results for PLCVT
Simulation algorithm
First numerical results

Histogram of perimeter length for $\sqrt{\lambda_p} = 0.0625$ and $\lambda_\ell = 0.0125$
Related cost functionals

Shortest path length

- Two Cox processes $X_H$ and $X_L$
- (Spatial) intensities $\lambda_H$ and $\lambda_L$
- Connected to same PVT $\tau_{X_p}$

- Distance from a point of $X_L$ to its nearest neighbor of $X_H$
  - Along the edges of $\tau_{X_p}$
  - (Random) shortest path length $c_{LH}$
  - Mean shortest path length $\mathbb{E}_{c_{LH}} = \mathbb{E}_{X_L} c(P(o, N(o)))$
Related cost functionals

*Shortest path length*

Realisation of $X_L$ and $X_H$ with serving zones
Related cost functionals

Shortest path length

\[ \mathbb{E}_{X_L} c(P(o, N(o))) = \frac{1}{\mathbb{E}_{X_H} \nu_1(L(\Xi^*_H))} \mathbb{E}_{X_H} \int_{L(\Xi^*_H)} c(P(u, o))du \]

- \( \Xi^*_H \) typical Voronoi cell of \( X_H \)
- \( L(\Xi^*_H) = \tau^{(1)}_{X_p} \cap \Xi^*_H \)
- \( h(\Xi^*_H) = \int_{L(\Xi^*_H)} c(P(u, o))du \Rightarrow \text{Palm representation for} \ \Xi^*_H \ \text{usable} \)

\[ \mathbb{E}_{X_L} c(P(o, N(o))) = \frac{\lambda_H}{8} \mathbb{E}_{X_p}[\nu_1(\partial \Xi^*_p)\mathbb{E}(\int_{L(\Xi^*_H(Z))} c(P(u, o))du|X_p)] \]

where \( Z \sim U(\partial \Xi^*_p) \)
Conclusion and outlook

- PVCVT
  - Model for road system
  - Palm representation of typical cell
  - Simulation algorithm
  - Palm representation of shortest path length

- Further steps
  - Systematic numerical analysis (scaling invariance)
  - Usage for estimation of cost functionals
  - Approximation formulae
  - Comparison to other geometry models (PLCVT, PDCVT, ...)

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