Simulation of typical Poisson-Voronoi-Cox-Voronoi cells

with applications to telecommunication network modelling

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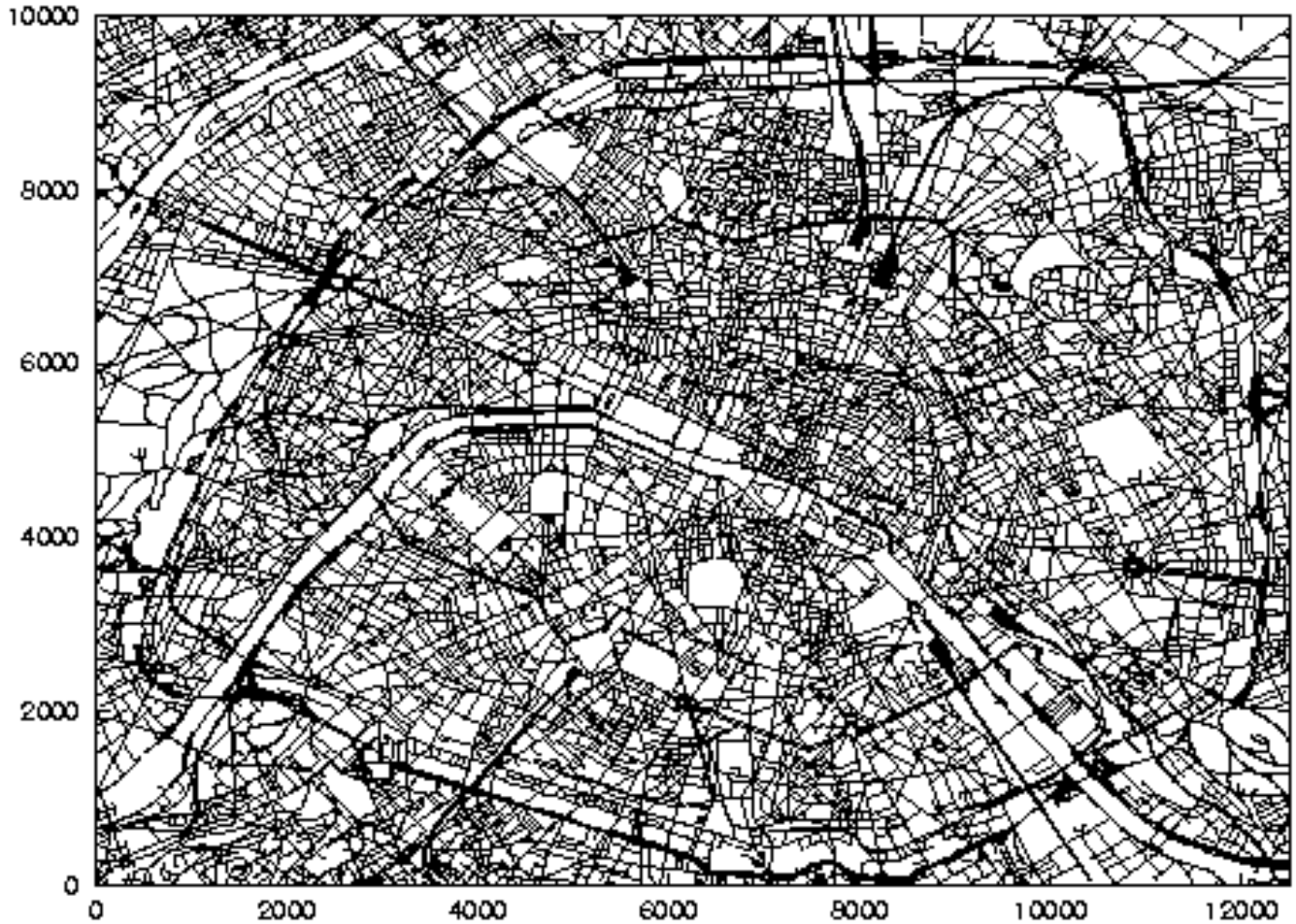
Motivation
Telecommunication networks

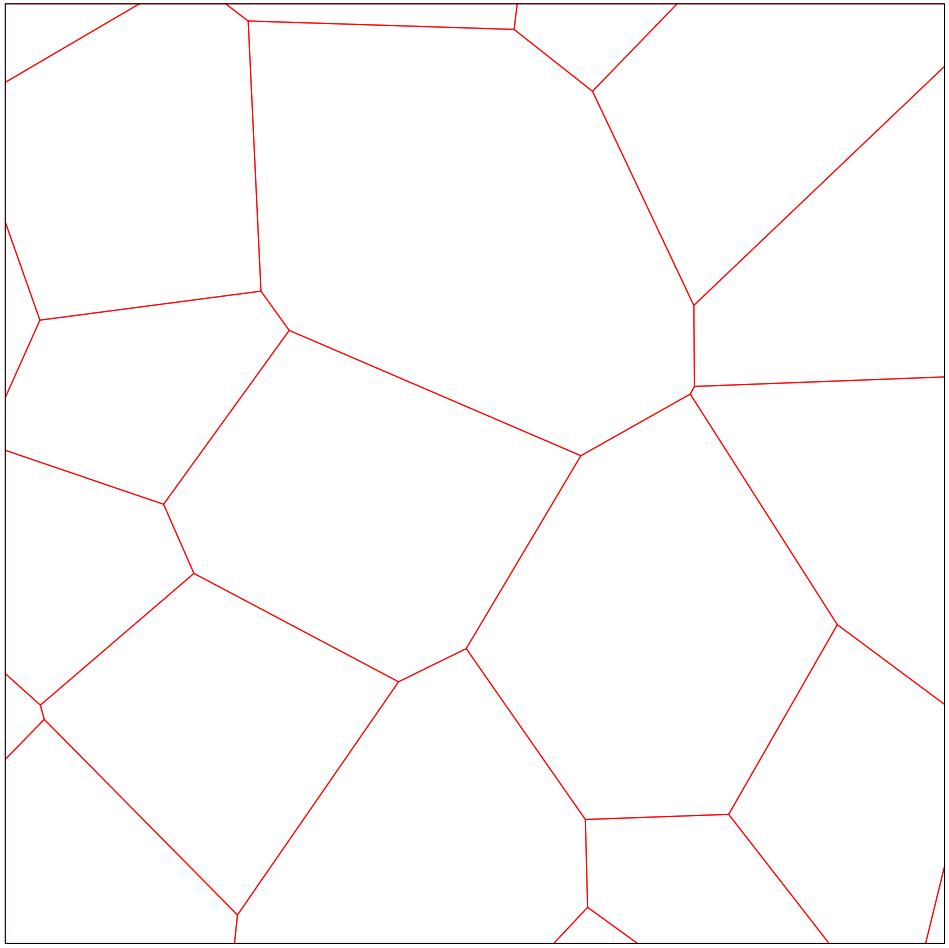


Infrastructure of Paris

Motivation
Agenda

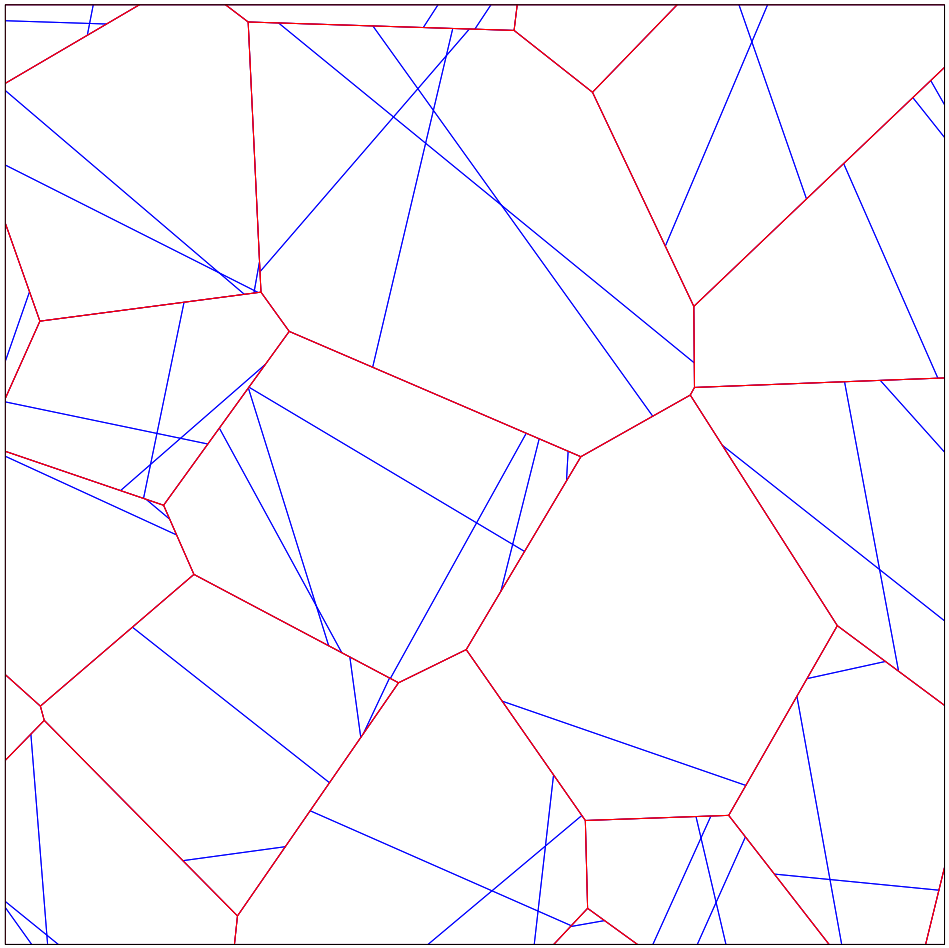
- Motivation
- Introduction of PVCVT
- Palm representation of the typical cell
- Simulation algorithm
- Related cost functionals
- Conclusion and outlook

Motivation
Modelling



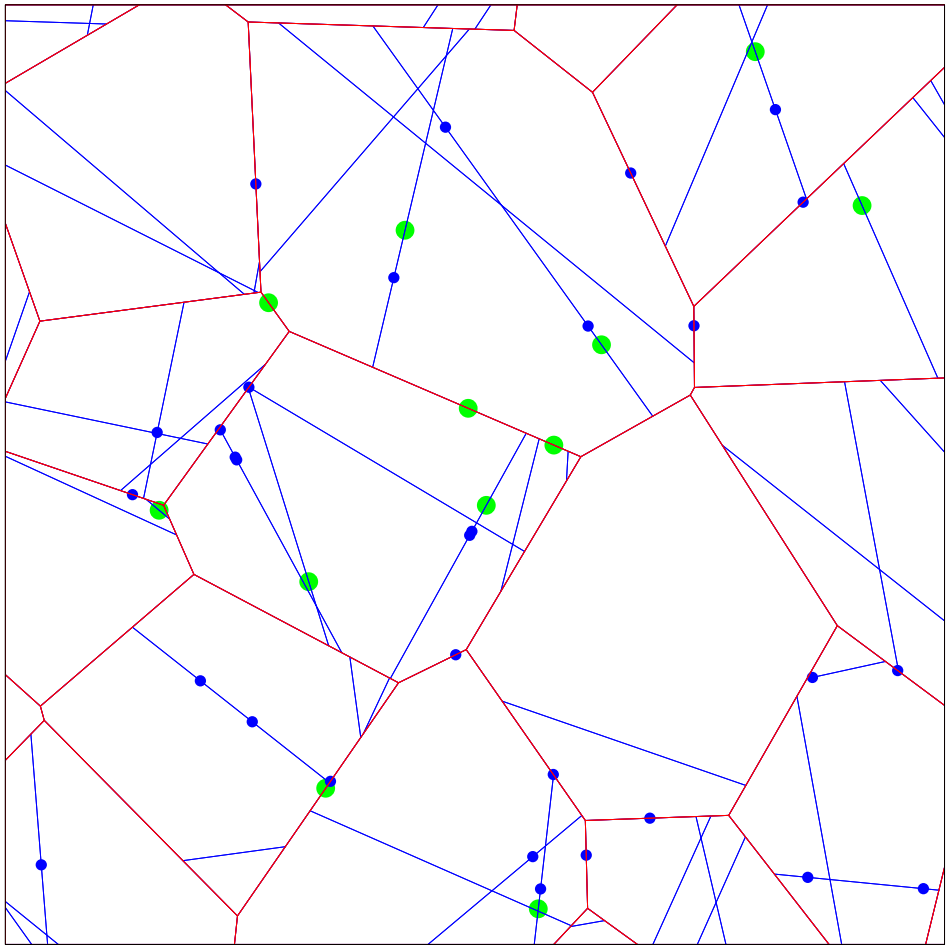
System of main roads

Motivation
Modelling



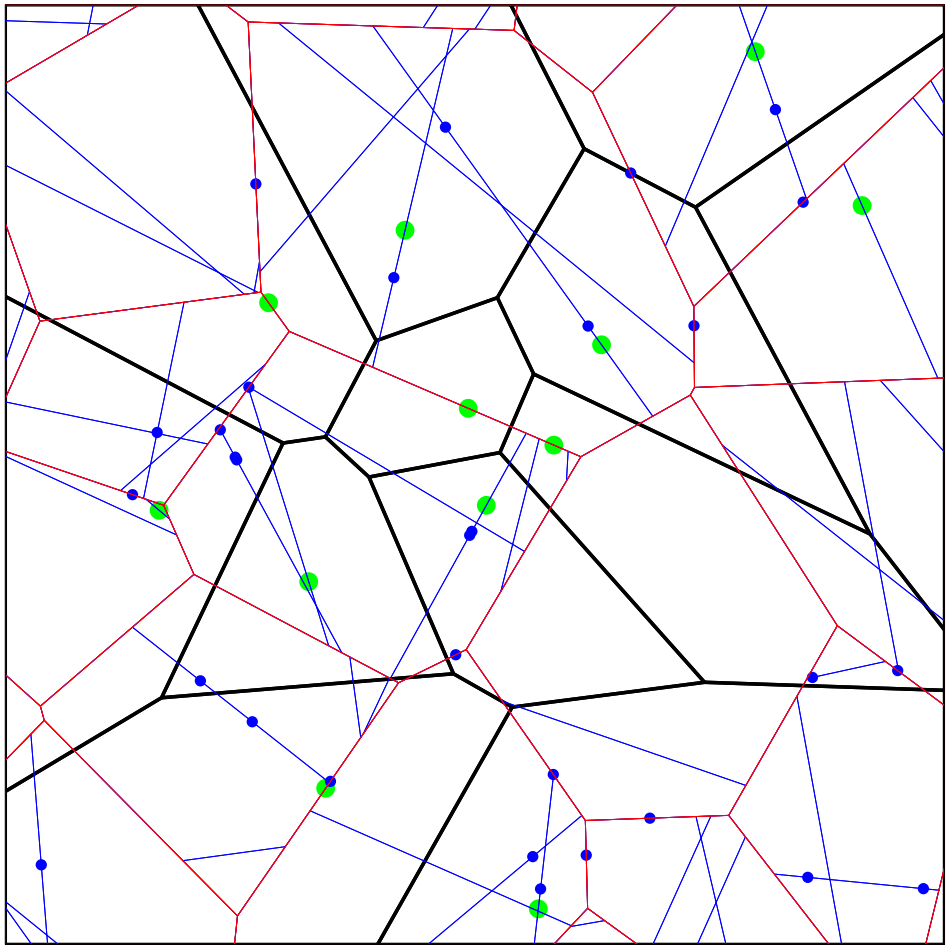
System of main roads and side streets

Motivation
Modelling



System of roads with network components

Motivation
Modelling

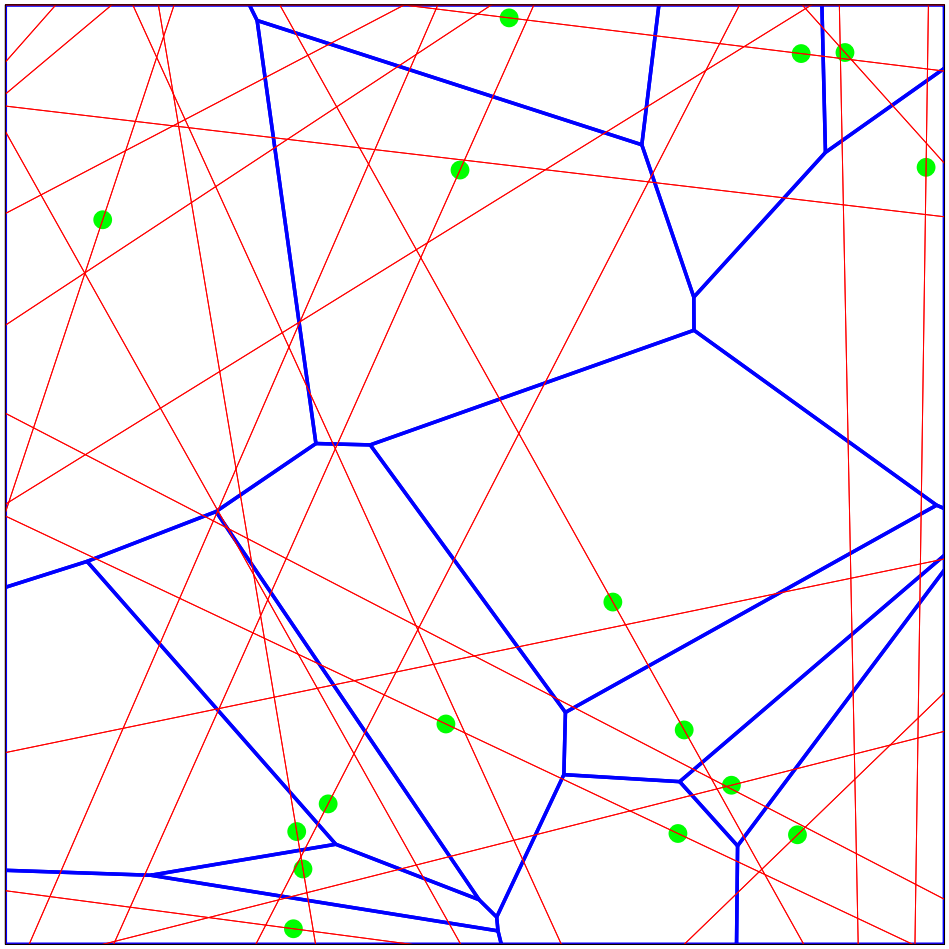


Serving zones

Motivation
Aims

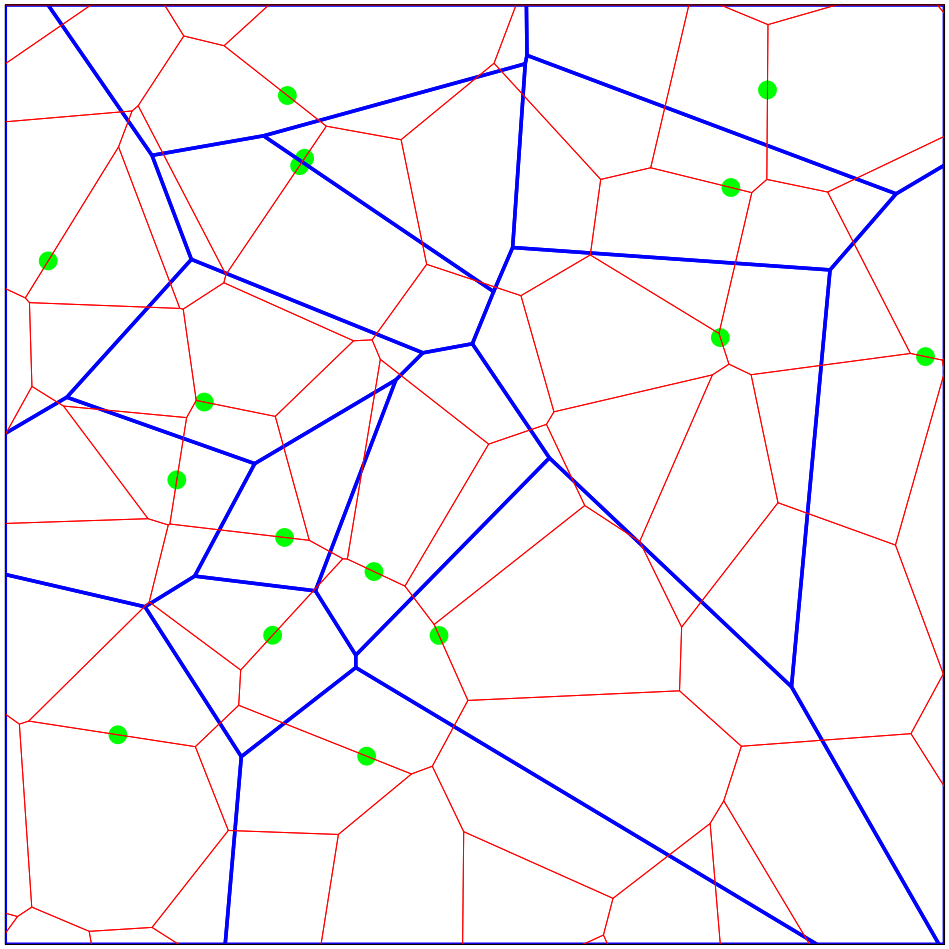
- Modelling by random-geometric networks
  - Description of networks by few parameters
  - Simulation of present and future network design scenarios
  - Cost analysis and risk evaluation
- Models necessary
  - Streets => geometry model
  - Network components => equipment model

Introduction of PVCVT
PLCVT



Geometry model PLT

Introduction of PVCVT
PVCVT



Geometry model PVT

Introduction of PVCVT
(Deterministic) Tessellation

- A sequence $\tau = \{C_n\}_{n\in\mathbb{N}}$ of convex compact polygons $C_n \in \mathbb{R}^2$ is called a (deterministic) tessellation of $\mathbb{R}^2$ if
  - $C_n \neq \emptyset \,\forall\, n \in \mathbb{N}$, $\mathrm{int} C_n \cap \mathrm{int} C_m = \emptyset \,\forall\, n \neq m$
  - $\bigcup\limits_{n\in\mathbb{N}} C_n = \mathbb{R}^2$
  - $\sum\limits_{n\in\mathbb{N}} \mathbf{1}_{\{C_n \cap K \neq \emptyset\}} < \infty$ for all compact sets $K \subset \mathbb{R}^2$
- The sets C_n are called the cells of τ

Introduction of PVCVT
Associated point and intensity

- Let C_n be a cell of a tessellation. Then the point $\alpha(C_n)$ is called associated point of C_n if
  - $\alpha(C_n) \in C_n$
  - $\alpha(C_n + x) = \alpha(C_n) + x$
- Let T be a stationary random tessellation. Then

$$\lambda_T := \mathbb{E}\#\{n : \alpha(\Xi_n) \in [0,1)^2\}$$

is called the intensity of T.

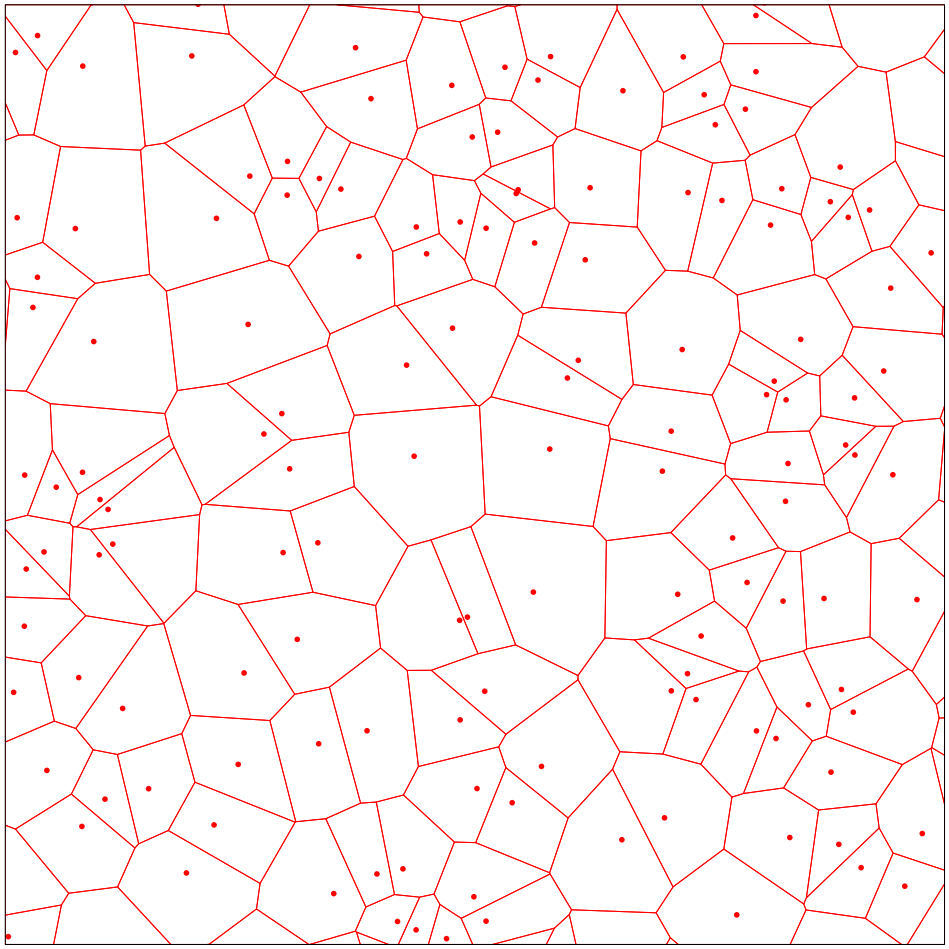
Introduction of PVCVT
PVT

- Cells of T are formed with respect to a point process $\{X_n\}_{n\in\mathbb{N}}$

$$\begin{aligned}\Xi_n &= \{x \in \mathbb{R}^2 : \| x - X_n \| \leq \| x - X_m \| \ \forall n \neq m\} \\ &= \bigcap_{m\in\mathbb{N}:m\neq n} H(X_n, X_m)\end{aligned}$$

  with half spaces
  $H(X_n, X_m) = \{x \in \mathbb{R}^2 : \| x - X_n \| \leq \| x - X_m \|\}$
- Possible associated point: $\alpha(\Xi_n) = X_n$
- X_p Poisson point process => τ_{X_p} Poisson-Voronoi tessellation (PVT)

Introduction of PVCVT
PVT

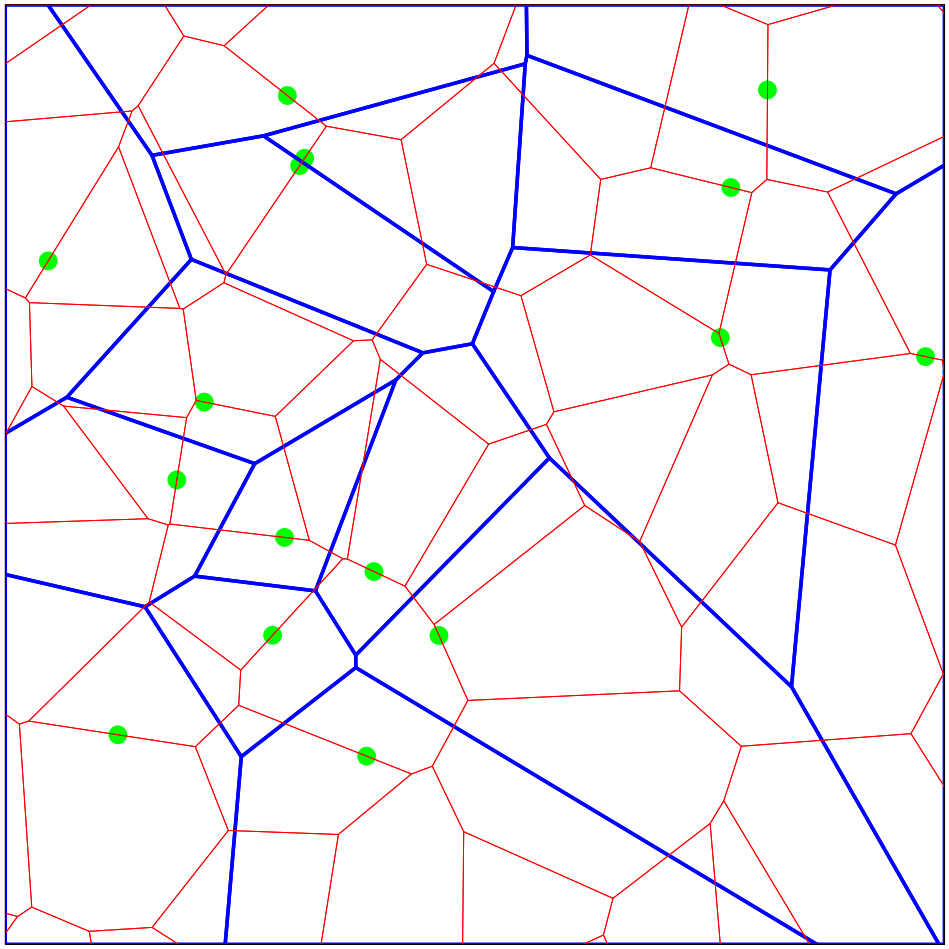


Realization of a Poisson process and corresponding Voronoi tessellation

Introduction of PVCVT
PVCVT

- Consider Cox process X_c with (random) intensity measure Λ_{X_c}
  - $\Lambda_{X_c}(B) = \lambda_\ell \nu_1(B \cap \tau^{(1)}_{X_p})$
  - $\tau^{(1)}_{X_p}$ process of edges of PVT τ_{X_p}
  - Linear Poisson processes on the the edges of τ_{X_p} with intensity λ_ℓ
- PVT induced by X_c => Poisson-Voronoi-Cox-Voronoi tessellation (PVCVT) τ_{X_c}
- Intensity $\lambda_c = 2\lambda_\ell \sqrt{\lambda_p}$

Introduction of PVCVT
PVCVT



Realization of a PVCVT

Palm representation of the typical cell
Aims

- Typical cell <=> cell chosen randomly out of all cells available
- According to Palm mark distribution
- Important
  - Serving zone (geometric characteristics)
  - Estimation of cost functionals
- Simulation algorithm based on suitable Palm representation
  - For generating Cox process
  - One-to-one correspondence to Voronoi tessellation

Palm representation of the typical cell
Palm distribution

Let τ_X be a random tessellation. The Palm distribution $\mathbb{P}^*_{\tau_X}$ is defined as

$$\mathbb{P}^*_{\tau_X}(A \times G) = \frac{\mathbb{E}\#\{n\colon \alpha(\Xi_n) \in [0,1)^2, T - \alpha(\Xi_n) \in A,\ \Xi_n - \alpha(\Xi_n) \in G\}}{\lambda_{\tau_X}}$$

where $A \in \mathcal{T}$, $G \in \mathcal{B}(\mathcal{F}) \cap \mathcal{P}^{(o)}$ with

- $\mathcal{P}^{(o)}$ family of all convex polytopes with associated point at o
- $\mathcal{F}$ family of all closed sets in $\mathbb{R}^2$

Palm representation of the typical cell
Palm mark distribution

- The Palm mark distribution $\mathbb{P}^*$ of τ_X is defined as

$$\mathbb{P}^*(G) = \frac{\mathbb{E}\#\{n : \alpha(\Xi_n) \in [0,1)^2, \Xi_n - \alpha(\Xi_n) \in G\}}{\lambda_{\tau_X}}$$

- A random polygon with distribution $\mathbb{P}^*$ is called the typical cell of the tessellation τ_X
- Local characterisation

$$\mathbb{P}^*(G) = \lim_{\epsilon \to 0} \mathbb{P}(\Xi^{(o)} - \alpha(\Xi^{(o)}) \in G \,\big|\, \alpha(\Xi^{(o)}) \in B(o,\epsilon))$$

Palm representation of the typical cell

- PVT τ_{X_p} generated by X_p
- PVCVT τ_{X_c} generated by X_c
- $h : \mathcal{P}^0 \longrightarrow \mathbb{R}_+$ measurable

$$\int h(\Xi_c^*)\mathbb{P}_{X_c}^* = \frac{1}{\mathbb{E}\nu_1(\partial\Xi_p^*)} \int \int_{\partial\Xi_p^*} h(\Xi_c(u))du\mathbb{P}_{X_p}^*$$

where

- $\mathbb{P}_{X_c}^*$, $\mathbb{P}_{X_p}^*$ palm mark distributions of τ_{X_c} and τ_{X_p}
- Ξ_c^*, Ξ_p^* typical cells of τ_{X_c} and τ_{X_p}
- $\Xi_c(u)$ cell of τ_{X_c} with nucleus $u \in \partial\Xi_p^*$ given X_c and τ_{X_p}

Palm representation of the typical cell

- $\mathbb{E}\nu_1(\partial\Xi_p^*) = 4/\sqrt{\lambda_p}$
- $\mathbb{E}_{X_p}\int_{\partial\Xi_p^*} h(\Xi_c(u))du = \mathbb{E}_{X_p}[\nu_1(\partial\Xi_p^*)\mathbb{E}h(\Xi_c(Z)|X_p)]$
- where $Z \in \partial\Xi_p^*$ a.s. and $Z \sim U(\partial\Xi_p^*)$

$$\int h(\Xi_c^*)\mathbb{P}_{X_c}^* = \frac{\sqrt{\lambda_p}}{4}\mathbb{E}_{X_p}[\nu_1(\partial\Xi_p^*)\mathbb{E}h(\Xi_c(Z)|X_p)]$$

- Representation induces simulation algorithm
- No direct simulation of the typical cell
- Information about characteristics w.r.t
  - Moments
  - Distribution

Simulation algorithm
General considerations

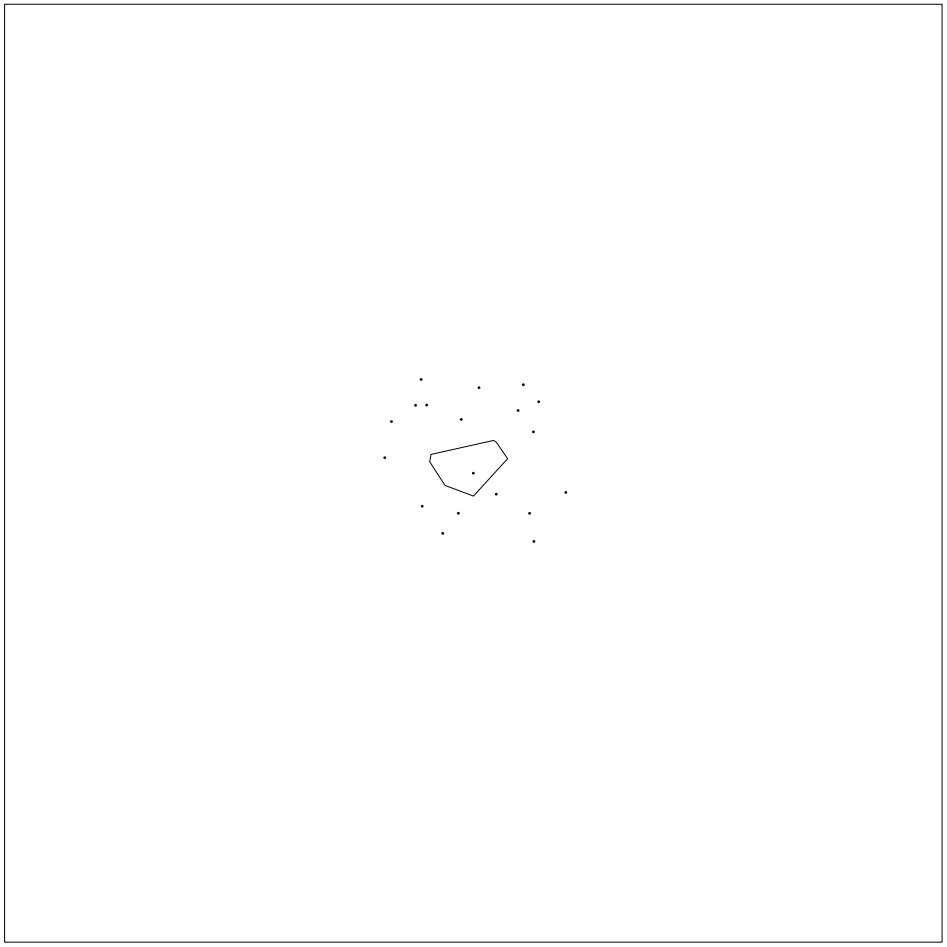
- Averaging over large sampling windows
  - Edge effects
  - Memory and runtime problems
- Direct simulation of typical cell avoids these problems
  - Simulation not clear/ very complicated
  - Usage of Palm representation

Simulation algorithm
Concept

Simulation algorithm for distributional properties of the typical cell of PVCVT

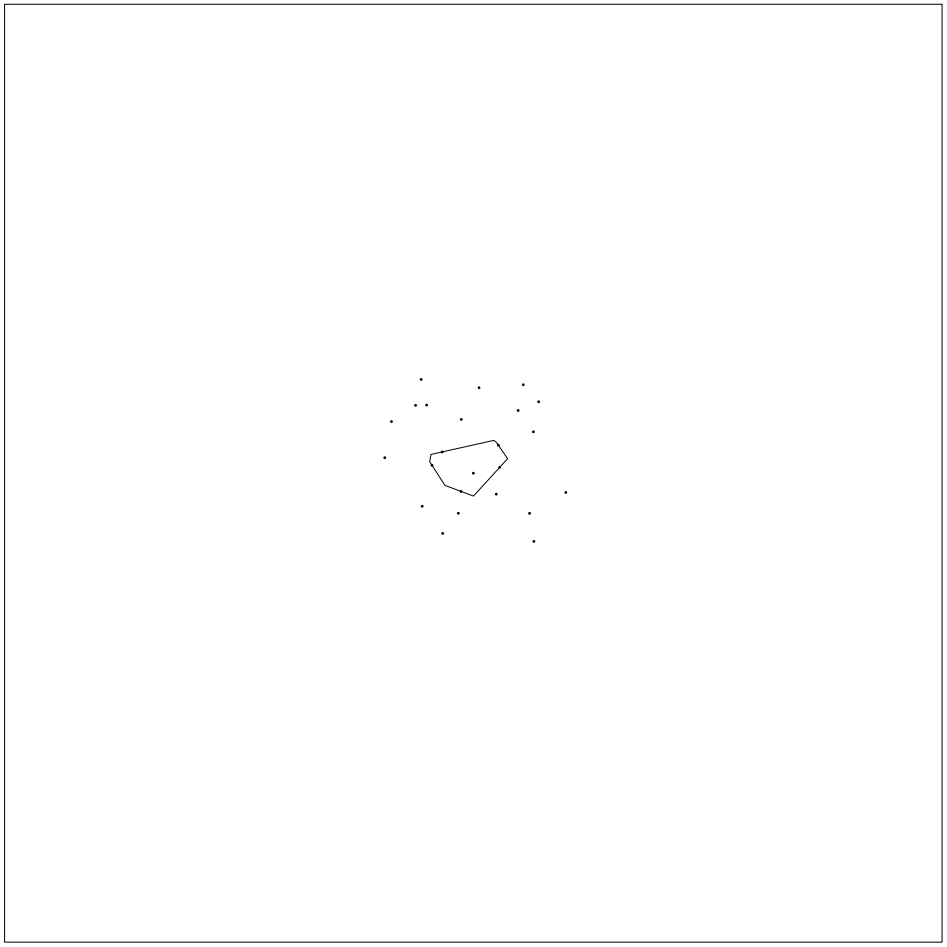
- Radial simulation
- Simulate PVT τ_{X_p} w.r.t. its Palm distribution
- Put points on the edges, one additional (random) point Z on $\partial\Xi_p^*$
- Construct cell $\Xi_c(Z)$ around Z
- Weight characteristic $h(\Xi_c(Z))$ by $\nu_1(\partial\Xi_p^*)\sqrt{\lambda_p}/4$
- Esimation by sample means of weighted characteristic

Simulation algorithm
Typical cell of PVT



Simulate $X_p \cup \{o\}$ radially until typical cell Ξ_p^* of τ_{X_p} can be constructed

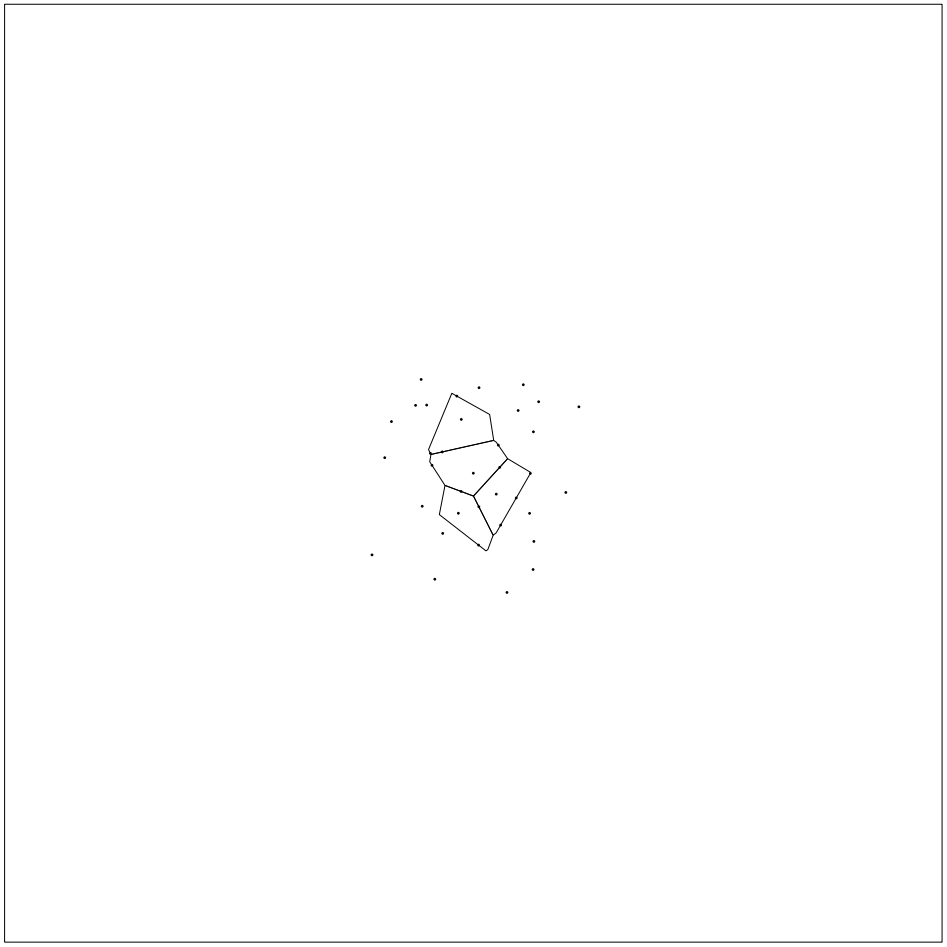
Simulation algorithm
First points on edges



Simulate $N+1$ points uniform random on $\partial\Xi_p^*$, where

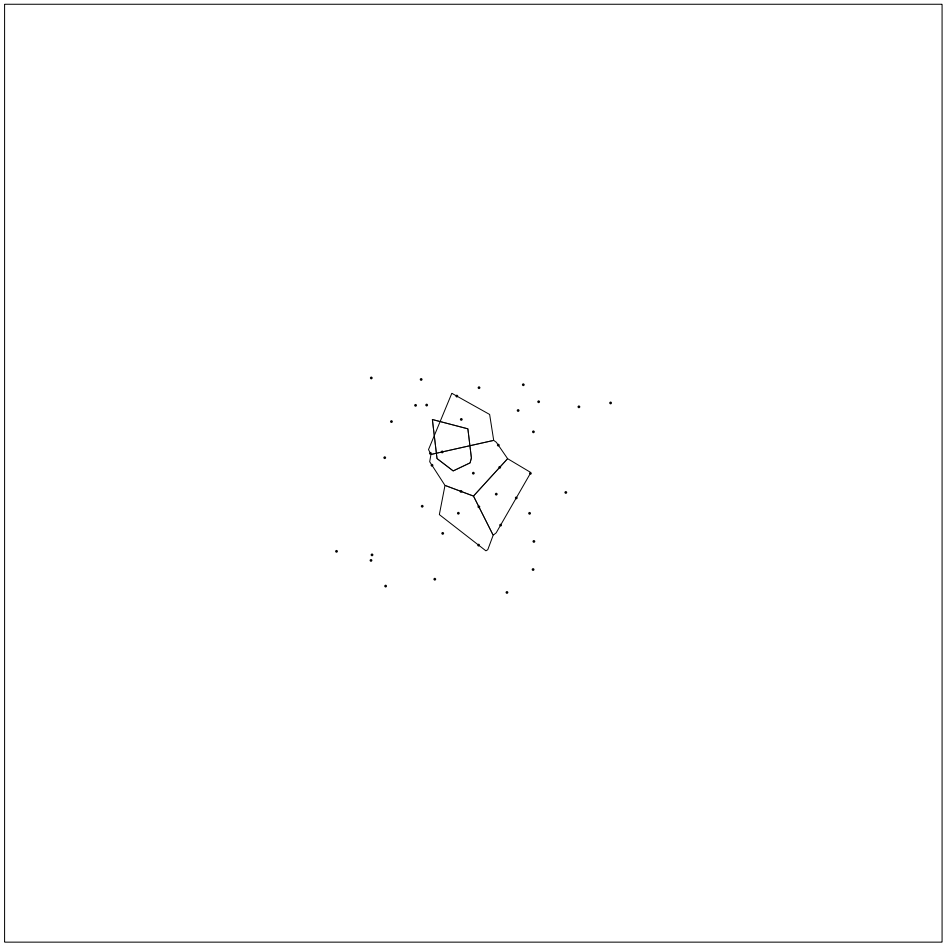
$$N \sim \mathrm{Poi}(\lambda_\ell \nu_1(\partial\Xi_p^*))$$

Simulation algorithm
Further points



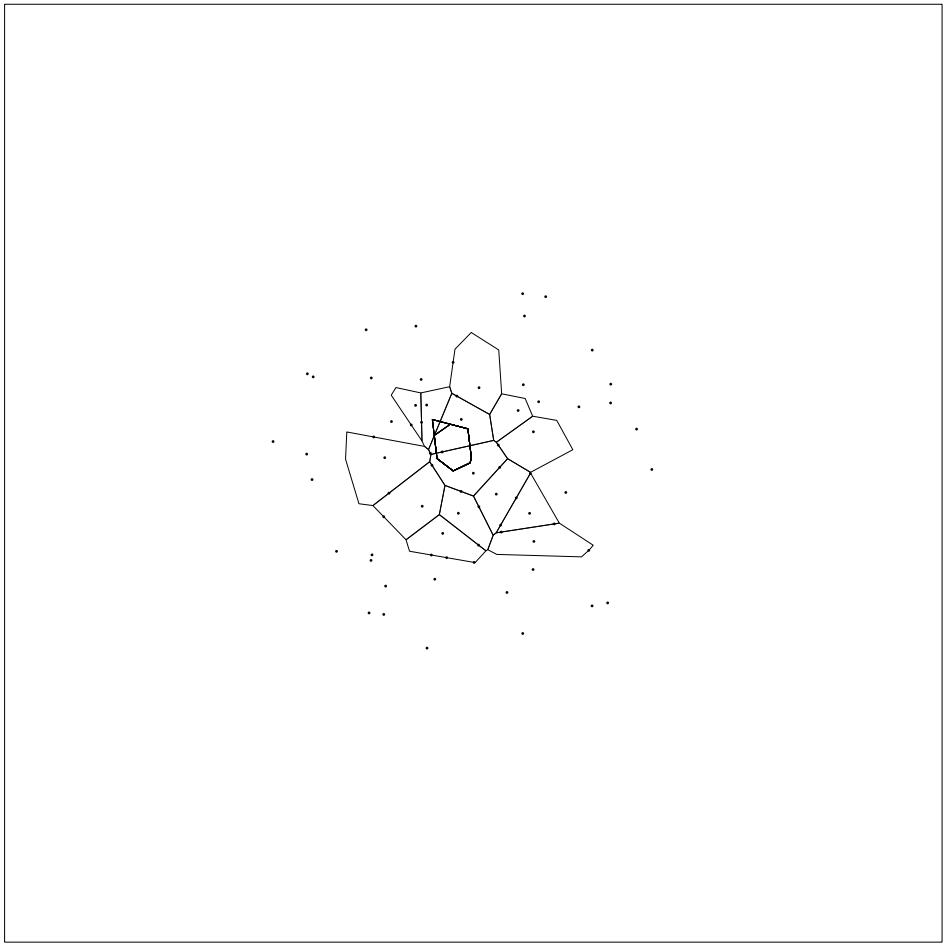
Simulate further cells of τ_{X_p} and place N_i points on the new edges s_i, $N_i \sim \mathrm{Poi}(\lambda_\ell \nu_1(s_i))$

Simulation algorithm
Initial cell



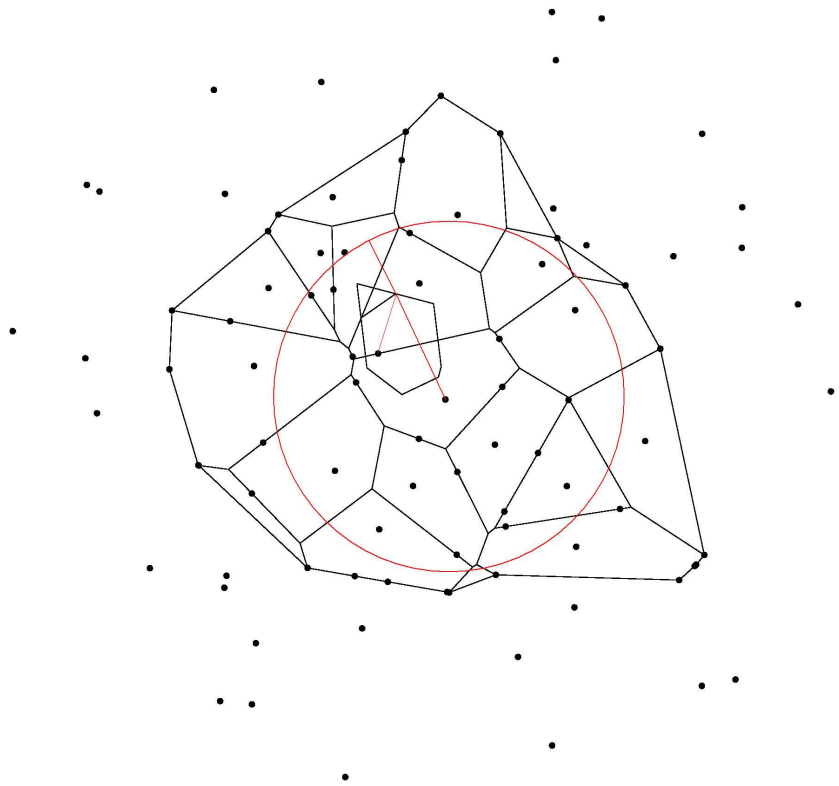
Construct initial cell

Simulation algorithm
Further intersections of initial cell



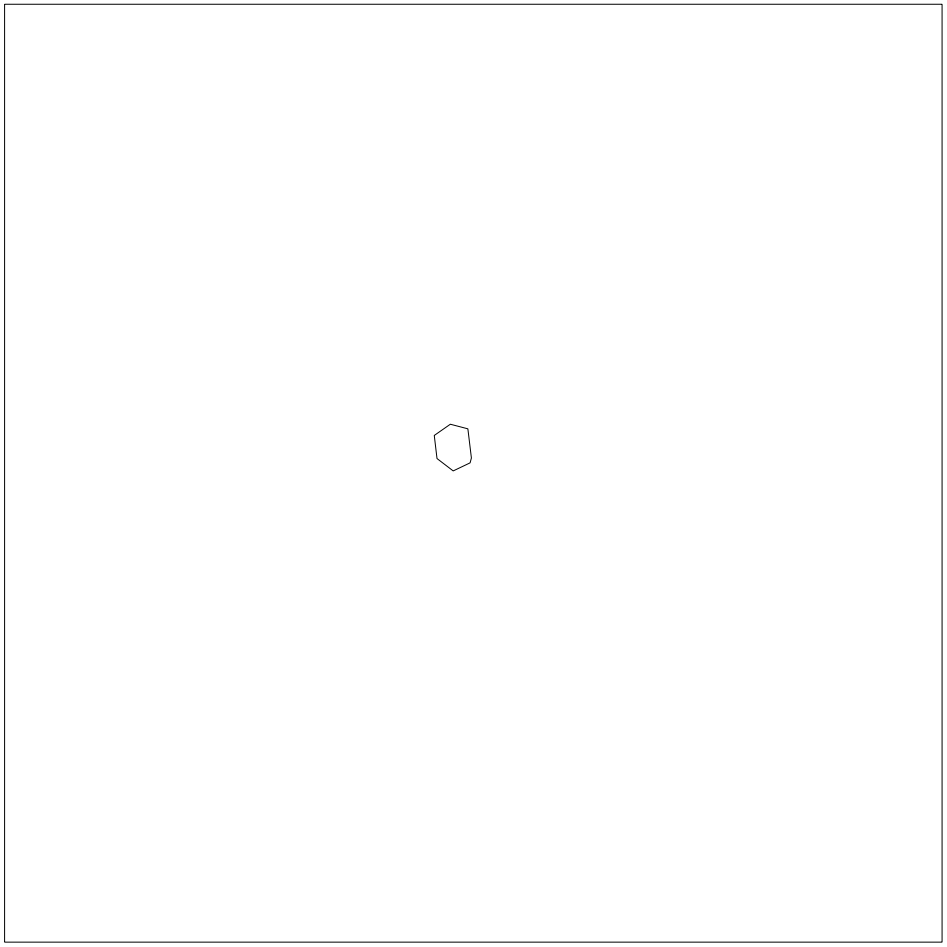
Simulate further points and cells, intersect initial cell with bisectors

Simulation algorithm
Stopping criterion



Stopping criterion

Simulation algorithm
Final cell

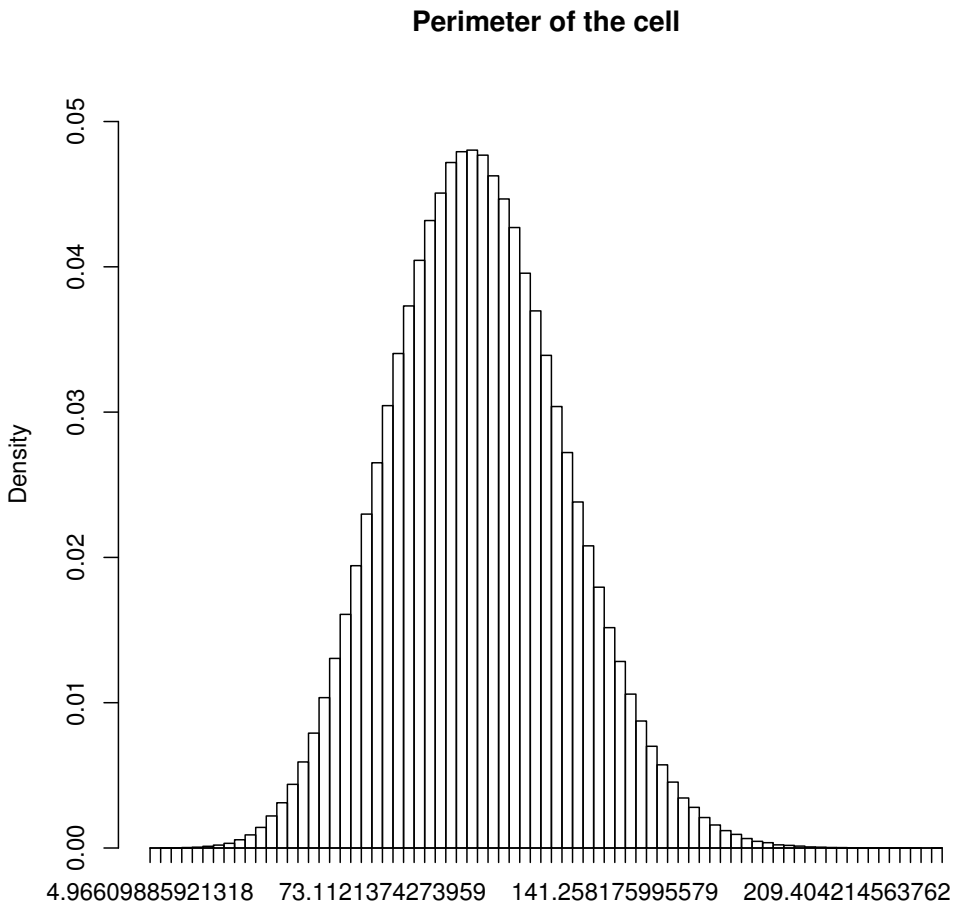


Realization of cell $\Xi_c(Z)$

Simulation algorithm
Functionals of interest

- Computation of distributions (and moments) of
  - Number of vertices
  - Perimeter
  - Area
- Computation of related cost functionals
  - Shortest path length
  - Subscriber line length
  - Capacity
- Comparison to results for PLCVT

Simulation algorithm
First numerical results

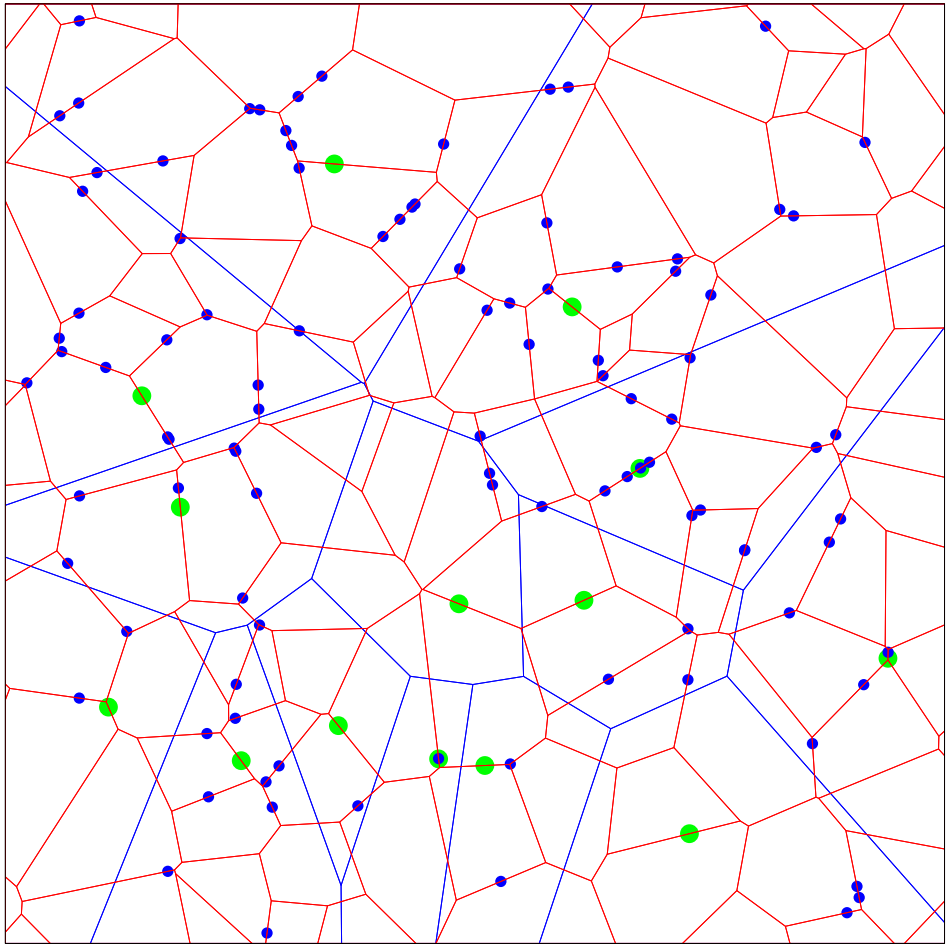


Histogram of perimer length for $\sqrt{\lambda_p} = 0.0625$ and $\lambda_\ell = 0.0125$

Related cost functionals
Shortest path length

- Two Cox processes X_H and X_L
  - (Spatial) intensities λ_H and λ_L
  - Connected to same PVT τ_{X_p}
- Distance from a point of X_L to its nearest neighbor of X_H
  - Along the edges of τ_{X_p}
  - (Random) shortest path length c_{LH}
  - Mean shortest path length $\mathbb{E}c_{LH} = \mathbb{E}_{X_L} c(P(o, N(o)))$

Related cost functionals
Shortest path length



Realisation of X_L and X_H with serving zones

Related cost functionals
Shortest path length

$$\mathbb{E}_{X_L} c(P(o, N(o))) = \frac{1}{\mathbb{E}_{X_H} \nu_1(L(\Xi_H^*))} \mathbb{E}_{X_H} \int_{L(\Xi_H^*)} c(P(u, o)) du$$

- Ξ_H^* typical Voronoi cell of X_H
- $L(\Xi_H^*) = \tau_{X_p}^{(1)} \cap \Xi_H^*$
- $h(\Xi_H^*) = \int_{L(\Xi_H^*)} c(P(u, o)) du$ => Palm representation for Ξ_H^* usable

$$\mathbb{E}_{X_L} c(P(o, N(o))) = \frac{\lambda_H}{8} \mathbb{E}_{X_p} [\nu_1(\partial \Xi_p^*) \mathbb{E}(\int_{L(\Xi_H^*(Z))} c(P(u, o)) du | X_p)],$$

where $Z \sim U(\partial \Xi_p^*)$

Conclusion and outlook

- PVCVT
  - Model for road system
  - Palm representation of typical cell
  - Simulation algorithm
  - Palm representation of shortest path length
- Further steps
  - Systematic numerical analysis (scaling invariance)
  - Usage for estimation of cost functionals
  - Approximation formulae
  - Comparison to other geometry models (PLCVT, PDCVT,...)

Literature

- C. Gloaguen, F. Fleischer, H. Schmidt, V. Schmidt (2006) Fitting of stochastic telecommunication network models, via distance measures and Monte-Carlo tests. *Telecommunication Systems* **31**(4), 353-377.
- C. Gloaguen, F. Fleischer, H. Schmidt, V. Schmidt (2006) Modelling and simulation of telecommunication networks: analysis of mean shortest path lengths. In: R. Lechnerova, I. Saxl, V. Benes (eds.) *Proc. Int. Conf. Spat. Stat. Stoch. Geom. (S4G)*, 11-25.
- C. Gloaguen, F. Fleischer, H. Schmidt, V. Schmidt (2005) Simulation of typical Cox-Voronoi cells, with a special regard to implementation tests. *Mathematical Methods of Operations Research* **62**, 357-373.