On the estimation of integrated covariance functions of stationary random fields

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Running Title: Estimation of integrated covariance functions

For stationary vector–valued random fields on $\mathbb{R}^d$ the asymptotic covariance matrix for estimators of the mean vector can be given by integrated covariance functions. In order to construct asymptotic confidence intervals and significance tests for the mean vector, nonparametric estimators of these integrated covariance functions are required. Integrability conditions are derived under which the estimators of the covariance matrix are mean-square consistent. For random fields induced by stationary Boolean models with convex grains, these conditions are expressed by sufficient assumptions on the grain distribution. Performance issues are discussed by means of numerical examples for the intrinsic volume densities of planar Boolean models with uniformly bounded grains.

Keywords: asymptotic unbiasedness; Boolean model; consistency; covariance matrix; empirical covariance; intrinsic volumes; nonparametric estimation; stationary random field.

1 Introduction

In various fields of practical application, such as medicine, biology, geology or material sciences, huge amounts of spatial data sets have to be categorized by means of certain characteristics of the underlying material. Assuming that these characteristics can be expressed by the mean vector of some stationary random field $Y = \{Y(x), x \in \mathbb{R}^d\}$, where $Y(x) = (Y_1(x), \ldots, Y_m(x))$ for $m \in \mathbb{N}$, we are interested in (asymptotic) confidence intervals and significance tests for the mean vector $\mu = \mathbb{E}\, Y(o)$ (with $o = (0, \ldots, 0)^\top$). On a sequence of expanding observation windows W_n, $n \in \mathbb{N}$ mean–square consistent estimators of μ are given by $\widehat{\mu}_n = (\widehat{\mu}_{n1}, \ldots, \widehat{\mu}_{nm})$, $n \geq 1$ with $\widehat{\mu}_{ni} = \int_{W_n} Y_i(x) G_i(W_n, x)\, dx$ for weighting functions G_i, $i = 1, .., m$. Under appropriate assumptions, it holds that $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ converges in distribution to a Gaussian random vector with mean vector zero and covariance matrix Σ as W_n tends to $\mathbb{R}^d$, where

$$\Sigma = \Big(\theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}\big(Y_i(o), Y_j(x)\big)\, dx\Big)_{i,j=1,..,m}$$

for certain constants $\theta_{ij} \in (0, \infty)$; see, e.g., the recently published manuscript Pantle *et al.* (2006) for a general class of stationary random fields induced by germ–grain models. In order to perform asymptotic significance tests for the mean vector μ (e.g., with the aim of an automated classification of the underlying material), the matrix of integrated covariance functions Σ has to be estimated consistently, since it is in general unknown or too complicated to be evaluated explicitly.

A nonparametric estimator $\widehat{\Sigma}_n$ of the asymptotic covariance matrix Σ is considered in Section 3, where

$$\widehat{\Sigma}_n = \Big(\int_{\mathbb{R}^d} \widehat{\mathrm{Cov}}_{nij}(x)\, \gamma_{ij}(W_n, x)\, dx \Big)_{i,j=1,..,m}, \qquad n \geq 1$$

for some weighting function $\gamma_{ij}(W_n, x)$ and a consistent estimator $\widehat{\mathrm{Cov}}_{nij}(x)$ of $\mathrm{Cov}(Y_i(o), Y_j(x))$ for fixed $x \in \mathbb{R}^d$. The construction principle employed is similar to the techniques used, e.g., in Böhm *et al.* (2004, Section 3) and Schmidt and Spodarev (2005, Section 3.5). Sufficient conditions for the asymptotic unbiasedness and for the mean-square consistency of $\widehat{\Sigma}_n$ are given in Lemma 1 and Theorem 1, respectively. Furthermore, a variant $\widehat{\Sigma}'_n$ of the estimator $\widehat{\Sigma}_n$ is introduced, motivated by the problem of an efficient computation of $\widehat{\Sigma}_n$. In Section 4, random fields induced by special random sets called *Boolean models* are discussed. For this case, sufficient conditions on the volume of the typical grain M_0 enlarged by some test set K can be formulated so that the assumptions of Theorem 1 are satisfied. Section 5 deals with an estimator of Σ derived from the empirical covariance of observations of $\widehat{\mu}_n$ on disjoint subwindows of W_n. The paper closes with a numerical comparison of the different estimates of the asymptotic covariance matrix. In the empirical part, we do not consider μ itself, but its linear transformation leading to the vector $v = (\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))$ of intrinsic volume densities of a Boolean model Ξ, see Section 6 for details.

2 Preliminaries

In the present section, we introduce some notation used throughout this paper and recall basic facts from stochastic geometry. Further details can be found for example in Adler and Taylor (2006), Schneider and Weil (2000) or Stoyan *et al.* (1995). In the second part, a class of estimators of the mean value of stationary random fields and their asymptotic properties are considered (compare, in particular, Ivanov and Leonenko (1989) and Pantle *et al.* (2006)).

2.1 Basic notations

Let $d \geq 2$ be an arbitrary fixed integer and let the d-dimensional Euclidean space $\mathbb{R}^d$ be equipped with the Borel σ-algebra $\mathcal{B}^d$. Denote the set of bounded Borel sets by $\mathcal{B}^d_0$ and write $o \in \mathbb{R}^d$ for the origin in $\mathbb{R}^d$. The Euclidean norm of a vector $x \in \mathbb{R}^d$ is denoted by $|x|$, whereas $|B|$ stands for the d-dimensional Lebesgue measure (or volume) of a set $B \in \mathcal{B}^d$. By ∂B we denote the boundary of a Borel set and by $\mathrm{int}(B)$ its interior. Furthermore, let $B_r(x) = \{y \in \mathbb{R}^d : |y - x| \leq r\}$ be the closed ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with radius $r > 0$. The *Minkowski sum* of two sets $B, B' \subset \mathbb{R}^d$ is given by $B \oplus B' = \{x + y : x \in B, y \in B'\}$, where we write $B + x$ instead of $B \oplus \{x\}$ for the *translation* of B by a vector $x \in \mathbb{R}^d$. Besides this, consider the *reflection* $\check{B} = \{-x : x \in B\}$ of B at the origin and denote the *Minkowski difference* of B and B' by $B \ominus B' = \{x : \check{B}' + x \subseteq B\}$.

Consider an arbitrary probability space $(\Omega, \mathcal{A}, P)$ and some $\mathcal{B}^d \otimes \mathcal{A}$-measurable mapping $Y : \mathbb{R}^d \times \Omega \to \mathbb{R}$ with mean function $\mu(x) = \mathbb{E}\, Y(x)$ and covariance function $\mathrm{Cov}(x, y) = \mathrm{Cov}(Y(x), Y(y))$ assuming that $\mathbb{E}\, Y^2(x) < \infty$ for any $x \in \mathbb{R}^d$. For simplicity, we use the notation $Y(x)$ instead of $Y(x, \cdot)$ and denote the random field $\{Y(x), x \in \mathbb{R}^d\}$ by Y as well. A random field Y is called *stationary* if its finite dimensional distributions are invariant with respect to translations, that is to say, the distribution of the random vectors $(Y(x_1 + h), \ldots, Y(x_m + h))$ and $(Y(x_1), \ldots, Y(x_m))$ coincide for all $x_1, \ldots, x_m, h \in \mathbb{R}^d$ and $m \in \mathbb{N}$. Stationarity implies, in particular, that $\mu(x)$ is constant and $\mathrm{Cov}(x, y)$ is a function depending on the difference $y - x$ only. Hence, set $\mu = \mathbb{E}\, Y(x)$ and $\mathrm{Cov}(x) = \mathrm{Cov}(y, y + x)$ for any $x, y \in \mathbb{R}^d$.

In Section 4 we will consider the case, where Y is defined as a functional of a *random closed set* (RACS) Ξ in $\mathbb{R}^d$. A random closed set is a $(\mathcal{A}, \sigma_{\mathcal{F}})$-measurable mapping from $(\Omega, \mathcal{A})$ into $(\mathcal{F}, \sigma_{\mathcal{F}})$, where $\mathcal{F} \subset \mathcal{B}^d$ denotes the family of all closed sets in $\mathbb{R}^d$, and $\sigma_{\mathcal{F}}$ is the σ-algebra generated by $\{F \in \mathcal{F} : F \cap C \neq \emptyset\}$ for arbitrary compact $C \subset \mathbb{R}^d$. Finally, let $\mathcal{K} \subset \mathcal{F}$ be the family of all convex and compact sets in $\mathbb{R}^d$, and define the σ-algebra $\sigma_{\mathcal{K}} = \{B \cap \mathcal{K} : B \in \sigma_{\mathcal{F}}\}$ on $\mathcal{K}$.

2.2 Estimating the mean of vector–valued random fields

Let $Y_1 = \{Y_1(x), x \in \mathbb{R}^d\}, \ldots, Y_m = \{Y_m(x), x \in \mathbb{R}^d\}$ be a set of $m \in \mathbb{N}$ random fields on the same probability space such that the finite dimensional distributions of the vector–valued random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ with $Y(x) = (Y_1(x), \ldots, Y_m(x))^\top$ are invariant with respect to translations. The random fields $Y_1, \ldots, Y_m$ are then called *jointly stationary.* For $i, j = 1, \ldots, m$, set $\mu_i = \mathbb{E}\, Y_i(o)$ and $\mathrm{Cov}_{ij}(x) = \mathrm{Cov}(Y_i(o), Y_j(x))$, $x \in \mathbb{R}^d$, where we assume that $\mathbb{E}\, Y_i^2(o) < \infty$, but μ_i and $\mathrm{Cov}_{ij}(x)$ are unknown. With regard to the estimation of $\mu = (\mu_1, \ldots, \mu_m)^\top$, consider a sequence $\{W_n\}$ of monotonously increasing, bounded Borel sets $W_n \subset \mathbb{R}^d, n \geq 1$ such that

$$\lim_{n\to\infty} |W_n| = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{|\partial W_n \oplus B_r(o)|}{|W_n|} = 0 \tag{1}$$

for any $r > 0$. Furthermore, suppose that the random fields $Y_i = \{Y_i(x), x \in \mathbb{R}^d\}$ are observable on sub-windows $W_{ni} = W_n \ominus \check{K}_i$ for some $K_i \in \mathcal{K}$, respectively. An unbiased estimator of the mean vector μ is then given by $\widehat{\mu}_n = (\widehat{\mu}_{n1}, \ldots, \widehat{\mu}_{nm})^\top, n \geq 1$ with

$$\widehat{\mu}_{ni} = \int_{W_n} Y_i(x)\, G_i(W_n, x)\, dx \tag{2}$$

for functionals $G_i : \mathcal{B}_0^d \otimes \mathbb{R}^d \to [0, \infty), i = 1, \ldots, m$, which are $\mathcal{B}^d$-measurable in the second component and satisfy

$$G_i(W, x) = 0 \quad \text{if } x \in \mathbb{R}^d \setminus W \ominus \check{K}_i, \qquad \text{and} \qquad \int_{\mathbb{R}^d} G_i(W, x)\, dx = 1 \tag{3}$$

for any $W \in \mathcal{B}_0^d$. Put $\Gamma_{nij}(x) = \int_{\mathbb{R}^d} G_i(W_n, y) G_j(W_n, y + x)\, dy$ for $i, j = 1, \ldots, m$. Note that $\Gamma_{nij}(x) = 0$ if $x \notin W_{ni} \oplus \check{W}_{nj}$. For any $n \geq 1$, it holds that

$$\mathrm{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj}) = \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x) \Gamma_{nij}(x)\, dx\,. \tag{4}$$

To study the asymptotic behaviour of $\widehat{\mu}_n$, we assume that there exist constants $c_1, \theta_{ij} \in (0,\infty)$ for all $i,j = 1,\ldots,m$ such that

$$\sup_{x\in\mathbb{R}^d} G_i(W_n,x) \le \frac{c_1}{|W_n|} \text{ for any } n \ge 1 \qquad \text{and} \qquad \lim_{n\to\infty} |W_n|\Gamma_{nij}(x) = \theta_{ij} \text{ for any } x \in \mathbb{R}^d . \tag{5}$$

Both conditions (3) and (5) hold, for instance, if $G_i(W_n,x) = 1\!\!\mathrm{I}(x \in W_n \ominus \check{K}_i) \,/\, |W_n \ominus \check{K}_i|, i = 1,\ldots,m$ for any $n \ge 1$ and $x \in \mathbb{R}^d$, where $1\!\!\mathrm{I}(B)$ denotes the indicator function of the set B.

Under appropriate mixing and integrability conditions one can show that

$$\begin{pmatrix} \sqrt{|W_n|}\,(\widehat{\mu}_{n1} - \mu_1) \\ \vdots \\ \sqrt{|W_n|}\,(\widehat{\mu}_{nm} - \mu_m) \end{pmatrix} \Longrightarrow \mathcal{N}_m(o,\Sigma)\,, \qquad n \to \infty\,,$$

where $\Longrightarrow$ denotes convergence in dtistribution and $\mathcal{N}_m(o,\Sigma)$ is an m-dimensional Gaussian random variable with mean zero and covariance matrix

$$\Sigma = \begin{pmatrix} \theta_{11}\int_{\mathbb{R}^d} \mathrm{Cov}_{11}(x)\,dx & \ldots & \theta_{1m}\int_{\mathbb{R}^d} \mathrm{Cov}_{1m}(x)\,dx \\ \vdots & \ddots & \vdots \\ \theta_{m1}\int_{\mathbb{R}^d} \mathrm{Cov}_{m1}(x)\,dx & \ldots & \theta_{mm}\int_{\mathbb{R}^d} \mathrm{Cov}_{mm}(x)\,dx \end{pmatrix}, \tag{6}$$

confer for example Ivanov and Leonenko (1989, Section 1.7), Pantle *et al.* (2006, Sections 4 and 5) and references therein. Regarding the symmetry of the covariance matrix Σ, note that $\mathrm{Cov}_{ij}(x) = \mathrm{Cov}_{ji}(-x)$ for all $x \in \mathbb{R}^d$ as well as $\Gamma_{nij}(x) = \Gamma_{nji}(-x)$, and consequently $\theta_{ij} = \theta_{ji}$. Since explicit formulae for σ_{ij}, $i,j = 1,\ldots,m$ are in general unknown, we are interested in the estimation of the integrated covariance functions $\int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\,dx$. At this, we assume that

$$\int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\,dx < \infty\,, \qquad i,j = 1,\ldots,m\,. \tag{7}$$

3 A weighted covariance estimator

Throughout the following, let $Y_1,\ldots,Y_m$ be a set of *jointly stationary* random fields with finite fourth moments and $\widehat{\mu}_n, n \ge 1$ the estimator of $\mathbb{E}\,Y(o)$ as defined in (2). The aim of this section is to establish a consistent estimator $\widehat{\Sigma}_n$ of the asymptotic covariance matrix Σ given in (6).

For any pair $i \le j$, choose an increasing sequence $\{U_{nij}\}$ of compact Borel sets with $U_{nij} \subseteq W_{ni} \oplus \check{W}_{nj}$ and $o \in U_{nij}$. Denote by $\mathrm{supp}(\mathrm{Cov}_{ij})$ the support of Cov_{ij} and assume that $\mathrm{supp}(\mathrm{Cov}_{ij}) \subseteq \lim_{n\to\infty} U_{nij}$. In addition, suppose that

$$\lim_{n\to\infty} |U_{nij}|^2 / \,|W_n| \;=\; 0\,, \qquad \forall\, i,j = 1,\ldots,m \tag{8}$$

and

$$\lim_{n\to\infty} \sup_{x\in U_{nij}} |\,\theta_{ij} - |W_n|\Gamma_{nij}(x)\,| = \; 0\,, \qquad \forall\, i,j = 1,\ldots,m. \tag{9}$$

Thus, we may assume, without loss of generality, that $|U_{nij}| > 0$ and $\Gamma_{nij}(x) > 0$ for all $x \in U_{nij}$, $i, j = 1, \ldots, m$ and any $n \in \mathbb{N}$. Finally, for $j < i$, put $U_{nji} = U_{nij}$ to preserve symmetry in the covariance matrix estimate. As an example, let $\mathrm{supp}(\mathrm{Cov}_{ij}) = \mathbb{R}^d$ and $W_n = nK_o$ for some $K_0 \in \mathcal{K}$ with $|K_0| > 0$ and $o \in \mathrm{int}(K_0)$. Define $\varrho_n = \sup\{r > 0 : B_r(o) \subseteq W_n\}$, and put $G_i(x) = 1\!\!1(x \in B_{\varrho_n - r_0}(o))/|B_{\varrho_n - r_0}(o)|$ for all i, where $r_0 > 0$ satisfies $K_i \subseteq B_{r_0}(o)$ for $i = 1, \ldots, m$ and $n \in \mathbb{N}$ is large enough ensuring that $\varrho_n > r_0$. Then, conditions (8) and (9) are fulfilled with $\theta_{ij} \equiv 1$ if $U_{nij} = B_{\sqrt{\varrho_n}\,\varepsilon_n - r_0}(o)$ for some sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$ and $\lim_{n\to\infty} \sqrt{n}\,\varepsilon_n = \infty$.

Based on the above-mentioned assumptions, define the estimator $\widehat{\Sigma}_n = (\widehat{\sigma}_{nij})$ by

$$\widehat{\sigma}_{nij} \;=\; \int_{U_{nij}} \widehat{\mathrm{Cov}}_{nij}(x)\, |W_n| \Gamma_{nij}(x)\, dx, \qquad n \geq 1 \tag{10}$$

with

$$\widehat{\mathrm{Cov}}_{nij}(x) \;=\; \int_{W_{ni} \cap (W_{nj} - x)} Y_i(y) Y_j(y + x)\, G_i(W_n, y) G_j(W_n, y + x)\, dy \cdot \Gamma^{-1}_{nij}(x) \;-\; \widehat{\mu}_{ni} \widehat{\mu}_{nj}. \tag{11}$$

Notice that $\widehat{\sigma}_{nij} = \widehat{\sigma}_{nji}$ for any $i, j = 1, \ldots, m, n \geq 1$. Following along the lines of the subsequent proofs, one observes that $\widehat{\mathrm{Cov}}_{nij}(x)$, considered separately for fixed $x \in \mathbb{R}^d$, is an asymptotically unbiased and consistent estimator of $\mathrm{Cov}(x)$. For the special case, where $\mathbb{E}\, Y(o) = 0$ and $G(W_n, x) = 1\!\!1(x \in W_n)/|W_n|$, further results on $\widehat{\mathrm{Cov}}_{nij}(x)$ can be found, for instance, in Ivanov and Leonenko(1989, Chapter 4).

3.1 Asymptotic properties

Lemma 1. *The estimator $\widehat{\Sigma}_n = (\widehat{\sigma}_{nij})$ with $\widehat{\sigma}_{nij}$ as defined in* (10) *and* (11) *is asymptotically unbiased for the covariance matrix $\Sigma = (\sigma_{ij})$ given in* (6).

Proof. Insert $\pm\mu_i\mu_j$ into the defining equation (11) of $\widehat{\mathrm{Cov}}_{nij}(x)$ for any $i, j = 1, \ldots, m$ and apply Fubini's theorem to obtain

$$\mathbb{E}\, \widehat{\sigma}_{nij} \;=\; \int_{U_{nij}} \mathrm{Cov}_{ij}(x)\, |W_n| \Gamma_{nij}(x)\, dx - |W_n| \mathrm{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj}) \int_{U_{nij}} \Gamma_{nij}(x)\, dx \tag{12}$$

By (4) and (5) one can derive that $\lim_{n\to\infty} |W_n| \mathrm{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj}) = \sigma_{ij} < \infty$ applying the dominated convergence theorem. Together with

$$0 \;\leq\; \limsup_{n\to\infty} \int_{U_{nij}} \Gamma_{nij}(x)\, dx \;\leq\; c_1 \cdot \lim_{n\to\infty} \frac{|U_{nij}|}{|W_n|} \;=\; 0\,,$$

the second expression in (12) converges to zero as $n \to \infty$. On the other hand, the first summand can be split up as follows

$$\begin{aligned} &\int_{U_{nij}} \mathrm{Cov}_{ij}(x)\, |W_n| \Gamma_{nij}(x)\, dx \\ &\quad = \;\theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, dx - \int_{U_{nij}} \mathrm{Cov}_{ij}(x) \big(\theta_{ij} - |W_n| \Gamma_{nij}(x)\big)\, dx - \theta_{ij} \int_{\mathbb{R}^d \setminus U_{nij}} \mathrm{Cov}_{ij}(x)\, dx\,. \end{aligned}$$

By (7) we have $\lim_{n\to\infty}\int_{\mathbb{R}^d\setminus U_{nij}} |\mathrm{Cov}_{ij}(x)|\,dx \;=\; 0$, and from (9) it follows that

$$\limsup_{n\to\infty}\Big|\int_{U_{nij}} \mathrm{Cov}_{ij}(x)(\theta_{ij}-|W_n|\Gamma_{nij}(x))\,dx\Big|$$
$$\le \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\,dx\cdot \lim_{n\to\infty}\sup_{x\in U_{nij}} \mid \theta_{ij}-|W_n|\Gamma_{nij}(x)\mid \;=\; 0\,,$$

□

According to Lemma 1, asymptotic unbiasedness of $\widehat{\Sigma}_n$ holds under the same integrability assumption (7) needed so that Σ is well defined. Additional assumptions are necessary, however, to ensure mean-square consistency. Consistency is understood with respect to the matrix norm $||A|| = \big(\sum_{i,j=1}^m a_{ij}^2\big)^{1/2}$ for a matrix $A=(a_{ij})\in\mathbb{R}^{m\times m}$.

Theorem 1. *Suppose that* $\mathbb{E}\,Y_i^4(o)<\infty$ *for* $i=1,\ldots,m$. *Then it holds that* $\lim_{n\to\infty}\mathbb{E}\,||\widehat{\Sigma}_n-\Sigma||^2=0$, *i.e.,* $\widehat{\Sigma}_n=(\widehat{\sigma}_{nij})$ *is mean-square consistent for* $\Sigma=(\sigma_{ij})$ *if there exist finite constants* κ_1,κ_2 *such that for any* $i,j=1,\ldots,m$

$$\frac{1}{|U_{nij}|^2}\int_{U_{nij}}\int_{U_{nij}}\int_{\mathbb{R}^d}\big|\mathrm{Cov}\big(Y_i(o)Y_j(x_1),Y_i(y)Y_j(x_2+y)\big)\big|\;dy\,dx_1dx_2\;\le\;\kappa_1 \tag{13}$$

and

$$\sup_{x_1,x_2\in\mathbb{R}^d}\int_{\mathbb{R}^d}\big|\mathbb{E}\big((Y_i(o)-\mu_i)(Y_i(y)-\mu_i)Y_j(x_1)Y_j(x_2)\big)\big|\,dy\;\le\;\kappa_2\,. \tag{14}$$

Proof. By Minkowski's inequality and since $\lim_{n\to\infty}\mathbb{E}\,\widehat{\sigma}_{nij}=\sigma_{ij}$ according to Lemma 1, it suffices to show that $\lim_{n\to\infty}\mathbb{E}\,(\widehat{\sigma}_{nij}-\mathbb{E}\,\widehat{\sigma}_{nij})^2=0$ for all $i,j=1,\ldots,m$. To this end, define

$$S_{n1}\;=\;\int_{U_{nij}}\int_{W_{ni}\cap(W_{nj}-x)}\Big(Y_i(y)Y_j(y+x)-\mathbb{E}\,(Y_i(y)Y_j(y+x))\Big)\,|W_n|G_i(W_n,y)G_j(W_n,y+x)\,dy\,dx$$

and

$$S_{n2}\;=\;-\big(\widehat{\mu}_{ni}\widehat{\mu}_{nj}-\mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big)\int_{U_{nij}}|W_n|\Gamma_{nij}(x)\,dx\,.$$

Hence, we find $\widehat{\sigma}_{nij}-\mathbb{E}\,\widehat{\sigma}_{nij}=S_{n1}+S_{n2}$ and have to verify that $\lim_{n\to\infty}\mathbb{E}\,S_{n1}^2=0$ and $\lim_{n\to\infty}\mathbb{E}\,S_{n2}^2=0$. For the first expression, one obtains

$$\mathbb{E}\left(\int_{U_{nij}}\int_{W_{ni}\cap(W_{nj}-x)}\Big[Y_i(y)Y_j(y+x)-\mathbb{E}(Y_i(y)Y_j(y+x))\Big]|W_n|G_i(W_n,y)G_j(W_n,y+x)\,dy\,dx\right)^2$$
$$\le\int_{U_{nij}}\int_{U_{nij}}\int_{W_n}\int_{W_n}\big|\mathrm{Cov}\big(Y_i(o)Y_j(x_1),Y_i(y_2-y_1)Y_j(x_2+y_2-y_1)\big)\big|$$
$$\times\,|W_n|^2\,G_i(W_n,y_1)G_j(W_n,x_1+y_1)G_i(W_n,y_2)G_j(W_n,x_2+y_2)\,dy_1\,dy_2\,dx_1\,dx_2$$

$$
\begin{aligned}
&= \int_{U_{nij}} \int_{U_{nij}} \int_{\mathbb{R}^d} \left|\mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\big)\right| \\
&\qquad\qquad \times |W_n|^2 \int_{W_n} G_i(W_n,y)G_j(W_n,x_1+y)G_i(W_n,y+u)G_j(W_n,x_2+y+u)\,du\,dy\,dx_1\,dx_2 \\
&\le c_1^3 \cdot \kappa_1 \cdot \frac{|U_{nij}|^2}{|W_n|}\,,
\end{aligned}
$$

where the second inequality follows from (5). Hence, the final expression converges to zero given (8). Furthermore, it holds that

$$
\begin{aligned}
\mathbb{E}\, S_{n2}^2 &= \mathbb{E}\big[\,\widehat{\mu}_{ni}\widehat{\mu}_{nj} - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big]^2 \cdot \left(\int_{U_{nij}} |W_n|\Gamma_{nij}(x)\,dx\right)^2 \\
&\le |W_n|\mathbb{E}\big[\,\widehat{\mu}_{ni}\widehat{\mu}_{nj} - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big]^2 \cdot c_1^2\,\frac{|U_{nij}|^2}{|W_n|}\,,
\end{aligned}
$$

which tends to zero as well if $\overline{\lim}_{n\to\infty}|W_n|\,\mathbb{E}\big(\widehat{\mu}_{ni}\widehat{\mu}_{nj} - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big)^2 < \infty$. The latter can be seen as follows. Inserting $\pm\mu_i\mu_j$ and $\pm\mu_i\widehat{\mu}_{nj}$ yields

$$
\begin{aligned}
|W_n|\,\mathbb{E}\big[\,\widehat{\mu}_{ni}\widehat{\mu}_{nj} - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big]^2 &= |W_n|\,\mathbb{E}\big[(\widehat{\mu}_{ni}-\mu_i)\,\widehat{\mu}_{nj} + \mu_i\,(\widehat{\mu}_{nj}-\mu_j) + \mu_i\mu_j - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big]^2 \\
&\le 3|W_n|\Big[\mathbb{E}\big([\widehat{\mu}_{ni}-\mu_i]^2\,\widehat{\mu}_{nj}^2\big) + \mu_i^2\,\mathbb{E}\big(\widehat{\mu}_{nj}-\mu_j\big)^2 + \mathrm{Cov}^2\big(\widehat{\mu}_{ni},\widehat{\mu}_{nj}\big)\Big],
\end{aligned}
$$

where $\lim_{n\to\infty}|W_n|\mathbb{E}\,(\widehat{\mu}_{nj}-\mu_j)^2 = \sigma_{jj} < \infty$ and $\lim_{n\to\infty}\sqrt{|W_n|}\mathrm{Cov}(\widehat{\mu}_{ni},\widehat{\mu}_{nj}) = 0$. From the definition of $\widehat{\mu}_{ni}, i = 1,...,m$ and Fubini's theorem it follows that

$$
\begin{aligned}
|W_n|\,\mathbb{E}\big([\widehat{\mu}_{ni}-\mu_i]^2\,\widehat{\mu}_{nj}^2\big) &= |W_n| \int_{W_n^4} \mathbb{E}\big([Y_i(v_1)-\mu_i]\,[Y_i(v_2)-\mu_i]\,Y_j(v_3)Y_j(v_4)\big) \\
&\qquad \times G_i(W_n,v_1)G_i(W_n,v_2)G_j(W_n,v_3)G_j(W_n,v_4)\,d(v_1,v_2,v_3,v_4) \\
&= |W_n| \int_{W_n}\int_{W_n-v}\int_{W_n-v}\int_{W_n-v} \mathbb{E}\big([Y_i(o)-\mu_i]\,[Y_i(y)-\mu_i]\,Y_j(x_1)Y_j(x_2)\big) \\
&\qquad \times G_i(W_n,v)G_i(W_n,y+v)G_j(W_n,x_1+v)G_j(W_n,x_2+v)\,dy\,dx_2\,dx_1\,dv \\
&\le c_1 \int_{W_n}\int_{W_n-v}\int_{W_n-v}\int_{\mathbb{R}^d} \left|\mathbb{E}\big([Y_i(o)-\mu_i]\,[Y_i(y)-\mu_i]\,Y_j(x_1)Y_j(x_2)\big)\right|dy \\
&\qquad \times G_i(W_n,v)G_j(W_n,x_1+v)G_j(W_n,x_2+v)\,dx_2\,dx_1\,dv \\
&\le c_1 \cdot \sup_{x_1,x_2\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left|\mathbb{E}\big([Y_i(o)-\mu_i)]\,[Y_i(y)-\mu_i]\,Y_j(x_1)Y_j(x_2)\big)\right|dy\,.
\end{aligned}
$$

The last two lines are obtained by (3) and (5), that is, in particular, that $G(W_n,\cdot)$ integrates to one over W_n. By condition (14), we obtain the desired result. □

As mentioned before, similar estimation methods for Σ are used in Böhm *et al.* (2004, Section 3) for 0-1-valued vector fields or Schmidt and Spodarev (2005, Section 3.5) for arbitrary stationary vector fields with uniform weights. Both variants assume that conditions (7) and (13) hold to prove mean-square consistency. In the case of 0-1-valued random fields no further assumptions are needed, since Y is uniformly bounded on $\mathbb{R}^d$. For the more general setting considered in Schmidt and Spodarev (2005), mean-square consistency is shown using different arguments in the proof, which lead to stronger integrability conditions. We point out that formula (56) in Schmidt and Spodarev (2005) is a sufficient condition for (13). In addition, we have to assume that

$$\int_{\mathbb{R}^{3d}} \Big| \mathbb{E}\big([Y_i(o)-\mu_i]\,[Y_j(x_1)-\mu_i]\,[Y_i(x_2)-\mu_i]\,[Y_j(x_3)-\mu_i]\big) \tag{15}$$
$$-\mathrm{Cov}_{ij}(x_1)\mathrm{Cov}_{ij}(x_2-x_3)-\mathrm{Cov}_{ii}(x_2)\mathrm{Cov}_{jj}(x_1-x_3)-\mathrm{Cov}_{ij}(x_3)\mathrm{Cov}_{ji}(x_1-x_2)\Big|\, d(x_1,x_2,x_3) \;\le\; \kappa$$

for some finite constant κ and $i,j=1,\ldots,m$. A general class of random fields $Y_i,\ i=1,\ldots,m$ satisfying the conditions of Theorem 1 is discussed in Section 4. As an example, it can easily be verified that (7) and (13)–(15) hold if the vector field Y is such that $\{Y(x), a \in A\}$ and $\{Y(y), y \in B\}$ are independent for any bounded $A, B \in \mathcal{B}_0^d$ with $\inf\{|x-y| : x \in A, y \in B\} > r_0$ for some $r_0 > 0$.

3.2 Modified weighted covariance estimator

In this section we discuss a modification of the estimator introduced in (10) and (11) motivated by the question of how to construct an efficient implementation for the computation of $\widehat{\Sigma}_n$. The mean–square error of $\widehat{\Sigma}_n$ estimating Σ tends to zero with expanding sampling window W_n, i.e., with the available information on Y. Simultaneously, a rise in the computation time needed to evaluate $\widehat{\mathrm{Cov}}_{nij}(x), x \in U_{nij}$ is usually observed. One way to use as many information from the observations in W_n as possible and to cut down the running time is to devide W_n into smaller subwindows and perform the estimation on each subwindow separately. Afterwards the estimated values can be averaged by means of the arithmetic mean for example.

To be more precise, we need some further definitions and notation. Let $\{V_n\}$ with $V_n \subset W_n, n \ge 1$ be a sequence of monotonously increasing, bounded Borel sets fulfilling (1) and $\{N(n)\}$ an increasing sequence of integers. For any $n \in \mathbb{N}$ choose some vectors $h_{n,1},\ldots,h_{n,N(n)} \in \mathbb{R}^d$ and define $V_{n,k} = V_n + h_{n,k}$, $k = 1,\ldots,N(n)$ subject to

$$\bigcup_{k=1}^{N(n)} V_{n,k} \subseteq W_n \qquad \text{and} \qquad G_i(V_{n,k},x) = G_i(V_n, x-h_{n,k}), \qquad x \in \mathbb{R}^d,\ i=1,\ldots,m, n \in \mathbb{N}. \tag{16}$$

Put $V_{ni} = V_n \ominus K_i$ and let $V_{ni,k}$ denote the translates of V_{ni} by $h_{n,k},\ i=1,\ldots,m, k=1,\ldots,N(n)$. Suppose that (5) holds for $\{V_{ni,k}\}$ with constant $\tilde{c}_1$ and limits $\tilde{\theta}_{ij} \in (0,\infty),\ i,j = 1\ldots,m$. Without loss of generality we may assume that $\tilde{c}_1 = c_1$ and $\theta_{ij} = \tilde{\theta}_{ij}$. Furthermore, denote by $\widehat{\mu}_{ni,k}$ and $\widehat{\mathrm{Cov}}_{nij,k}(x)$ the estimators of $\mu_i = \mathbb{E}\, Y_i(o)$ and $\mathrm{Cov}_{ij}(x)$ as given in (2) and (11), respectively, but based on observations within $V_{ni,k}$ only. By means of these definitions, we introduce the modified estimator

$$\widehat{\sigma}'_{nij} \;=\; \frac{1}{N(n)} \sum_{k=1}^{N(n)} \widehat{\sigma}_{nij,k} \qquad \text{with} \qquad \widehat{\sigma}_{nij,k} = \int_{U_{nij}} \widehat{\mathrm{Cov}}_{nij,k}(x) |V_n| \Gamma_{V_{nij,k}}(x)\, dx, \tag{17}$$

where $\Gamma_{V_{nij,k}}(x) = \int_{\mathbb{R}^d} G_i(V_{n,k}, y) G_j(V_{n,k}, y+x)\, dy$ and U_{nij} is chosen as before, but restricted to V_n, i.e., $U_{nij} \subseteq V_{ni} \oplus \check{V}_{nj}$, $\lim_{n\to\infty} |U_{nij}|^2/|V_n| = 0$ and $\lim_{n\to\infty} \sup_{x\in U_{nij}} |\theta_{ij} - |V_n|\Gamma_{V_{nij}}(x)| = 0$ for $\Gamma_{V_{nij}}(x) = \int_{\mathbb{R}^d} G_i(V_n, y) G_j(V_n, y+x)\, dy$.

Corollary 1. *The estimator $\widehat{\Sigma}'_n = (\widehat{\sigma}'_{nij})$ with $\widehat{\sigma}'_{nij}$ defined in* (17) *is asymptotically unbiased for Σ. If, in addition, the conditions of Theorem* 1 *are fulfilled, then it holds that $\lim_{n\to\infty} \mathbb{E}\, ||\widehat{\Sigma}'_n - \Sigma||^2 = 0$.*

Proof. Notice that each estimator $\widehat{\sigma}_{nij,k}$, $k \in \{1, \dots, N(n)\}$ has the same asymptotic properties as $\widehat{\sigma}_{nij}$ considered in (10). According to (16), we have

$$\widehat{\mu}_{ni,k} = \int_{V_{n,k}} Y_i(x)\, G_i(V_{n,k}, x)\, dx = \int_{V_n + h_{n,k}} Y_i(x)\, G_i(V_n, x - h_{n,k})\, dx = \int_{V_n} Y_i(x + h_{n,k})\, G_i(V_n, x)\, dx$$

and similarly

$$\begin{aligned} \widehat{\mathrm{Cov}}_{nij,k}(x) &= \int_{V_{ni,k} \cap (V_{nj,k} - x)} \left[Y_i(y) Y_j(y+x) - \widehat{\mu}_{ni,k}\, \widehat{\mu}_{nj,k} \right] G_i(V_{n,k}, y) G_j(V_{n,k}, y+x)\, dy\, \Gamma^{-1}_{V_{nij,k}}(x) \\ &= \int_{V_{ni} \cap (V_{nj} - x)} \left[Y_i(y + h_{n,k}) Y_j(y + h_{n,k} + x) - \widehat{\mu}_{ni,k}\, \widehat{\mu}_{nj,k} \right] \\ &\qquad\qquad \times G_i(V_n, y) G_j(V_n, y+x)\, dy\, \Gamma^{-1}_{V_{nij}}(x) \end{aligned}$$

for any $x \in U_{nij}$, since $\Gamma_{V_{nij,k}}(x) = \Gamma_{V_{nij}}(x)$ for any $k \in \{1, \dots, N(n)\}$. Thus, one observes that

$$\mathbb{E}\, \widehat{\sigma}_{nij,k} = \int_{U_{nij}} \mathrm{Cov}_{ij}(x) |V_n| \Gamma_{V_{nij}}(x)\, dx - \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x) |V_n| \Gamma_{V_{nij}}(x) \cdot \int_{U_{nij}} \Gamma_{V_{nij}}(x)\, dx\,,$$

Corresponding equations hold for the second moments and for the mean-squared distance $\mathbb{E}(\widehat{\sigma}_{nij,k} - \sigma_{ij})^2$, as can easily be derived from the proof of Theorem 3.1. The average $\widehat{\sigma}'_{nij}$ in (17) is therefore an asymptotically unbiased estimator of σ_{nij} given (7), and it is mean-square consistent if conditions (13) and (14) are satisfied. □

To give an idea, why the subdivision of the sampling window is useful, we briefly explain an efficient algorithm that can be used to compute $\widehat{\Sigma}_n$ and $\widehat{\Sigma}'_n$. The mentioned algorithm is called the *Fast Fourier Transform* (in short FFT), see Brigham (1992, Chapter 10), Koch *et al.* (2003) and references therein. The Fourier transform of a integrable function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as the integral

$$F(y) = \int_{\mathbb{R}^d} f(x) e^{2\pi i <x,y>}\, dx, \qquad y \in \mathbb{R}^d,$$

where $< x, y > = \sum_{\ell=1}^d x_\ell y_\ell$ is the scalar product of $x, y \in \mathbb{R}^d$. Under appropriate assumptions, we get the inverse formula

$$f(x) = \int_{\mathbb{R}^d} F(y) e^{-2\pi i <x,y>}\, dy, \qquad x \in \mathbb{R}^d.$$

As we will see below, we are interested in the evaluation of the convolution of two functions f and h, i.e.,

$$(f * h)(x) \;=\; \int_{\mathbb{R}^d} f(y) \cdot h(x-y)\, dy\,.$$

Let F and H be the Fourier transforms of f and h, respectively. Then it follows from the *convolution theorem*, confer Brigham (1992, p. 78), that

$$(f * h)(x) \quad = \quad \int_{\mathbb{R}^d} F(y) \cdot H(y)\, e^{-2\pi i \, <x,y>}\, dy. \tag{18}$$

The FFT algorithm can now be applied to approximate the Fourier transforms (forward or inverse) on the right hand side of (18). To use this for our estimation problem, we note that $\widehat{\mathrm{Cov}}_{nij}(-x)\Gamma_{nij}(-x)$ can be written as

$$\widehat{\mathrm{Cov}}_{nij}(-x)\Gamma_{nij}(-x) \;=\; \int_{W_{ni}\cap(W_{nj}+x)} Y_i(y)Y_j(y-x)G_i(W_n,y)G_j(W_n,y-x)\,dy - \widehat{\mu}_{ni}\widehat{\mu}_{nj}\Gamma_{nji}(x)\,.$$

Defining $\widehat{\mathrm{C}}_{ij}(x) = \int_{W_{ni}\cap(W_{nj}+x)} Y_i(y)Y_j(y-x)G_i(W_n,y)G_j(W_n,y-x)\,dy$ we see that

$$\widehat{\mathrm{C}}_{ij}(x) \quad = \quad \int_{\mathbb{R}^d} f_i(y)\cdot h_j(x-y)\,dy \;=\; (f_i * h_j)(x), \qquad x \in \check{U}_{nij} \tag{19}$$

with $f_i(z) = Y_i(z)G_i(W_n,z)$ and $h_j(z) = f_j(-z)$ for $z \in \mathbb{R}^d$. Besides this, we have

$$\Gamma_{nji}(x) \quad = \quad \int_{\mathbb{R}^d} f'_i(y)\cdot h'_j(x-y)\,dy \;=\; (f'_i * h'_j)(x), \qquad x \in \check{U}_{nij}$$

for $f'_i(z) = G_i(W_n,z)$ and $h'_j(z) = f'_j(-z), z \in \mathbb{R}^d$.

Suppose now, that the integral (19) is discretized into a sum of n measurement points $x_1,\ldots,x_n$ within $W_{ni}\cap W_{nj}$, then the computation of one Fourier transform using FFT has complexity $O(n\log(n))$. If the measurement points and V_n are chosen in such a way that approximately $n/N(n)$ points lie inside of $V_{ni,k}\cap V_{nj,k}$ for each $k=1,...,N(n)$, then the same computation using the subdivision of the observation window requires only $O(n\log(n/N(n)))$ operations. A discussion about the balance between accuracy of the estimation and computation time can be found in Section 6.

4 Random fields associated with the Boolean model

The aim of this section is to give a simple sufficient condition for the integrability conditions (13) and (14) for a special class of jointly stationary random fields. As mentioned at the beginning, these fields are functionals of particular stationary random closed sets.

4.1 Boolean model

Let $X = \{X_l\}$ be a stationary Poisson point process on $\mathbb{R}^d$ with finite intensity $\lambda > 0$. To each *germ* X_l affix an independent copy M_l of a non-empty compact and convex random set M_0 called the *typical grain*. The sequence $M = \{M_l\}$ of grains has to be independent of the point process X. A Boolean model Ξ with convex grains is defined as the set-theoretic union of the translated RACS $M_l + X_l$, i.e.,

$$\Xi = \bigcup_{l=1}^{\infty} (M_l + X_l)\,. \tag{20}$$

The right-hand side of (20) is almost surely closed and different from $\mathbb{R}^d$ if

$$\mathbb{E}\,|M_0 \oplus \check{K}| < \infty \qquad \text{for any } K \in \mathcal{K}\,; \tag{21}$$

see Heinrich (1992). Moreover, a Boolean model Ξ with convex grains M_l can be represented as the union set of a Poisson particle process $\Psi = \{\Psi_l\}$ on $\mathcal{K}$, where particles Ψ_l are defined as $\Psi_l = M_l + X_l$, confer Schneider and Weil (2000, Section 4.4). The intensity measure of Ψ is denoted by Λ, where $\Lambda(B) = \mathbb{E}\,\Psi(B)$ is the expected number of particles belonging to B for $B \in \sigma_{\mathcal{K}}$. Let $g_{\Psi(B)}(s) = \mathbb{E}\left(s^{\Psi(B)}\right)$, $s \in \mathbb{R}$ be the generating function of $\Psi(B)$. Then it holds that

$$g_{\Psi(B)}(s) = e^{(s-1)\,\Psi(B)}, \qquad s \in \mathbb{R}\,. \tag{22}$$

An application of Campbell's theorem for independently marked point processes in $\mathbb{R}^d$ (see, e.g. Schneider and Weil (2000, Section 3)) yields

$$\begin{aligned}\Lambda(B) \;&=\; \mathbb{E} \sum_{\psi\,\in\Psi} \mathbb{1}(\psi \in B) \;=\; \mathbb{E} \sum_{y\in X, M_0 \in M} \mathbb{1}((M_0 + y) \in B) \\ &=\; \lambda \int_{\mathbb{R}^d} \mathbb{E}\,\mathbb{1}((M_0 + y) \in B)\,dy \end{aligned}\tag{23}$$

for all $B \in \sigma_{\mathcal{K}}$. Considering $\mathcal{K}_K = \{K' \in \mathcal{K} : K' \cap K \neq \emptyset\}$, we find that $\Lambda(\mathcal{K}_K) = \lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|$ so that $g_{\Psi(\mathcal{K}_K)}(s) < \infty$ for any $s \in \mathbb{R}$ given that (21) is fulfilled.

4.2 Associated random fields

Let $f : \mathcal{R} \to \mathbb{R}$ be a functional on the convex ring $\mathcal{R}$, which is the family of all finite unions of sets from $\mathcal{K}$. We say, f is a *valuation* on $\mathcal{R}$ if f is measurable and additive, i.e.,

$$f(K_1 \cup K_2) = f(K_1) + f(K_2) - f(K_1 \cap K_2)$$

for any $K_1, K_2 \in \mathcal{R}$ and $f(\emptyset) = 0$. Furthermore, we assume that f is *conditionally bounded* on $\mathcal{K}$, meaning that for any $K \in \mathcal{K}$ there exists a finite bound $c(K)$ such that for all $K' \in \mathcal{K}$ with $K' \subseteq K$ it holds that

$$|f(K')| \;\leq\; c(K)\,.$$

For any fixed convex body $K \in \mathcal{K}$ and for any Boolean model Ξ satisfying (21) the random set $\Xi \cap K$ belongs to $\mathcal{R}$ with probability one. Consequently, we may consider the random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ defined by

$$Y(x) = f\left((\Xi - x) \cap K\right) , \qquad x \in \mathbb{R}^d . \tag{24}$$

Since Ξ is stationary, the random field Y is stationary. Notice that by (22) all moments of $Y(x), x \in \mathbb{R}^d$ are finite, since one can show that

$$\mathbb{E}\,|Y^p(x)| \;\leq\; c^p(K) e^{(2^p-1)\lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|} \qquad \text{for any } p > 0\,, \tag{25}$$

see for example Lemma 2.1 of Pantle *et al.* (2006). Finally, consider $m \in \mathbb{N}$ pairs $(f_i, K_i), i = 1, \ldots, m$ of conditionally bounded valuations $f_i : \mathcal{R} \to \mathbb{R}$ and convex test sets $K_i \in \mathcal{K}$. Then the vector field $Y = \{Y(x), x \in \mathbb{R}^d\}$ with $Y(x) = (Y_1(x), \ldots, Y_m(x))$ and $Y_i = \{f_i((\Xi - x) \cap K_i), x \in \mathbb{R}^d\}$ is of the form discussed in Section 3.

4.3 Integrability of mixed moments

In the following, we show that the vector field Y constructed above obeys the integrability conditions (13) and (14) under second moment conditions on the volume of the dilated primary grain $M_0 \oplus \check{K}$, $K \in \mathcal{K}$. To do this, we use the representation of Ξ as the union set of the generating Poisson particle process $\Psi = \{\Psi_l\}$ on $\mathcal{K}$ with intensity measure Λ as introduced above. In Lemma 4.1 of Pantle *et al.* (2006) it has already been shown that (7), i.e. $\int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, dx < \infty$, holds under the assumption $\mathbb{E}\,|M_0 \oplus \check{K}_i|^2 < \infty$ for $i = 1, \ldots, m$. The subsequent theorem yields that (13) and (14) are satisfied under the very same condition.

Theorem 2. *If* $\mathbb{E}\,|M_0 \oplus \check{K}_i|^2 < \infty$ *for* $i = 1, \ldots, m$*, then there exist constants* $\kappa_1, \kappa_2 < \infty$ *such that*

$$\sup_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left|\mathrm{Cov}\big(Y_i(o) Y_j(x_1), Y_i(y) Y_j(x_2 + y)\big)\right| dy \;\leq\; \kappa_1$$

and

$$\sup_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left|\mathbb{E}\big([Y_i(o) - \mu_i]\,[Y_i(y) - \mu_i]\, Y_j(x_1)\, Y_j(x_2)\big)\right| dy \;\leq\; \kappa_2$$

for all $i, j = 1, ..., m$.

Proof. We use a similar construction as in the proof of Lemma 4.1 in Pantle *et al.* (2006). For simplicity, consider the case $i = j$ and omit the indices in the following. If $i \neq j$ the proof is analogous. With $\mathcal{K}_C = \{K' \in \mathcal{K} : K' \cap C \neq \emptyset\}$ for any $C \subset \mathbb{R}^d$ define $\mathcal{K}^* = \mathcal{K}^*(x_1, x_2, y) = (\mathcal{K}_K \cup \mathcal{K}_{K+x_1}) \cap (\mathcal{K}_{K+y} \cup \mathcal{K}_{K+x_2+y})$ for $x_1, x_2, y \in \mathbb{R}^d$. Now, consider the event $A = \{\Psi(\mathcal{K}^*) > 0\}$ and its complement A^c, where $\Psi(B)$ is the random number of particles of Ψ belonging to a set $B \subseteq \mathcal{K}$. Then, it holds that

$$\begin{aligned}
&\left|\mathrm{Cov}\big(Y(o)Y(x_1), Y(y)Y(x_2 + y)\big)\right| \\
&\quad= \left|\mathbb{E}\big(\,Y(o)Y(x_1)\,(\mathbb{1}_A + \mathbb{1}_{A^c}) \cdot \big[Y(y)Y(x_2 + y) - \mathbb{E}\,(Y(y)Y(x_2 + y))\big]\big)\right| \\
&\quad= \Big|\mathbb{E}\big(\,Y(o)Y(x_1)\,\mathbb{1}_A \cdot \big[Y(y)Y(y + x_2) - \mathbb{E}\,(Y(y)Y(y + x_2))\big]\big) \\
&\qquad+ \mathbb{E}\big(\,Y(o)Y(x_1)\,\mathbb{1}_{A^c} \cdot \big[Y(y)Y(y + x_2) - \mathbb{E}\,(Y(y)Y(y + x_2))\big]\big)\Big|.
\end{aligned} \tag{26}$$

In the first step, we investigate the summand (26). Define $Y_B(x) = f\big((\bigcup_{l:\Psi_l\in B}\Psi_l - x)\cap K\big)$ for any $B\subseteq\mathcal{K}$ and $x\in\mathbb{R}^d$. By additivity of f we see that

$$\begin{aligned} Y(x) &= Y_B(x) + Y_{\mathcal{K}\setminus B}(x) - f\big((\bigcup_{l:\Psi_l\in B}(\Psi_l) - x)\ \cap (\bigcup_{l:\Psi_l\in\mathcal{K}\setminus B}\Psi_l - x)\ \cap K\big) \\ &= Y_{B\cap\mathcal{K}_{K+x}}(x) + Y_{\mathcal{K}_{K+x}\setminus B}(x) - f\big((\bigcup_{l:\Psi_l\in B\cap\mathcal{K}_{K+x}}\Psi_l - x)\ \cap (\bigcup_{l:\Psi_l\in\mathcal{K}_{K+x}\setminus B}\Psi_l - x)\ \cap K\big) \end{aligned}$$

Thus, we have $Y(x)\mathbb{I}(\Psi(\mathcal{K}^*)=0) = Y_{\mathcal{K}_{K+x}\setminus\mathcal{K}^*}(x)\mathbb{I}(\Psi(\mathcal{K}^*)=0)$ almost surely and obtain the following equations

$$Y(o)Y(x_1)\cdot\mathbb{I}_{A^c} = Y_{\mathcal{K}_K\setminus\mathcal{K}^*}(o)Y_{\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*}(x_1)\cdot\mathbb{I}_{A^c}$$

and

$$Y(y)Y(y+x_2)\cdot\mathbb{I}_{A^c} = Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2)\cdot\mathbb{I}_{A^c}.$$

The random variables $Y_{\mathcal{K}_K\setminus\mathcal{K}^*}(o)Y_{\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*}(x_1)\cdot\mathbb{I}_{A^c}$ and $Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2)$ are independent, since the particles Ψ_l involved are mutually independent. As a result we get

$$\begin{aligned} &\mathbb{E}\big(Y(o)Y(x_1)\mathbb{I}_{A^c}\cdot\big[Y(y)Y(y+x_2) - \mathbb{E}\,(Y(y)Y(y+x_2))\big]\big) \\ &= \mathbb{E}(Y_{\mathcal{K}_K\setminus\mathcal{K}^*}(o)Y_{\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*}(x_1)\,\mathbb{I}_{A^c})\cdot\big[\,\mathbb{E}(Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2)) - \mathbb{E}\,(Y(y)Y(y+x_2)]\big) \\ &= \mathbb{E}(Y(o)Y(x_1)\,\mathbb{I}_{A^c})\cdot\mathbb{E}\big(\,[Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2) - Y(y)Y(y+x_2)]\,\mathbb{I}_A\big). \end{aligned}$$

Insert the above equations into (26) to obtain by triangle inequality

$$\begin{aligned} &\big|\mathrm{Cov}\big(Y(o)Y(x_1), Y(y)Y(x_2+y)\big)\big| \\ &\le \big|\mathbb{E}\big(\,Y(o)Y(x_1)\,\mathbb{I}_A\cdot\big[Y(y)Y(y+x_2) - \mathbb{E}\,(Y(y)Y(y+x_2))\big]\big)\big| \\ &\quad + \mathbb{E}\big|Y(o)Y(x_1)\big|\cdot\big|\mathbb{E}\big([Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2) - Y(y)Y(y+x_2)]\mathbb{I}_A\big)\big|\,. \end{aligned}$$

For the first summand in the upper bound we can conclude that

$$\begin{aligned} &\big|\mathbb{E}\big(Y(o)Y(x_1)\,\mathbb{I}_A\cdot\big[Y(y)Y(y+x_2) - \mathbb{E}\,(Y(y)Y(y+x_2))\big]\,\big)\big| \\ &\le c^2(K)\,\mathbb{E}\big(2^{\Psi(\mathcal{K}_K)+\Psi(\mathcal{K}_{K+x_1})}\cdot[c^2(K)\,2^{\Psi(\mathcal{K}_{K+y})+\Psi(\mathcal{K}_{K+y+x_2})} + \mathbb{E}\,|Y(y)Y(y+x_2)|\,]\cdot\mathbb{I}_A\big). \end{aligned}$$

Since Ψ is a Poisson process, the random variables $\Psi(B)$ and $\Psi(B')$ are independent for any two disjount sets $B, B'\subseteq\mathcal{K}$, whence

$$\begin{aligned} &\mathbb{E}\big(2^{\Psi(\mathcal{K}_K)+\Psi(\mathcal{K}_{K+x_1})+\Psi(\mathcal{K}_{K+y})+\Psi(\mathcal{K}_{K+y+x_2})}\cdot\mathbb{I}_A\big) \\ &= \mathbb{E}\,2^{\Psi(\mathcal{K}_K\setminus\mathcal{K}^*)+\Psi(\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*)}\cdot\mathbb{E}\,2^{\Psi(\mathcal{K}_{K+y}\setminus\mathcal{K}^*)+\Psi(\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*)}\cdot\mathbb{E}\big(2^{4\Psi(\mathcal{K}^*)}\,\mathbb{I}_A\big) \end{aligned}$$

and

$$\mathbb{E}\,\big(2^{\Psi(\mathcal{K}_K)+\Psi(\mathcal{K}_{K+x_1})}\cdot\mathbb{I}_A\big) = \mathbb{E}\,2^{\Psi(\mathcal{K}_K\setminus\mathcal{K}^*)+\Psi(\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*)}\cdot\mathbb{E}\,\big(2^{2\,\Psi(\mathcal{K}^*)}\,\mathbb{I}_A\big)\,.$$

For the second summand of the considered upper bound, we analogously obtain

$$
\begin{aligned}
&\left|\mathbb{E}\big(\big[Y(y)Y(y+x_2) - Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2)\big]\,\mathbb{1}_A\big)\right| \\
&\quad\le\ c^2(K)\,\mathbb{E}\,2^{\Psi(\mathcal{K}_{K+y})+\Psi(\mathcal{K}_{K+y+x_2})}\,\mathbb{E}\big([2^{2\Psi(\mathcal{K}^*)}+1]\,\mathbb{1}_A\big)\,.
\end{aligned}
$$

Due to the fact that $c(K) < \infty$, $\mathbb{E}\,|Y(x)Y(y)| < \infty$ and $\mathbb{E}\,2^{\Psi(\mathcal{K}_{K+x}\setminus\mathcal{K}^*)+\Psi(\mathcal{K}_{K+y}\setminus\mathcal{K}^*)} \le g_{\Psi(\mathcal{K}_K)}(4) < \infty$ for any $x, y \in \mathbb{R}^d$, we may concentrate on $\mathbb{E}\,(s^{\Psi(\mathcal{K}^*)}\,\mathbb{1}_A)$, say, for arbitrary $s \in \mathbb{R}_+$. This quantity, however, obeys the estimate

$$
\begin{aligned}
\mathbb{E}\,(s^{\Psi(\mathcal{K}^*)}\mathbb{1}_A) &= \mathbb{E}\,(s^{\Psi(\mathcal{K}^*)}) - \mathbb{E}\,(\mathbb{1}_{A^c}) = e^{(s-1)\Lambda(\mathcal{K}^*)} - e^{-\Lambda(\mathcal{K}^*)} \\
&\le s\,e^{(s-1)\Lambda(\mathcal{K}^*)}\Lambda(\mathcal{K}^*) \le s\,e^{(s-1)\lambda\,\mathbb{E}\,|M_0\oplus\check{K}|}\Lambda(\mathcal{K}^*)
\end{aligned}
$$

using that $1 - e^{-x} \le x$ for any $x \in \mathbb{R}_+$ and $\Lambda(\mathcal{K}^*) \le \Lambda(\mathcal{K}_K) + \Lambda(\mathcal{K}_{K+x_1}) = 2\lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|$ by (23). It remains to show that $\Lambda(\mathcal{K}^*) = \Lambda(\mathcal{K}^*(x_1, x_2, y))$ is integrable with respect to $y \in \mathbb{R}^d$ and that the integral admits an upper bound uniformly in $x_1, x_2 \in \mathbb{R}^d$. Employing equation (23) and Fubini's theorem we get

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \Lambda(\mathcal{K}^*(x_1,x_2,y))\,dy \\
&= \lambda\,\mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}\big[z \in \check{M}_0 \oplus K \cup (\check{M}_0 \oplus (K+x_1))\big]\,\mathbb{1}\big[z-y \in (\check{M}_0\oplus K)\cup(\check{M}_0\oplus(K+x_2))\big]\,dy\,dz \\
&\le \lambda\,\mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(\mathbb{1}[z \in \check{M}_0 \oplus K] + \mathbb{1}[z - x_1 \in \check{M}\oplus K]\big) \\
&\qquad\qquad \times\ \big(\mathbb{1}[z-y \in \check{M}_0 \oplus K] + \mathbb{1}[z-y \in \check{M}_0 \oplus (K+x_2)]\big)\,dy\,dz \\
&= 4\lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|^2 < \infty.
\end{aligned}
$$

for arbitrary $x_1, x_2 \in \mathbb{R}^d$. Combining all estimates, we conclude that there exists a constant $\kappa_1 = \kappa_1(M_0, \lambda, K)$, which depends on M_0 and K through $|M_0 \oplus \check{K}|$ and $c(K)$, such that for all $x_1, x_2 \in \mathbb{R}^d$ it holds $\int_{\mathbb{R}^d} |\mathrm{Cov}(Y(o)Y(x_1), Y(y)Y(x_2+y))| \le \kappa_1$.

The second assertion of Theorem 2 can be shown following the same idea. Let $\mathcal{K}^{**} = \mathcal{K}^{**}(x_1, x_2, y) = \mathcal{K}_{K+y} \cap (\mathcal{K}_K \cup \mathcal{K}_{K+x_1} \cup \mathcal{K}_{K+x_2})$ for $x_1, x_2, y \in \mathbb{R}^d$ and consider the event $\bar{A} = \{\Psi(\mathcal{K}^{**}) > 0\}$. Then, by the same arguments as before, it follows that

$$
\begin{aligned}
\big|\mathbb{E}\big([Y(o)-\mu]\,[Y(y)-\mu]\,Y(x_1)\,Y(x_2)\big)\big| &\le \big|\mathbb{E}\big([Y(o)-\mu]\,[Y(y)-\mu]\,Y(x_1)\,Y(x_2)\,\mathbb{1}_{\bar{A}}\big)\big| \\
&\quad + \mathbb{E}\big|[Y(o)-\mu]\,Y(x_1)\,Y(x_2)\big|\big|\mathbb{E}\big([Y_{\mathcal{K}_{K+y}}(y) - Y(y)]\mathbb{1}_{\bar{A}}\big)\big| \\
&\le \kappa(M_0,\lambda,K)\cdot\Lambda(\mathcal{K}^{**}(x_1,x_2,y))
\end{aligned}
$$

for some finite bound $\kappa(M_0, \lambda, K)$ depending on K and M_0 through $c(K)$ and $\mathbb{E}\,|M_0 \oplus \check{K}|$ only. Furthermore, we have for any $x_1, x_2 \in \mathbb{R}^d$ that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \Lambda(\mathcal{K}^{**}(x_1,x_2,y))\,dy \\
&= \lambda\,\mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}\big[y - z \in M_0 \oplus \check{K}\big]\,\mathbb{1}\big[z \in (\check{M}_0 \oplus K)\cup(\check{M}_0\oplus(K+x_1))\cup(\check{M}_0\oplus(K+x_2))\big]\,dy\,dz \\
&\le 3\lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|^2 < \infty.
\end{aligned}
$$

□

5 The empirical covariance

The estimator $\widehat{\sigma}_{nij}$ introduced in (10) makes use of the fact that σ_{ij} is basically the integral of $\mathrm{Cov}_{ij}(x)$ over $\mathbb{R}^d$. Thus, estimating the covariance and considering the integral over an unboundedly increasing window is a quite intuitive approach. Another possibility is to employ that $\sigma_{ij} = \lim_{n\to\infty} |W_n|\mathrm{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj})$. Provided that there exist samples $\sqrt{|W_n|}\widehat{\mu}_{ni,1}, \ldots, \sqrt{|W_n|}\widehat{\mu}_{ni,N(n)}$ and $\sqrt{|W_n|}\widehat{\mu}_{nj,1}, \ldots, \sqrt{|W_n|}\widehat{\mu}_{nj,N(n)}$ for some $N(n) \in \mathbb{N}$, a natural estimate is given by the empirical covariance of these samples. Initially, we assumed that there exists only one value $\widehat{\mu}_{ni}$ and $\widehat{\mu}_{nj}$, respectively. Yet, using the idea of Section 3.2, enables us to generate a sample of size $N(n)$.

Let $\{V_n\}$, $\{N(n)\}$ and $h_{n,1}, \ldots, h_{n,N(n)}$ be given as in Section 3.2 with $\lim_{n\to\infty} N(n) = \infty$. In addition to condition (16), we assume that there exists some $r > 0$ such that

$$V_{n,k} \cap V_{n,\ell} \subset \partial V_{n,k} \oplus B_r(o) \qquad \text{for } k, \ell \in \{1, \ldots, N(n)\} \text{ with } k \neq \ell. \tag{27}$$

Assuming this setup, define a new estimator $\widetilde{\Sigma}_n = (\widetilde{\sigma}_{nij})$ by the formula

$$\widetilde{\sigma}_{nij} = \frac{|V_n|}{N(n)-1} \sum_{k=1}^{N(n)} \big(\widehat{\mu}_{ni,k} - \overline{\mu}_{ni}\big)\big(\widehat{\mu}_{nj,k} - \overline{\mu}_{nj}\big), \tag{28}$$

where $\widehat{\mu}_{ni,k} = \int_{V_{n,k}} Y_i(x) G_i(V_{n,k}, x)\, dx$ and $\overline{\mu}_{ni} = \frac{1}{N(n)} \sum_{k=1}^{N(n)} \widehat{\mu}_{ni,k}$ for $i = 1, \ldots, m$.

The estimator $\widetilde{\Sigma}_n$ is asymptotically unbiased under the same assumptions as considered for $\widehat{\Sigma}_n$, but mean-square consistency requires integrability condition (15).

Lemma 2. *The estimator $\widetilde{\Sigma}_n$ defined in (28) is asymptotically unbiased for Σ as $n \to \infty$.*

Proof. Set $N = N(n)$ and employ the following representation

$$\widetilde{\sigma}_{nij} = \frac{|V_n|}{N} \sum_{k=1}^{N} \big(\widehat{\mu}_{ni,k} - \mu_i\big)\big(\widehat{\mu}_{nj,k} - \mu_j\big) - \frac{|V_n|}{N(N-1)} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^{N} \big(\widehat{\mu}_{ni,k} - \mu_i\big)\big(\widehat{\mu}_{nj,\ell} - \mu_j\big). \tag{29}$$

This formula can be derived by elementary transformation of $\sum_{k=1}^{N} \big(\widehat{\mu}_{ni,k} - \overline{\mu}_{ni} \pm \mu_i\big)\big(\widehat{\mu}_{nj,k} - \overline{\mu}_{nj} \pm \mu_j\big)$. With regard to the expectation of the first summand in (29) we get

$$\begin{aligned}
\frac{|V_n|}{N} \sum_{k=1}^{N} \mathbb{E}\,\big(\widehat{\mu}_{ni,k} - \mu_i\big)\big(\widehat{\mu}_{nj,k} - \mu_j\big) &= \frac{|V_n|}{N} \sum_{k=1}^{N} \int_{V_{n,k}} \int_{V_{n,k}} \mathrm{Cov}_{ij}(y-x) G_i(V_{n,k}, x) G_j(V_{n,k}, y)\, dx\, dy \\
&= \frac{|V_n|}{N} \sum_{k=1}^{N} \int_{V_n} \int_{V_n} \mathrm{Cov}_{ij}(y-x) G_i(V_n, x) G_j(V_n, y)\, dx\, dy \\
&= \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, |V_n| \Gamma_{V_{nij}}(x)\, dx\,,
\end{aligned}$$

where we used condition (16) in the second line. The last expression converges to σ_{ij} as $n \to \infty$ by the dominated convergence theorem, given (7) and (5). It remains to show that the expectation of the second expression in (29) tends to zero as $n \to \infty$. In fact, one obtains for any $k \neq \ell$ that

$$\begin{aligned}\mathbb{E}\left(\widehat{\mu}_{ni,k} - \mu_i\right)\left(\widehat{\mu}_{nj,\ell} - \mu_j\right) &= \int_{V_{n,k}} \int_{V_{n,\ell}} \mathrm{Cov}_{ij}(y-x) G_i(V_{n,k}, x) G_j(V_{n,\ell}, y)\, dx\, dy \\ &\leq c_1^2 \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)| \frac{|V_{n,k} \cap (V_{n,\ell} - x)|}{|V_n|^2}\, dx\,.\end{aligned}$$

By (27), it holds that $|V_{n,k} \cap (V_{n,\ell} - x)| \leq |(\partial V_n \oplus B_{|x|+r}(o)) \cap V_n|$ for each $l \neq k, l, k = 1, \ldots, N$. Thus, assuming (1) and (7) we have

$$\mathbb{E}\Big(\frac{|V_n|}{N(N-1)} \sum_{\substack{k,\ell=1\\ k \neq \ell}}^{N} \left(\widehat{\mu}_{ni,k} - \mu_i\right)\left(\widehat{\mu}_{nj,\ell} - \mu_j\right)\Big) \leq c_1^2 \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)| \frac{|(\partial V_n \oplus B_{|x|+r}(o)) \cap V_n|}{|V_n|}\, dx \to 0$$

as n tends to infinity. □

Theorem 3. *If condition* (15) *holds, then* $\widetilde{\Sigma}_n$ *is a mean-square consistent estimator of* Σ.

Proof. By means of Lemma 2, it suffices to show that $\lim_{n\to\infty} \mathbb{E}\,(\widetilde{\sigma}_{nij} - \mathbb{E}\,\widetilde{\sigma}_{nij})^2 = 0$. To simplify notation, write $\omega_{ij,kn}(x,y) = G_i(V_{n,k}, x) G_j(V_{n,k}, y)$ for all $i, j = 1, ..., m, x, y \in \mathbb{R}^d$. As in the proof of Lemma 2 consider the two summands of formula (29) separately, using the fact that the covariance of two random variables converges to zero whenever the variances do. For the variance of the first summand we observe that

$$\begin{aligned}&\mathbb{E}\Big(\frac{|V_n|}{N} \sum_{k=1}^{N} \Big(\left(\widehat{\mu}_{ni,k} - \mu_i\right)\left(\widehat{\mu}_{nj,k} - \mu_j\right) - \int_{V_{n,k}} \int_{V_{n,k}} \mathrm{Cov}_{ij}(y-x)\, \omega_{ij,kn}(x,y)\, dx\, dy\Big)\Big)^2 \\ &= \frac{|V_n|^2}{N^2} \sum_{k,\ell=1}^{N} \int_{V_{n,k}^2} \int_{V_{n,\ell}^2} \Big(\mathbb{E}\left((Y_i(v_1) - \mu_i)(Y_j(v_2) - \mu_j)(Y_i(v_3) - \mu_i)(Y_j(v_4) - \mu_j)\right) \\ &\qquad - \mathrm{Cov}_{ij}(v_2 - v_1)\mathrm{Cov}_{ij}(v_4 - v_3)\Big)\, \omega_{ij,kn}(v_1, v_2)\, \omega_{ij,\ell n}(v_3, v_4)\, d(v_1, v_2)\, d(v_3, v_4)\,.\end{aligned}$$

For any $v_1, v_2, v_3, v_4 \in \mathbb{R}^d$ set $c_{ij}^{(4)}(v_1, v_2, v_3, v_4) = \mathbb{E}\left((Y_i(v_1) - \mu_i)(Y_j(v_2) - \mu_j)(Y_i(v_3) - \mu_i)(Y_j(v_4) - \mu_j)\right)$ $- \mathrm{Cov}_{ij}(v_2 - v_1)\mathrm{Cov}_{ij}(v_4 - v_3) - \mathrm{Cov}_{ii}(v_3 - v_1)\mathrm{Cov}_{jj}(v_4 - v_2) - \mathrm{Cov}_{ij}(v_4 - v_1)\mathrm{Cov}_{ij}(v_2 - v_3)$. Then it follows that

$$\begin{aligned}&\int_{V_{n,k}^2} \int_{V_{n,\ell}^2} \Big(\mathbb{E}\left((Y_i(v_1) - \mu_i)(Y_j(v_2) - \mu_j)(Y_i(v_3) - \mu_i)(Y_j(v_4) - \mu_j)\right) - \mathrm{Cov}_{ij}(v_2 - v_1)\mathrm{Cov}_{ij}(v_4 - v_3)\Big) \\ &\qquad \times \omega_{ij,kn}(v_1, v_2)\, \omega_{ij,\ell n}(v_3, v_4)\, d(v_1, v_2, v_3, v_4) \\ &= \int_{V_{n,k}^2} \int_{V_{n,\ell}^2} c_{ij}^{(4)}(v_1, v_2, v_3, v_4)\, \omega_{ij,kn}(v_1, v_2)\, \omega_{ij,\ell n}(v_3, v_4)\, d(v_1, v_2, v_3, v_4) \\ &\quad + \int_{V_{n,k}^2} \int_{V_{n,\ell}^2} \Big(\mathrm{Cov}_{ii}(v_3 - v_1)\mathrm{Cov}_{jj}(v_4 - v_2) + \mathrm{Cov}_{ij}(v_4 - v_1)\mathrm{Cov}_{ij}(v_2 - v_3)\Big) \\ &\qquad \times \omega_{ij,kn}(v_1, v_2)\, \omega_{ij,\ell n}(v_3, v_4)\, d(v_1, v_2, v_3, v_4)\end{aligned}$$

Using the same arguments as in the proof of Lemma 2, one obtains

$$\frac{|V_n|^2}{N^2}\sum_{k,\ell=1}^{N}\int_{V_{n,k}^2}\int_{V_{n,\ell}^2}\Big(\mathbb{E}\left((Y_i(v_1)-\mu_i)(Y_j(v_2)-\mu_j)(Y_i(v_3)-\mu_i)(Y_j(v_4)-\mu_j)\right)$$
$$-\,\mathrm{Cov}_{ij}(v_2-v_1)\mathrm{Cov}_{ij}(v_4-v_3)\Big)\,\omega_{ij,kn}(v_1,v_2)\,\omega_{ij,\ell n}(v_3,v_4)\,d(v_1,v_2)\,d(v_3,v_4)$$
$$\leq\ \frac{c_1^4}{|V_n|}\cdot\int_{\mathbb{R}^{3d}}|\,c_{ij}^{(4)}(o,x_1,x_2,x_3)|\,d(x_1,x_2,x_3)+$$

$$+\,\frac{c_1^4}{N^2}\sum_{k,\ell=1}^{N}\int_{\mathbb{R}^d}|\,\mathrm{Cov}_{ii}(x)|\,\frac{|V_{ni,k}\cap(V_{ni,l}-x)|}{|V_n|}\,dx\int_{\mathbb{R}^d}|\,\mathrm{Cov}_{jj}(x)|\,dx$$
$$+\,\frac{c_1^4}{N^2}\sum_{k,\ell=1}^{N}\int_{\mathbb{R}^d}|\,\mathrm{Cov}_{ij}(x)|\,\frac{|V_{ni,k}\cap(V_{nj,l}-x)|}{|V_n|}\,dx\int_{\mathbb{R}^d}|\,\mathrm{Cov}_{ij}(x)|\,dx$$
$$\leq\ \frac{c_1^4}{|V_n|}\cdot\kappa+c_1^4\int_{\mathbb{R}^d}|\,\mathrm{Cov}_{ii}(x)|\cdot\left(\frac{1}{N}+\frac{|(\partial V_n\oplus B_{|x|+r}(o))\cap V_n|}{|V_n|}\right)dx\,\int_{\mathbb{R}^d}|\,\mathrm{Cov}_{jj}(x)|\,dx$$
$$+\,c_1^4\int_{\mathbb{R}^d}|\,\mathrm{Cov}_{ij}(x)|\cdot\left(\frac{1}{N}+\frac{|(\partial V_n\oplus B_{|x|+r}(o))\cap V_n|}{|V_n|}\right)dx\,\int_{\mathbb{R}^d}|\,\mathrm{Cov}_{ij}(x)|\,dx\,.$$

The last expression tends to zero, since $\lim_{n\to\infty}|V_n|=\lim_{n\to\infty}N(n)=\infty$. Besides that, it can now easily be derived that the variance of the second summand of (29) obeys similar estimates and, consequently, congerges to zero as well. □

If $Y(x)=\mathbb{1}(x\in\Xi)$, where Ξ is a Boolean model with compact typical grain M_0, then it is known from Lemma 7 of Heinrich (2005) that a sufficient assumption for the conditions of Theorem 3 is given by $\mathbb{E}\,|M_0|^4<\infty$. Notice that $\widehat{\Sigma}_n$, on the contrary, is mean–square consistent in this particular case provided that M_0 is compact and convex and $\mathbb{E}\,|M_0|^2<\infty$, compare Theorem 2.

6 Numerical examples for the Boolean model

We conclude with a numerical comparison of the three estimators $\widehat{\Sigma}_n,\widehat{\Sigma}_n'$ and $\widetilde{\Sigma}_n$ with respect to 'goodness' and computational effort. Let Ξ be a stationary Boolean model with compact and convex grains such that (21) is satisfied. Then, the intrinsic volumes $V_j(\Xi\cap K), j=0,...,d$ of $\Xi\cap K$ are well defined for any convex body $K\in\mathcal{K}$, see e.g. Schneider (1993) or Schneider and Weil (2000) on intrinsic volumes. In the plane, i.e., $d=2$, for instance, $V_2(\Xi\cap K)$ is the usual area, $2V_1(\Xi\cap K)$ is the boundary length and $V_0(\Xi\cap K)$ is the *Euler-Poincaré characteristic* of the set $\Xi\cap K$. In the following, we consider the so–called *intrinsic volume densities* of a Boolean model Ξ and the corresponding integrated covariance functions. For any sequence $\{K_n\}$ of convex bodies $K_n=nK_0$ with $K_0\in\mathcal{K}$ such that $|K_0|>0$ and $o\in\mathrm{int}(K_0)$, the limits

$$\overline{V}_j(\Xi)=\lim_{n\to\infty}\frac{\mathbb{E}\,V_j(\Xi\cap K_n)}{|K_n|}\,,\qquad j=0,\dots,d \tag{30}$$

exist and are called the intrinsic volume densities of Ξ. For some intrinsic volume densities, estimators of several types are considered in the literature. The following indirect estimation method has been proposed in Spodarev and Schmidt (2005). Let $Y_i(x) = V_0(\Xi \cap B_{r_i}(x))$ for $m > d+1$ distinct radii $r_i, i = 1, ..., m$. From Spodarev and Schmidt (2005, Section 4.3) it follows that

$$\widehat{v}_n = \left(A^\top_{r_1,\dots,r_m} A_{r_1,\dots,r_m}\right)^{-1} A^\top_{r_1,\dots,r_m} \widehat{\mu}_n$$

is a least-squares estimator for $v = (\overline{V}_0(\Xi), \dots, \overline{V}_d(\Xi))$ minimizing $|A_{r_1,\dots,r_m}\widehat{\mu}_n - v|$ on $\mathbb{R}^{d+1}$, where $\widehat{\mu}_n$ is defined in (2) and $A_{r_1,\dots,r_m}$ is a specific $m \times (d+1)$–dimensional matrix of rank $d+1$; see Spodarev and Schmidt (2005) for details. We also refer to the article Guderlei *et al.* (2006) for related implementation issues. A major advantage of this estimation method for our purpose is that the values $Y_i(x)$ can be determined for each point x inside the observation window W explicitly with acceptable runtime. To assess the quality of the estimates of Σ, the transformed estimators $\widehat{C}_n, \widehat{C}'_n$ and $\widetilde{C}_n$ are compared, where

$$\widehat{C}_n \;=\; \left(A^\top_{r_1,\dots,r_m} A_{r_1,\dots,r_m}\right)^{-1} A^\top_{r_1,\dots,r_m} \widehat{\Sigma}_n \, A_{r_1,\dots,r_m} \left(A^\top_{r_1,\dots,r_m} A_{r_1,\dots,r_m}\right)^{-1}, \quad n \geq 1$$

and $\widehat{C}'_n, \widetilde{C}_n$ are defined analogously. In other words, we compare the estimated values of the asymptotic covariance matrix C of $\sqrt{|W_n|}(\widehat{v}_n - v)$.

In the sequel, set $d = 2$ and skip the index n for simplicity. Several Boolean models Ξ with uniformly bounded grains were simulated in the observation window $W = [-1500, 1500]^2$. The intensity of the underlying point process is chosen so that $\overline{V}_2(\Xi) = 0.5$ for each setting. Notice that the values of $\overline{V}_1(\Xi)$ and $\overline{V}_0(\Xi)$ are also known for these models (see Stoyen *et al.* (1995, p. 76)). For the auxiliary vector $Y = (Y_1, \dots, Y_m)$, put $m = 16$ and $r_{i+1} = 4.2 + 1.3\,i$, $i = 0, \dots, 15$; confer Guderlei *et al.* (2006) on the choice of these parameters. Since $Y_i(x)$ and $Y_j(y)$ are independent if $|x - y| > r_i + r_j + 2r_{M_0}$, $x, y \in \mathbb{R}^2$, where $2r_{M_0}$ is the maximal (deterministic) diameter of the typical grain M_0, we put $U_{nij} = B_{r_i + r_j + 2r_{M_0}}(o)$. Any integral in the definitions of the estimators is discretized using observations on the grid $W \ominus B_r(o) \cap \Delta\mathbb{Z}^2$ for some $\Delta \in \mathbb{N}$ and $r = \max\{r_1, \dots, r_m\}$. For the computation of $\widehat{\Sigma}$, we choose the discrete weighting functions $g_i(W, x) = \mathbb{1}(x \in (W \ominus B_r(o)) \cap \Delta_1\mathbb{Z}^2)/\mathrm{card}((W \ominus B_r(o)) \cap \Delta_1\mathbb{Z}^2)$ with $\Delta_1 = 10$. Smaller values of grid mesh size Δ_1 lead to slightly more accurate results, measured with respect to the estimates of v. The running time, however, is unreasonably higher. The evaluation of $\widehat{\Sigma}'$ and $\widetilde{\Sigma}$ is performed both on $N = 9$ non–overlapping subwindows V_k of size 1000×1000, which we call the first partition, and on $N = 36$ subwindows $\widetilde{V}_k$ of size 500×500 each, which is the second partition. Again, uniform weights are assigned to each observation in $V_k \ominus B_r(o) \cap \Delta_1\mathbb{Z}^2$ with $\Delta_1 = 10$ as before and in $\widetilde{V}_k \ominus B_r(o) \cap \Delta_2\mathbb{Z}^2$ with $\Delta_2 = 5$. A finer mesh size Δ_2 is chosen for the second partition due to the reduced size of the subwindows. For each Boolean model, $k = 200$ simulations were performed. Typical simulation results are shown in the Tables 1–6 including the relative standard deviation δ (in percent) from the corresponding averaged values. The underlying Boolean model has typical grain $M_0 = B_R(o)$ with $R \sim U(20, 40)$ and intensity $\lambda = -\log(0.5)/\pi\,\mathbb{E}(R^2)$. On average, a Pentium IV (2.4 GHz) requires about 25 minutes for the evaluation of $\widehat{C}$ on $W = [-1500, 1500]^2$. It takes approximately 7 minutes to compute $\widehat{C}'$ on the first partition and 5 minutes on the second partition. As expected, since only very elementary operations are needed, the running time for $\widetilde{C}$ is even shorter with 3 minutes on average on both partitions.

Since analytic formulae for C are not available, a table of empirical co-/variances from 1000 independent samples of $\sqrt{|W|}(\widehat{v} - v)$ are displayed as reference values in Table 1. With respect to these reference values, $\widehat{C}$ provides the best results in most cases (see Table 2). The estimates $\widehat{C}'$ on the first partition (displayed in Table 3) are comparable to those of $\widehat{C}$ in domain and variability. Taking the required computational time into account, this variant seems to be the most favorable. When considering $\widehat{C}'$ on the second partition (see Table 5), we find that the variation of the estimates increases notably. At this, recall that in the case of $\widehat{C}'$ the subdivision of the observation window is only designed to enhance an efficient evaluation. On the contrary, the results for $\widetilde{C}$ get better the finer the subpartition is (compare Tables 4 and 6). The fluctuations in the estimated values $\widetilde{C}$ are still much higher than those of the other two estimators. The deviation from the reference values increases in several components on the second partition. This is not surprising, since here the estimation of v is performed on relatively smaller (sub)windows compared to the reference values. Experiments showed that the application of $\widetilde{C}$ is advisable only if the observation window W is sufficiently large so that it can be decomposed into 'sufficiently many' disjoint, but 'not too small' subwindows. In addition to the right choice of parameters m and $r_i, i = 1, \ldots, m$, the question of an adequate size and number of the subwindows makes $\widetilde{C}$ rather critical for application. A property not observable from the tables is the positive semidefiniteness of the covariance matrix estimates. By definition, the matrix $\widetilde{C}$ (resp. $\widetilde{\Sigma}$) is always positive semidefinite and proved to be positive definite in all simulated examples. Using $\widehat{C}$ on W or $\widehat{C}'$ on the first partition at most 2% of the estimates were not positive semidefinite, whereas 6–10% of the realizations of $\widehat{C}'$ on the second partition failed to be positive semidefinite. We also remark that all samples $\widehat{v}_1, ..., \widehat{v}_k$ were tested on multivariate normality using the test proposed in Henze and Zirkler (1990) with significance level $\alpha = 0.05$ and scaling parameter $\beta = 0.5, 1.0$ and 3.0, respectively. None of the tests lead to rejection of the multivariate normality assumption.

Note that there exists a simple direct estimation method for the area density $p = \overline{V}_2(\Xi)$ considered separately, see Böhm *et al.* (2004). The random field used here is given by $Y(x) = \mathbb{1}(x \in \Xi), x \in \mathbb{R}^d$. For this method, an explicit formula exists for the asymptotic variance σ_{pp} of $\sqrt{|W|}(\int_W Y(x)G(W_n, x)dx - p)$, where $\sigma_{pp} \approx 678.097$ in the considered example. The corresponding estimates $\widehat{\sigma}_{pp}, \widehat{\sigma}'_{pp}$ and $\widetilde{\sigma}_{pp}$ are attached to each table for comparison.

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Table 1: Empirical co-/variances of 1000 independent samples. Reference values for the unknown asymptotic covariance matrix.

	$\overline{c}_{0j}$	$\overline{c}_{1j}$	$\overline{c}_{2j}$
$\overline{c}_{i0}$	**1.83568e-4**	**-1.84006e-3**	**-0.16793**
$\overline{c}_{i1}$		**0.22882**	**6.78238**
$\overline{c}_{i2}$			**678.0453**

Table 2: Average value for $\widehat{C}$ (out of 200 runs) on observation window $W = [-1500, 1500]^2$. Average estimate of the variance of the area density $\widehat{\sigma}_{pp} \approx 678.9269$ with relative standard deviation $\delta \approx 4.70\,\%$.

	$\widehat{c}_{0j}$	$\widehat{c}_{1j}$	$\widehat{c}_{2j}$
$\widehat{c}_{i0}$ δ	**1.83297e-4** 21.95 %	**-1.74102e-3** -67.95 %	**-0.19659** -16.31 %
$\widehat{c}_{i1}$ δ		**0.22250** 22.25 %	**6.62421** 16.00 %
$\widehat{c}_{i2}$ δ			**675.4805** 5.08 %

Table 3: Average value for $\widehat{C}'$ (out of 200 runs) on sub–windows $V_k = [-500, 500)^2 + h_k,\ h_k = (k_1\,1000, k_2\,1000)^\top,\ k = (k_1, k_2), k_1, k_2 = -1, 0, 1$. Average estimate of the variance of the area density $\widehat{\sigma}'_{pp} \approx 646.3963$ with relative standard deviation $\delta \approx 5.14\,\%$.

	$\widehat{c}'_{0j}$	$\widehat{c}'_{1j}$	$\widehat{c}'_{2j}$
$\widehat{c}'_{i0}$ δ	**1.89746e-4** 23.83 %	**-1.70143e-3** -77.50 %	**-0.20282** -19.16 %
$\widehat{c}'_{i1}$ δ		**0.21844** 27.32 %	**6.98146** 19.48 %
$\widehat{c}'_{i2}$ δ			**683.80956** 5.21 %

Table 4: Average value for $\widetilde{C}$ (out of 200 runs) on observation windows $V_k = [-500, 500)^2 + h_k$, $h_k = (k_1\, 1000, k_2\, 1000)^\top$, $k = (k_1, k_2), k_1, k_2 = -1, 0, 1$. Average estimate of the variance of the area density $\widetilde{\sigma}_{pp} \approx 648.0608$ with relative standard deviation $\delta \approx 51.38\,\%$.

	$\widetilde{c}_{0j}$	$\widetilde{c}_{1j}$	$\widetilde{c}_{2j}$
$\widetilde{c}_{i0}$ δ	**1.95305e-4** 46.25 %	**-1.97664e-3** -134.48 %	**-0.20151** -71.37 %
$\widetilde{c}_{i1}$ δ		**0.23634** 50.10 %	**6.60805** 76.05 %
$\widetilde{c}_{i2}$ δ			**685.6371** 52.33 %

Table 5: Average value for $\widehat{C}'$ (out of 200 runs) on observation window $\widetilde{V}_k = [-250, 250)^2 + \widetilde{h}_k$, $\widetilde{h}_k = (k_1\, 500, k_2\, 500)^\top$, $k = (k_1, k_2), k_1, k_2 = -2, -1, 0, 1, 2$. Average estimate of the variance of the area density $\widehat{\sigma}'_{pp} \approx 588.9602$ with relative standard deviation $\delta \approx 6.20\,\%$.

	$\widehat{c}'_{0j}$	$\widehat{c}'_{1j}$	$\widehat{c}'_{2j}$
$\widehat{c}'_{i0}$ δ	**1.68884e-4** 29.42 %	**-1.53425e-3** -130.07 %	**-0.20651** -20.65 %
$\widehat{c}'_{i1}$ δ		**0.21747** 25.18 %	**5.61730** 35.55 %
$\widehat{c}'_{i2}$ δ			**704.3347** 5.80 %

Table 6: Average value for $\widetilde{C}$ (out of 200 runs) on observation windows $\widetilde{V}_k = [-250, 250)^2 + \widetilde{h}_k$, $\widetilde{h}_k = (k_1\, 500, k_2\, 500)^\top$, $k = (k_1, k_2), k_1, k_2 = -2, -1, 0, 1, 2$. Average estimate of the variance of the area density $\widetilde{\sigma}_{pp} \approx 651.1494$ with relative standard deviation $\delta \approx 23.95\,\%$.

	$\widetilde{c}_{0j}$	$\widetilde{c}_{1j}$	$\widetilde{c}_{2j}$
$\widetilde{c}_{i0}$ δ	**2.08729e-4** 22.51 %	**-1.94155e-3** -69.36 %	**-0.23333** -36.39%
$\widetilde{c}_{i1}$ δ		**0.26688** 17.89 %	**8.53916** 30.93 %
$\widetilde{c}_{i2}$ δ			**810.7263** 21.69 %

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