LIMIT THEOREMS FOR STATIONARY TESSELLATIONS
WITH RANDOM INNER CELL STRUCTURES

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Abstract
Stationary and ergodic tessellations $X = \{\Xi_n\}_{n \geq 1}$ in $\mathbb{R}^d$ are considered, where $X$ is observed in a bounded and convex sampling window $W_0 \subset \mathbb{R}^d$. It is assumed that the cells $\Xi_n$ of $X$ possess random inner structures, examples of which include point patterns, fibre systems, and tessellations. These inner cell structures are generated both independently of each other and independently of the tessellation $X$ by generic stationary random sets which are connected with a stationary random vector measure $J_0$ acting on $\mathbb{R}^d$. In particular, the asymptotic behavior of a multivariate random functional is studied, which is determined both by $X$ and the individual cell structures contained in $W_0$, as $W_0 \uparrow \mathbb{R}^d$. It turns out that by this functional an unbiased estimator for the intensity vector associated with $J_0$ is provided. Furthermore, under natural restrictions, strong laws of large numbers and a multivariate central limit theorem of the normalized functional are proven. Finally, some numerical examples and applications are discussed in detail, for which the inner structures of the cells of $X$ are induced by iterated Poisson-type tessellations.

Keywords: Stationary random tessellations; inner random cell structure; cumulative random functionals; ergodic theorem; central limit theorem; Poisson nesting; model choice; telecommunication networks

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1. Introduction

Let $X = \{\Xi_n\}_{n \geq 1}$ be a stationary and ergodic tessellation in $\mathbb{R}^d$. For each $n \geq 1$, consider a certain ($d$-dimensional) random vector $\alpha(\Xi_n)$, which is called an associated point of the cell $\Xi_n$ of $X$. It is well known that the tessellation $X$ can be regarded as a stationary and ergodic marked point process $\{n \geq 1 : \alpha(\Xi_n), \Xi_n^0\} = \{n \geq 1 \in \mathbb{R}^d\}$, where the shifted cells $\Xi_n^0 = \Xi_n - \alpha(\Xi_n)$ contain the origin $\alpha(\Xi_n)$; see, e.g., Section 6.1 of [22]. Furthermore, for each $n \geq 1$, consider a vector $J_n = (J_n^{[1]}, \ldots, J_n^{[m]})$ of $m \geq 1$ stationary random measures in $\mathbb{R}^d$. Assume that the sequence $(J_n)_{n \geq 1}$ is independent of $X$ and that it consists of i.i.d. copies of some generic random vector measure $J_0 = (J_0^{[1]}, \ldots, J_0^{[m]})$.

For each $n \geq 1$, the random measures $J_n^{[1]}, \ldots, J_n^{[m]}$ describe the inner structure of the $n$-th cell $\Xi_n$ of the tessellation $X$. In particular, in the planar case $d = 2$, examples

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of such random measures are the number of vertices and the number or the total length of the edges, which are generated by some (component) tessellation $X_n$ within the cell $\Xi_n$ of the (initial) tessellation $X$. Figure 1(a) shows the case where $X_n$ is a so-called PLT/PVT-nesting, which means that Poisson line tessellations (PLT) iterated by Poisson–Voronoi tessellations (PVT) are inscribed into the cells of $X$. Similarly in Figure 1(b), PLT/PLT-nestings inscribed into the cells of $X$ are shown. However, the random measures $J^{(1)}_n, \ldots, J^{(m)}_n$ need not necessarily be induced by tessellations. Another type of examples is shown in Figure 2, where the inner structure of $\Xi_n$ is determined by point processes.

![Figure 1](image)

**Figure 1:** Inner structure of the cells of $X$ induced by iterated tessellations

Suppose that only a single realization of the tessellation $X$ as well as of the vector measures $J_1, J_2, \ldots$ is available. This realization is restricted to some (presumably large) sampling window $W_\varrho$ and the support of $J_n$ is observable only in $\Xi_n \cap W_\varrho$ for $n \geq 1$. The region $W_\varrho$ is assumed to have the form $W_\varrho = \varrho W$ with scaling factor $\varrho > 0$ (which increases unboundedly) and with a convex body $W \subset \mathbb{R}^d$ containing the closed ball $B(o, r)$ centered at the origin with fixed radius $r > 0$. The main subject investigated in the present paper is the vector of cumulative functionals $Z_\varrho = (Z^{(1)}_\varrho, \ldots, Z^{(m)}_\varrho)^\top$, where the components $Z^{(i)}_\varrho$ of $Z_\varrho$ are given by

$$Z^{(i)}_\varrho = \sum_{n \geq 1} J^{(i)}_n (\Xi_n \cap W_\varrho),$$

for each $i = 1, \ldots, m$. In the first step, the expectation vector $\mathbb{E}Z_\varrho$ and the covariance matrix $\text{Cov}(Z_\varrho)$ of $Z_\varrho$ as well as the asymptotic covariance matrix $K = \lim_{\varrho \to \infty} \text{Cov}(\tilde{Z}_\varrho)$ of the vector of normalized functionals

$$\tilde{Z}_\varrho = \left( \frac{Z^{(1)}_\varrho - \lambda^{(1)} W_\varrho}{\sqrt{|W_\varrho|}}, \ldots, \frac{Z^{(m)}_\varrho - \lambda^{(m)} W_\varrho}{\sqrt{|W_\varrho|}} \right)^\top$$

(1.2)
are determined. Here, $|B|$ denotes the $d$-dimensional Lebesgue measure of the bounded Borel set $B \subset \mathbb{R}^d$ and $\lambda^{(i)} = \mathbb{E}J_{\theta}^{(i)}([0,1]^d)$ is the intensity of the stationary random measure $J_{\theta}^{(i)}$; see Theorem 3.1. In the next step, under some mild integrability conditions, Theorem 4.2 yields the following strong law of large numbers

$$\frac{1}{|W_{\theta}|} \sum_{n \geq 1} J_{n}^{(i)}(\Xi_n \cap W_{\theta}) \xrightarrow{\mathbb{P}} \lambda^{(i)} \quad \text{for} \quad i = 1, \ldots, m,$$

showing that the vector $Z_{\theta}/|W_{\theta}|$ is a strongly consistent (and unbiased) estimator for the intensity vector $(\lambda^{(1)}, \ldots, \lambda^{(m)})^T$ of the stationary vector measure $J_{\theta}$. The proof of Theorem 4.2 relies on the ergodicity of the tessellation $X = \{\Xi_n\}_{n \geq 1}$ and the conditional independence of the random vectors $J_1(\Xi_1 \cap W_{\theta}), J_2(\Xi_2 \cap W_{\theta}), \ldots$ given the tessellation $X$. For this purpose some estimates are needed which show that the contribution of those cells of $X$ hitting the boundary $\partial W_{\theta}$ is asymptotically negligible as $\theta \to \infty$; see Lemma 4.1.

![Inner structure of the cells of X induced by point processes](image)

**Figure 2:** Inner structure of the cells of $X$ induced by point processes

In the third step, the following multivariate central limit theorem

$$\hat{Z}_{\theta} \Rightarrow N(o,K), \quad \text{i.e.,} \quad \lim_{\theta \to \infty} \sup_{x \in \mathbb{R}^m} |\mathbb{P}(\hat{Z}_{\theta} \leq x) - \Phi_K(x)| = 0$$

is derived, where "$\Rightarrow$" means convergence in distribution and $\Phi_K$ denotes the distribution function of the $(m$-dimensional) Gaussian vector $N(o,K)$ with zero mean components and covariance matrix $K$; see Theorem 5.1.

Our results can be applied to stochastic modelling and statistical analysis of complex network structures. In [9] the so-called stochastic subscriber line model (SSLM) is described, which is an example of a stochastic-geometric model of telecommunication networks. Figure 3 shows a realization of the SSLM, where the urban infrastructure along which the cable trench system is built is represented by a random tessellation. Within each cell, subscribers are located according to some point process and line
segments represent dead end streets. Along the streets 1-level and 2-level stations are displayed, where each subscriber is connected to its closest 2-level station via a 1-level station.

![Diagram of stochastic subscriber line model]

**Figure 3:** Realization of the stochastic subscriber line model.

In the context of the SSLM, the results of the present paper provide a theoretical basis for statistical analysis of the morphological structure of spatial telecommunication data and help fit appropriate tessellation models.

Notice that there exists a number of papers investigating problems closely related to the topics of our work. For example, [11] derives a central limit theorem for a class of random measures associated with germ–grain models, while [1] and [19] investigate central limit theorems for Poisson–Voronoi and Poisson line tessellations in $\mathbb{R}^2$, respectively. In [10], normal approximations are given for some mean-value estimates of absolutely regular tessellations. Asymptotic properties of estimators for the volume fraction and other specific intrinsic volumes of stationary random sets are examined in [3], [4], [16], and [21], for example. Simulation studies on the typical cell of stationary tessellations can be found, e.g., in [13].

The present paper is organized as follows. In Section 2, a short introduction on basic notions and notations of stochastic geometry in general is given. Section 3 is devoted to first and second order moments for functionals of stationary random measures associated with the cells of random tessellations. The strong law of large numbers and the multivariate central limit theorem mentioned above are derived in Sections 4 and 5, respectively. Some numerical examples are discussed in Section 6, where functionals are considered which describe several intracellular structures of the cells of tessellations.
in \( \mathbb{R}^2 \). Finally, in Section 7 an outlook on further research perspectives is given.

2. Basic notions and notation

In this section, the basic notation used in the present paper is introduced and a brief account of some relevant notions of stochastic geometry is given. For a detailed discussion of the subject, it is referred to the literature, for example [22] and [23].

The abbreviations \( \mathbb{B}, \partial \mathbb{B}, \) and \( \mathbb{B}^c \) are used to denote the interior, the boundary, and the complement of a set \( \mathbb{B} \subset \mathbb{R}^d \), respectively. For arbitrary sets \( \mathbb{B}, \mathbb{B}' \subset \mathbb{R}^d \), we will consider the operations of translation \( \mathbb{B} + x = \{ y + x : y \in \mathbb{B} \} \) for \( x \in \mathbb{R}^d \), reflection \( \mathbb{B} = -\mathbb{B} = \{ -x : x \in \mathbb{B} \} \), scaling \( \mathbb{B}_q = \{ qx : x \in \mathbb{B} \} \) for any constant \( q \in [0, \infty) \), and Minkowski-addition \( \mathbb{B} \oplus \mathbb{B}' = \{ x + x' : x \in \mathbb{B}, x' \in \mathbb{B}' \} \). Furthermore, let \( b(x, r) = \{ y \in \mathbb{R}^d : ||x - y|| \leq r \} \) denote the ball of radius \( r \geq 0 \) centered at \( x \in \mathbb{R}^d \), where ||x - y|| is the length of the vector \( x - y \).

By \( \mathcal{F}, \mathcal{K}, \) and \( \mathcal{C} \), the families of all closed sets, compact sets, and convex bodies (compact and convex sets) in \( \mathbb{R}^d \) are denoted, respectively. For any \( C \in \mathcal{C} \), the volume \( |C \oplus b(o, r)| \) of the so-called parallel set \( C \oplus b(o, r) \) is given by Steiner’s formula

\[
|C \oplus b(o, r)| = \sum_{k=0}^{d} \binom{d}{k} M_k(C) r^k, \quad r \geq 0,
\]

(2.1)

where \( M_k(C) \) denotes the \( k \)-th Minkowski-functional of the convex body \( C \); \( 0 \leq k \leq d \). Note that in particular \( M_0(C) = |C| \) and \( M_d(C) = 1 - \delta_d(C) \). Clearly, for each \( q > 0 \) and for any convex body \( C \in \mathcal{C} \), the volume \( |C_q| \) equals \( q^d |C| \), and the diameter \( D(C_q) \) of the scaled set \( C_q \) is given by \( qD(C) \), where \( D(C) = \sup \{ ||x - y|| : x, y \in C \} \). Furthermore, for any \( C \in \mathcal{C} \), the following isodiametric inequality holds (see e.g. [12])

\[
M_k(C) \leq \omega_d \left( \frac{D(C)}{2} \right)^{d-k}, \quad 0 \leq k \leq d,
\]

(2.2)

where \( \omega_d \) denotes the volume of the unit ball \( b(o, 1) \). Recall that a random closed set \( \Xi \in \mathbb{R}^d \) is a measurable mapping \( \Xi : \Omega \rightarrow \mathcal{F} \) from some probability space \((\Omega, \mathcal{A}, \mathbb{P})\) into the measurable space \((\mathcal{F}, \mathcal{B}(\mathcal{F}))\), where \( \mathcal{B}(\mathcal{F}) \) denotes the smallest \( \sigma \)-algebra of subsets of \( \mathcal{F} \) that contains all sets \( \{ F \in \mathcal{F}, F \cap K \neq \emptyset \} \) for any \( K \in \mathcal{K} \). Particularly, the random closed set \( \Xi \) is called a random compact set or a random convex body if \( \mathbb{P}(\Xi \in \mathcal{K}) = 1 \) or \( \mathbb{P}(\Xi \in \mathcal{C}) = 1 \), respectively.

A tessellation in \( \mathbb{R}^d \) is a countable family \( \tau = \{ C_n \}_{n \geq 1} \) of convex bodies \( C_n \in \mathcal{C} \) such that \( \text{int } C_n \neq \emptyset \) for all \( n \), \( \text{int } C_n \cap \text{int } C_m = \emptyset \) for all \( n \neq m \), \( \bigcup_{n \geq 1} C_n = \mathbb{R}^d \), and \( \sum_{n \geq 1} 1_{\{ C_n \cap K \neq \emptyset \}} < \infty \) for any \( K \in \mathcal{K} \). Notice that the sets \( C_n \), called the cells of \( \tau \), are polytopes in \( \mathbb{R}^d \). The family of all tessellations in \( \mathbb{R}^d \) is denoted by \( \mathcal{T} \). A random tessellation \( \{ \Xi_n \}_{n \geq 1} \) in \( \mathbb{R}^d \) is a sequence of random convex bodies \( \Xi_n \) such that \( \mathbb{P}(\{ \Xi_n \}_{n \geq 1} \in \mathcal{T}) = 1 \). Notice that a random tessellation \( \{ \Xi_n \}_{n \geq 1} \) can also be considered as a marked point process \( \sum_{n \geq 1} \delta_{[\alpha(\Xi_n), \Xi_n]} \), where \( \alpha : \mathcal{C}' \rightarrow \mathbb{R}^d, \mathcal{C}' = \mathcal{C} \setminus \{ \emptyset \} \), is a measurable mapping such that \( \alpha(C) \in \mathcal{C} \) and \( \alpha(C) + x = \alpha(C) + x \) for any \( C \in \mathcal{C}' \) and \( x \in \mathbb{R}^d \), and where \( \Xi_0 = \Xi_n - \alpha(\Xi_n) \) is the centered cell corresponding to \( \Xi_n \) which contains the origin. The point \( \alpha(C) \in \mathbb{R}^d \) is called the associated point of \( C \) and can be chosen, for example, to be the lexicographically smallest point of \( C \).
Suppose that the marked point process $\sum_{n \geq 1} \delta_{(\alpha(\Xi_n), \Xi_n)}$ is stationary with positive and finite intensity $\lambda = \mathbb{E}(\# \{ n : \alpha(\Xi_n) \in [0,1]^d \})$. By $P^0$ we denote the set of all convex polytopes with their associated point at the origin. Then, the Palm mark distribution $P^0$ of $X$ is given by

$$P^0(B) = \lambda^{-1} \mathbb{E}(\# \{ n : \alpha(\Xi_n) \in [0,1]^d, \Xi_n \in B \}, \quad B \in \mathcal{B}(\mathcal{F}) \cap P^0. \quad (2.3)$$

Notice that a random polytope $\Xi^* : \Omega \to P^0$, whose distribution coincides with $P^0$, is called the typical cell of $X$. Furthermore, it holds

$$\lambda^{-1} = \int_{P^0} |C| P^0(dC), \quad (2.4)$$

i.e., the mean volume $\mathbb{E}[\Xi^*] = \int_{P^0} |C| P^0(dC)$ of the typical cell $\Xi^*$ is equal to $\lambda^{-1}$.

A (deterministic) iterated tessellation $\tau = \{ C_{n \geq 1} \}$ in $\mathbb{R}^d$ consists of an initial tessellation $\tau = \{ C_n \}_{n \geq 1}$ in $\mathbb{R}^d$ and a sequence $(\tau_n)_{n \geq 1}$ of component tessellations $\tau_n = \{ C_{n \geq 1} \}$. Hence, in order to define the notion of a random iterated tessellation, we can proceed as follows; see [14]. Let $\Xi$ be a random convex body in $\mathbb{R}^d$, where int $\Xi \neq \emptyset$, and let $X = \{ \Xi_n \}_{n \geq 1}$ be a random tessellation in $\mathbb{R}^d$. Then, the mapping $Y(\cdot \mid \Xi) : \Omega \to N(\mathcal{F})$ defined by $Y(B \mid \Xi) = \sum_{n \geq 1} \delta_{\Xi_n \cap B}$, where $F = \mathcal{F} \setminus \{ \emptyset \}$ and $\eta(B) = \sum_{n \geq 1} \eta(B \cap \Xi_n) \delta_{\Xi_n \cap B}$, is a point-process representation of an iterated random tessellation $X$ in $\mathbb{R}^d$ with initial tessellation $X$ and component tessellations $X_1, X_2, \ldots$. Clearly, the point process $Y$ is stationary and isotropic, respectively, provided that both the initial tessellation $X$ and the component tessellations $X_1, X_2, \ldots$ possess these properties. Moreover, $Y$ is ergodic if $X$ is ergodic.

3. Expectation vector and covariance matrix

Let $X = \{ \Xi_n \}_{n \geq 1}$ be an arbitrary stationary and ergodic tessellation in $\mathbb{R}^d$. Recall that the tessellation $X$ can be equivalently described as a stationary and ergodic marked point process $\sum_{n \geq 1} \delta_{(\alpha(\Xi_n), \Xi_n)}$, where $\Xi_n^0 = \Xi_n - \alpha(\Xi_n)$ and $\alpha(\Xi_n)$ denotes the associated point of $\Xi_n$. The intensity $\lambda = \mathbb{E}(\# \{ n : \alpha(\Xi_n) \in [0,1]^d \})$ is assumed to be positive and finite. For each individual cell $\Xi_n$ of $X$, we consider an $m$-dimensional vector $J_n = (J_{n,1}, \ldots, J_{n,m})$ of stationary random measures in $\mathbb{R}^d$, which describe the inner structure of $\Xi_n$. We assume that the sequence $(J_n)_{n \geq 1}$ is independent of $X$ and consists of i.i.d. copies of a generic stationary random vector measure $J_0 = (J_{0,1}, \ldots, J_{0,m})$.

Throughout this paper, we assume that only a single realization of the tessellation $X$ and of the random vectors $J_n(\Xi_n \cap W_n)$, $n \geq 1$, can be observed in an (unboundedly
increasing) sampling window \( W_\rho = \rho W \uparrow \mathbb{R}^d \) (as \( \rho \uparrow \infty \)), where the convex body \( W \) satisfies the inclusion \( b(o,r) \subseteq W \subseteq b(o,R) \) for some fixed \( 0 < r < R < \infty \). To begin with, we determine the expectation vector \( \mathbb{E}Z_\rho \) of the random vector \( Z_\rho = (Z^1_\rho, \ldots, Z^m_\rho)^\top \) defined by (1.1). Furthermore, we derive conditions under which the covariance matrix \( \operatorname{Cov}(Z_\rho) \) and the limit \( \lim_{\rho \to \infty} \operatorname{Cov}(\tilde{Z}_\rho) \) exist, where \( \tilde{Z}_\rho \) is the normalized vector of functionals introduced in (1.2).

**Theorem 3.1.** If \( \lambda^{(i)} = \mathbb{E}J_0^{(i)}([0,1]^d) < \infty \) for each \( i = 1, \ldots, m \), then

\[
\mathbb{E}Z_\rho = |W_\rho| (\lambda^{(1)}, \ldots, \lambda^{(m)})^\top.
\]

Under the additional assumption that

\[
\int_{\mathbb{R}^d} \mathbb{E}(J_0^{(i)}(C))^2 P^0(dC) < \infty, \quad i = 1, \ldots, m,
\]

the covariance matrix \( \operatorname{Cov}(Z_\rho) = \left( \operatorname{Cov}(Z^i_\rho, Z^j_\rho) \right)_{i,j=1}^m \) exists with entries taking the form

\[
\operatorname{Cov}(Z^i_\rho, Z^j_\rho) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \operatorname{Cov}(J_0^{(i)}(C \cap (W_\rho - x)), J_0^{(j)}(C \cap (W_\rho - x))) \, dx \, P^0(dC).
\]

Moreover, the asymptotic covariance matrix \( K = \lim_{\rho \to \infty} \operatorname{Cov}(\tilde{Z}_\rho) = \left( \sigma_{ij}^2 \right)_{i,j=1}^m \) exists with entries

\[
\sigma_{ij}^2 = \int_{\mathbb{R}^d} \operatorname{Cov}(J_0^{(i)}(C), J_0^{(j)}(C)) \, P^0(dC), \quad i,j = 1, \ldots, m.
\]

**Proof.** In view of the independence of \( X \) and the sequence \( (J_n)_{n \geq 1} \) we may write for any \( i = 1, \ldots, m \) that

\[
\mathbb{E}Z^{(i)}_\rho = \mathbb{E} \left( \sum_{n \geq 1} J_n^{(i)}(\Xi_n \cap W_\rho) \right) = \sum_{n \geq 1} \mathbb{E} \left( J_n^{(i)}(\Xi_n \cap W_\rho) \right),
\]

where \( \mathbb{E} \) denotes the conditional expectation given the tessellation \( X \). By stationarity of \( J_0^{(i)} \), the expectation \( \mathbb{E} \left( J_n^{(i)}(\Xi_n \cap W_\rho) \right) \) equals \( \lambda^{(i)} |\Xi_n \cap W_\rho| \). Since the interior of the cells \( \Xi_n, n \geq 1 \) fills the space \( \mathbb{R}^d \) up to a set of Lebesgue measure zero, we have

\[
\sum_{n \geq 1} |\Xi_n \cap W_\rho| = |W_\rho|,
\]

which proves (3.1). To derive (3.3) we first carry out all rearrangements without regarding the existence of the integrals and expectations involved and after that we will check their correctness. Using the independence of \( X \) and the sequence \( (J_n)_{n \geq 1} \) once more and combining this with (3.1), we get

\[
\operatorname{Cov}(Z^{(i)}_\rho, Z^{(j)}_\rho) = \mathbb{E} \left( \sum_{n \geq 1} \mathbb{E}_X \left( J_n^{(i)}(\Xi_n \cap W_\rho) J_n^{(j)}(\Xi_n \cap W_\rho) \right) \right) + \]
\[
\mathbb{E} \left( \sum_{n, \ell \geq 1} \mathbb{E}_X \left( J_n^{(i)}(\Xi_n \cap W_\ell) J_\ell^{(j)}(\Xi_\ell \cap W_n) \right) \right) - \lambda^{(i)} \lambda^{(j)} |W_\ell|^2.
\]

The difference of the two expressions in the latter line is just equal to
\[
- \mathbb{E} \left( \sum_{n \geq 1} \mathbb{E}_X \left( J_0^{(i)}(\Xi_n \cap W_\ell) \right) \mathbb{E}_X \left( J_\ell^{(j)}(\Xi_\ell \cap W_n) \right) \right),
\]
since, for \( n \neq \ell \), we have
\[
\mathbb{E}_X \left( J_n^{(i)}(\Xi_n \cap W_\ell) J_\ell^{(j)}(\Xi_\ell \cap W_n) \right) = \mathbb{E}_X J_n^{(i)}(\Xi_n \cap W_\ell) \mathbb{E}_X J_\ell^{(j)}(\Xi_\ell \cap W_n)
\]
by the assumed independence of \( J_n^{(i)} \) and \( J_\ell^{(j)} \). Thus, denoting by \( \text{Cov}_X \) the conditional covariance given the tessellation \( X = \{ \Xi_n \}_{n \geq 1} \), we find that
\[
\text{Cov}(Z_n^{(i)}, Z_\ell^{(j)}) = \mathbb{E} \left( \sum_{n \geq 1} \text{Cov}_X \left( J_0^{(i)}(\Xi_n \cap W_\ell), J_\ell^{(j)}(\Xi_\ell \cap W_n) \right) \right). \tag{3.6}
\]

Finally, writing \( \Xi_n = \Xi_n^0 + \alpha(\Xi_n) \) and applying Campbell’s theorem to the stationary marked point process \( \sum_{n \geq 1} \delta_{\alpha(\Xi_n), \Xi_n^0} \), we get
\[
\text{Cov}(Z_n^{(i)}, Z_\ell^{(j)}) = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{Cov} \left( J_0^{(i)}((C + x) \cap W_\ell), J_\ell^{(j)}((C + x) \cap W_n) \right) P^0(dC) \, dx
\]
\[
= \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{Cov} \left( J_0^{(i)}(C \cap (W_\ell - x)), J_\ell^{(j)}(C \cap (W_n - x)) \right) P^0(dC) \, dx,
\]
where in the latter equality we used Fubini’s theorem and the invariance of the covariance \( \text{Cov}(J_0^{(i)}(A), J_0^{(j)}(B)) \) under diagonal shifts, i.e., \( \text{Cov}(J_0^{(i)}(A + x), J_0^{(j)}(B + x)) = \text{Cov}(J_0^{(i)}(A), J_0^{(j)}(B)) \) for any \( A, B \in \mathcal{C} \) and for any \( x \in \mathbb{R}^d \). To complete the proof of (3.3), we justify the steps and changes of integration above by showing that our integrability condition (3.2) ensures the existence of the second moment
\[
\mathbb{E} \sum_{n \geq 1} \left( J_n^{(i)}(\Xi_n \cap W_\ell) \right)^2 = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E}(J_0^{(i)}(C \cap (W_\ell - x)))^2 \, dx P^0(dC)
\]
for each \( i = 1, \ldots, m \). By Fubini’s theorem, we have
\[
\int_{\mathbb{R}^d} \left( J_0^{(i)}(C \cap (W_\ell - x)) \right)^2 \, dx = \int_{C} \int_{\mathbb{R}^d} |(W_\ell - y) \cap (W_\ell - z)| J_0^{(i)}(dy) J_0^{(i)}(dz)
\]
\[
\leq |W_\ell| \left( J_0^{(i)}(C) \right)^2 \tag{3.7}
\]
and therefore, by means of (3.2),
\[
\mathbb{E} \sum_{n \geq 1} \left( J_n^{(i)}(\Xi_n \cap W_\ell) \right)^2 \leq \lambda |W_\ell| \int_{\mathbb{R}^d} \mathbb{E} \left( J_0^{(i)}(C) \right)^2 P^0(dC) < \infty.
\]
for \( i = 1, \ldots, m \). The existence of the other expressions is seen by the inequality of Cauchy–Schwarz, which proves (3.3). In order to show (3.4), notice that the properties of the family of convex sets \((W_0^*)\) entail that
\[
\lim_{{\eta \to \infty}} \frac{|(W_0^* - y) \cap (W_0^* - z)|}{|W_0^*|} = 1 \quad \text{for any fixed } y, z \in \mathbb{R}^d.
\]
Hence, bounding the mixed second moments \( \mathbb{E}(J_0^{(i)}(C \cap (W_0^* - x)) J_0^{(j)}(C \cap (W_0^* - x))) \) from above, quite similar to (3.7), and using the dominated convergence theorem, we get
\[
\lim_{{\eta \to \infty}} \int \frac{\mathbb{E}(J_0^{(i)}(C \cap (W_0^* - x)) J_0^{(j)}(C \cap (W_0^* - x)))}{|W_0^*|} \, dx = \mathbb{E}(J_0^{(i)}(C) J_0^{(j)}(C))
\]
for any \( C \in \mathbb{P}^0 \) and \( i, j = 1, \ldots, m \). The previous relation remains true for the first-order moments resulting from the covariance formula (3.6). Thus, applying Campbell’s and Fubini’s theorems to (3.6), together with the dominated convergence theorem, we finally obtain that
\[
\lim_{{\eta \to \infty}} \frac{\text{Cov}(Z_0^{(i)}, Z_0^{(j)})}{|W_0^*|} = \lambda \int \lim_{{\eta \to \infty}} \int \frac{\text{Cov}(J_0^{(i)}(C \cap (W_0^* - x)) \cdot J_0^{(j)}(C \cap (W_0^* - x)))}{|W_0^*|} \, dx \, P^0(dC)
\]
for any \( i, j = 1, \ldots, m \), where the expression on right-hand side coincides with \( \sigma^2_{ij} \) as defined in (3.4).

Note that the second part of Theorem 3.1 implies that the asymptotic variance of the scalar product \( t^\top \tilde{Z}_0 = \sum_{i=1}^{m} t_i \tilde{Z}_0^{(i)} \) exists for any \( t = (t_1, \ldots, t_m)^\top \in \mathbb{R}^m \), more precisely,
\[
\lim_{{\eta \to \infty}} \text{Var}(t^\top \tilde{Z}_0) = t^\top K t = \sum_{i,j=1}^{m} t_i t_j \sigma^2_{ij}, \quad (3.8)
\]
where \( K = (\sigma^2_{ij})_{i,j=1}^{m} \) is given by (3.4).

We conclude this section with a discussion of the integrability conditions in (3.2). Our aim is to put separate conditions on the random measures \( J_0^{(i)} \) and the typical cell \( \Xi^* \) of \( X \) which together imply (3.2).

**Lemma 3.1.** The following inequalities
\[
(\lambda^{(i)})^2 \mathbb{E} |\Xi^*|^2 \leq \int \mathbb{E}(J_0^{(i)}(C))^2 P^0(dC) \leq \mathbb{E} |\Xi^*| \otimes b(|\xi|, \sqrt{d})^2 \mathbb{E}(J_0^{(i)}([0,1]^d))^2 \tag{3.9}
\]
hold for each \( i = 1, \ldots, m \). Consequently, (3.2) is satisfied whenever
\[
\mathbb{E} M_k^2(\Xi^*) < \infty \quad \text{and} \quad \mathbb{E}(J_0^{(i)}([0,1]^d))^2 < \infty \tag{3.10}
\]
for any \( k = 0, \ldots, d-1 \) and \( i = 1, \ldots, m \).
Proof. Taking into account that \( \mathbb{E}J_0^{(i)}(C) = \lambda^{(i)}|C| \), the first inequality in (3.9) immediately follows from Jensen’s inequality \( (\mathbb{E}J_0^{(i)}(C))^2 \leq \mathbb{E}(J_0^{(i)}(C))^2 \). The obvious set-theoretic inclusions

\[
C \subseteq \bigcup_{z \in \mathbb{Z}^d \cap ([0,1]^d + z) \neq \emptyset} ([0,1]^d + z) \subseteq C \oplus b(o, \sqrt{d}),
\]

which are true for any subset \( C \) of \( \mathbb{R}^d \), and Steiner’s formula (2.1) imply that

\[
\#\{z \in \mathbb{Z}^d : C \cap ([0,1]^d + z) \neq \emptyset\} \leq |C \oplus b(o, \sqrt{d})| = \sum_{k=0}^d \binom{d}{k} M_k(C) d^{k/2} \quad (3.11)
\]

for each \( C \in \mathcal{C} \). Thus, using the monotonicity and translation-invariance of the set function \( \mathbb{E}(J_0^{(i)}(\cdot))^2 \) and the elementary inequality \( (a_1 + \cdots + a_N)^2 \leq N(a_1^2 + \cdots + a_N^2) \), we arrive at

\[
\mathbb{E}(J_0^{(i)}(C))^2 \leq \#\{z \in \mathbb{Z}^d_C\} \sum_{z \in \mathbb{Z}^d_C} \mathbb{E}(J_0^{(i)}([0,1]^d + z))^2 \leq |C \oplus b(o, \sqrt{d})|^2 \mathbb{E}(J_0^{(i)}([0,1]^d))^2,
\]

where \( \mathbb{Z}^d_C = \{z \in \mathbb{Z}^d : C \cap ([0,1]^d + z) \neq \emptyset\} \). This proves the second inequality in (3.9). Furthermore, by (3.11), \( \mathbb{E}|\Xi^* \oplus b(o, \sqrt{d})|^2 < \infty \) is true if the first integrability condition in (3.10) is satisfied.

In some cases, it might not be possible to directly check whether or not the first integrability condition in (3.10) is satisfied. This is due to the fact that it is sometimes difficult to determine the second moment \( \mathbb{E}M_k^2(\xi^*) \) of the \( k \)-th Minkowski-functional \( M_k(\xi^*) \) of the typical cell \( \xi^* \) of \( X \). However, the isodiametric inequality (2.2) implies that \( \mathbb{E}M_k^2(\xi^*) < \infty \) holds for each \( k = 0, \ldots, d-1 \) provided that

\[
\mathbb{E}D^{2d}(\xi^*) < \infty. \quad (3.12)
\]

4. Laws of large numbers

Recall that the individual ergodic theorem applied to the (stationary and ergodic) marked point process \( \sum_{n \geq 1} q_{\alpha(\Xi_n)}z_{\Xi_n} \) reads as follows; see [6], p.339. For any real-valued integrable function \( h \in L^1(P^0, B(\mathcal{F}) \cap \mathcal{P}^0, P^0) \), we have

\[
\frac{1}{|W_\theta|} \sum_{n \geq 1} \mathbb{I}_{W_\theta}(\alpha(\Xi_n)) h(\Xi_n^0) \xrightarrow{\mathbb{P} \rightarrow \infty} \lambda \mathbb{E}h(\iota^*) = \lambda \int h(C)P^0(dC). \quad (4.1)
\]

However, in the context of this paper as in many other statistical applications of the spatial ergodic theorem (4.1), we have to consider spatial averages over cells of \( X = \{\Xi_n\}_{n \geq 1} \) which belong to the sampling window \( W_\theta \) only partly. Such boundary effects are being taken into account by the subsequent result, which provides the strong consistency of a (not necessarily unbiased) estimator for \( \lambda \mathbb{E}g(\iota^*) \) in the case of a non-random, translation-invariant, and isotonic functional on \( \mathcal{C} \).
Theorem 4.1. Let $g : \mathcal{C} \to [0, \infty)$ be a $\mathcal{B}(\mathcal{F})$-measurable, non-negative set-function such that $g(C) \leq g(C')$ for $C \subseteq C'$ and $g(C) = g(C + x)$ for any $C \in \mathcal{C}$ and $x \in \mathbb{R}^d$. If the typical cell $\Xi^*$ of $X$ satisfies

$$\mathbf{E}D^d(\Xi^*) < \infty \quad \text{and} \quad \mathbf{E}g(\Xi^*) < \infty,$$

then

$$\frac{1}{|W|} \sum_{n \geq 1} \mathbb{I}_{\{\Xi_n \cap W \neq \emptyset\}} g(\Xi_n \cap W) \xrightarrow{\text{a.s.}} g \mathbb{E}g(\Xi^*).$$

The proof of Theorem 4.1 is postponed to Section 4.2. Notice however that Theorem 4.1 is not completely new. For instance, in the planar case $d = 2$, one can find it in [5] applied to some particular functionals $g(\Xi_n \cap W)$ of the cells $\Xi_n \cap W$, whereas we consider a general class of isotonic and translation-invariant functionals $g : \mathcal{C} \to [0, \infty)$. Furthermore, a straightforward application of Theorem 4.1 to each of the particular functionals

$$g_1(C) = \mathbf{E}\left(J_0^{(i)}(C) \cdot J_0^{(j)}(C)\right) \quad \text{and} \quad g_2(C) = |C|^2, \quad 1 \leq i, j \leq m, \quad C \in \mathcal{C},$$

yields the following result.

Corollary 4.1. Let $J_0 = (J_0^{(1)}, \ldots, J_0^{(m)})^\top$ be a vector of stationary random measures on $\mathbb{R}^d$ being independent of $X$ and satisfying both (3.2) (or (3.10)) and $\mathbf{E}D^d(\Xi^*) < \infty$. Then, for any $t = (t_1, \ldots, t_m)^\top \in \mathbb{R}^m$,

$$\frac{1}{|W|} \sum_{n \geq 1} \mathbb{E}X(t^\top J_0(\Xi_n \cap W))^2 \xrightarrow{\text{a.s.}} \lambda \mathbb{E}(t^\top J_0(\Xi^*))^2$$

and

$$\frac{1}{|W|} \sum_{n \geq 1} |\Xi_n \cap W|^2 \xrightarrow{\text{a.s.}} \lambda \mathbb{E}|\Xi^*|^2.$$
negligible as \( \rho \to \infty \). For this to show we define the families of events \((A_\rho)_{\rho > \rho_0}\) and \((B_\rho)_{\rho > \rho_0}\) by

\[
A_\rho = \bigcap_{n \geq 1} \left\{ \{ \Xi_n + \alpha(\Xi_n) \} \cap W_\rho = \emptyset \right\} \cup \{ \alpha(\Xi_n) \in W_{\rho + q(\rho)} \} \tag{4.7}
\]
and

\[
B_\rho = \bigcap_{n \geq 1} \left\{ \{ \Xi_n + \alpha(\Xi_n) \} \subseteq W_\rho \right\} \cup \{ \alpha(\Xi_n) \notin W_{\rho - q(\rho)} \} , \tag{4.8}
\]

where \( \rho_0 \geq 0 \) is some constant and the function \( q : (\rho_0, \infty) \to (0, \infty) \) is such that

\[
q(\rho) < \rho \quad \text{for each} \quad \rho > \rho_0 .
\]

**Lemma 4.1.** Under the assumption that \( \mathbb{E}D^d(\Xi^*) < \infty \), there exists a non-decreasing function \( q : (\rho_0, \infty) \to (0, \infty) \) satisfying \( q(\rho) < \rho \) for \( \rho > \rho_0 \), \( q(\rho) \to \infty \), and \( \frac{\mathbb{E}r}{q} \downarrow 0 \) as \( \rho \to \infty \) such that

\[
\lim_{\rho \to \infty} \mathbb{P}(\bigcup_{k \geq \rho} A_k^c) = 0 \quad \text{and} \quad \lim_{\rho \to \infty} \mathbb{P}(\bigcup_{k \geq \rho} B_k^c) = 0 . \tag{4.9}
\]

**Proof.** To begin with we recall the well-known fact from analysis that the integrability of \( D^d(\Xi^*) \) implies the existence of a convex function \( H : [0, \infty) \to [0, \infty) \) strictly increasing on its support \( (x_0, \infty) \) (for some \( x_0 \geq 0 \)) such that \( H(x)/x \) is non-decreasing for \( x > 0 \) with \( \lim_{x \to \infty} H(x)/x = \infty \) and \( \mathbb{E}H(D^d(\Xi^*)) < \infty \); see e.g. Theorem 1.22 in [7]. Furthermore, for some \( r > 0 \) such that \( b(\rho, r) \subseteq W \), the value \( q(\rho) \) ( \( > x_0^{1/d}/r \) ) is defined as the unique solution of the equation

\[
\rho^d = H(r^d q^d(\rho)) \quad \text{for any} \quad \rho > 0 . \tag{4.10}
\]

It is easily checked that the function \( \rho \mapsto q(\rho) \) possesses the required properties for \( \rho > \rho_0 = \inf \{ x > 0 : H(r^d q^d(x)) > q^d(x) \} \). Note that \( b(\rho, r \rho(\rho)) \subseteq W(\rho) \) and, by the convexity of \( W \),

\[
W_\rho \oplus b(\rho, r \rho(\rho)) \subseteq W_\rho \oplus W_\rho(\rho) \subseteq W_{\rho + \rho(\rho)} .
\]

Thus, for any \( C \in C \) with \( o \in C \), we have \( D(C) > r \rho(\rho) \) provided that \( W_{\rho + \rho(\rho)}(\rho) \cap (W_\rho \oplus C) \neq \emptyset \). By the definition of \( A_\rho \) and the latter implication, this yields that

\[
\mathbb{P}\left( \bigcup_{k \geq \rho} A_k^c \right) = \mathbb{P}\left( \bigcup_{n \geq 1, k \geq \rho} \{ \alpha(\Xi_n) \in W_{\rho + q(\rho)} \cap (W_k \oplus \Xi_n^0) \} \right) \\
\leq \mathbb{P}\left( \bigcup_{n \geq 1, k \geq \rho} \left( \{ \alpha(\Xi_n) \in W_k \oplus b(\rho, D(\Xi_n^0)) \} \cap \{ D(\Xi_n^0) > r(\rho) \} \right) \right) \\
= \mathbb{P}\left( \bigcup_{n \geq 1, k \geq \rho} \{ \alpha(\Xi_n) \in W_k \oplus b(\rho, D(\Xi_n^0)) \} \right) \\
\quad \cap \{ r(\rho) < D(\Xi_n^0) \leq r(\rho + 1) \} \right) . \tag{4.11}
\]
The last equality becomes clear by the fact that for any two sequences of events \( \{ E_k \}_{k \geq \ell} \) and \( \{ E'_k \}_{k \geq \ell} \) with \( E_k \subseteq E_{k+1} \) and \( E'_k \supseteq E'_{k+1} \) for \( k \geq \ell \), the identity
\[
\bigcup_{k \geq \ell} E_k \cap E'_k = \bigcup_{k \geq \ell} (E_k \setminus E'_{k+1})
\]
holds. Thus, the subadditivity of \( P \), the inequality \( P(U \geq 1) \leq E[|U|] \) and Campbell's theorem for stationary marked point processes imply that
\[
P\left( \bigcup_{k \geq \ell} A^\circ_k \right) \leq \sum_{k \geq \ell} \sum_{n \geq 1} \left( \sum_{s=0}^{d} \binom{d}{s} k^{d-s} M_s(W) \right) E\left( D^s(\Xi^*) \mathbb{I}_{[r(q(k), r(q(k+1))]}(D(\Xi)) \right) 
\]
\[
= \lambda \sum_{k \geq \ell} \int_{\mathbb{R}^d} \left| W_k \oplus b(o, D(C)) \right| \mathbb{I}_{[r(q(k), r(q(k+1))]}(D(\Xi)) P^d(dC). 
\]
Since \( W_k = kW \in \mathcal{C} \), we now are in a position to apply Steiner’s formula (2.1) which together with the homogeneity relation \( M_s(W_k) = k^{d-s} M_s(W) \) for the Minkowski–functionals \( M_s \) (see e.g. [12]) and the monotonicity of the function \( H(\cdot) \) leads to
\[
P\left( \bigcup_{k \geq \ell} A^\circ_k \right) \leq \lambda \sum_{k \geq \ell} \sum_{s=0}^{d} \binom{d}{s} k^{d-s} M_s(W) \mathbb{E}\left( H(D^s(\Xi^*)) \mathbb{I}_{[r(q(k), r(q(k+1))]}(D(\Xi^*)) \right) 
\]
\[
= \lambda \sum_{k \geq \ell} \sum_{s=0}^{d} \binom{d}{s} M_s(W) \frac{k^{d-s} q^s(k+1)}{H(r^d q^d(k))} 
\]
\[ \times \mathbb{E}\left( H(D^s(\Xi^*)) \mathbb{I}_{[r(q(k), r(q(k+1))]}(D(\Xi^*)) \right) \right). 
\]
By (4.10), we have \( H(r^d q^d(k)) = k^d \) and \( q(k+1) \leq k^d \) for all \( k \geq \ell \) (\( \ell \) large enough) and therefore,
\[
\frac{k^{d-s} q^s(k+1)}{H(r^d q^d(k))} \leq \left( \frac{1}{k^d} \right)^s r^s \leq 2^s r^s \quad \text{for} \quad k \geq \ell.
\]
This yields
\[
P\left( \bigcup_{k \geq \ell} A^\circ_k \right) \leq \lambda \sum_{s=0}^{d} \binom{d}{s} (2 r)^s M_s(W) \mathbb{E}\left( H(D^s(\Xi^*)) \mathbb{I}_{[r(q(k), 1]}(D(\Xi^*)) \right) \rightarrow 0.
\]
To show the second assertion in (4.9) we first realize that
\[
P\left( \bigcup_{k \geq \ell} B^\circ_k \right) = P \left( \bigcup_{n \geq 1} \bigcup_{k \geq \ell} \left\{ \alpha(\Xi_n) \in W_k \cap (W_k^{\circ} \oplus \Xi_n) \right\} \right).
\]
In analogy to the considerations above one can show that, for any \( C \in \mathcal{C} \) with \( o \in C \), we have \( D(C) > r(q) \) provided that \( (W_k^{\circ} \oplus C) \cap W_{q-o} \neq \emptyset \). Together with \( W_k \cap (W_k^{\circ} \oplus C) \), this gives that
\[
P\left( \bigcup_{k \geq \ell} B^\circ_k \right) \leq P \left( \bigcup_{n \geq 1} \bigcup_{k \geq \ell} \left\{ \alpha(\Xi_n) \in W_k \cap (D(\Xi_n) > r(q(k))) \right\} \right).
\]
Since the right-hand side of the latter inequality is not larger than the bound on the right-hand side of (4.11), we immediately get that $P\left(\bigcup_{k \geq \ell} B_k^c\right) \to 0$.

4.2. Proof of Theorem 4.1

For notational simplicity we put

$$
\Delta_k = \frac{1}{|W_k|} \sum_{n \geq 1} \mathbf{1}_{\{(\Xi_n + \alpha(\Xi_n)) \cap W_k \neq \emptyset\}} g((\Xi_n + \alpha(\Xi_n)) \cap W_k) \quad \text{for} \quad k \geq \ell.
$$

By rewriting the almost sure convergence, the assertion of Theorem 4.1 is equivalent to

$$
P\left(\sup_{k \geq \ell} |\Delta_k - \lambda E g(\Xi^*)| \geq \delta\right) = P\left(\bigcup_{k \geq \ell} \left\{|\Delta_k - \lambda E g(\Xi^*)| \geq \delta\right\} \cap A_k \cap B_k\right) \to 0 \quad (4.12)
$$

for any $\delta > 0$; see e.g. Lemma 6.8 in [20]. Furthermore,

$$
P\left(\bigcup_{k \geq \ell} \left\{|\Delta_k - \lambda E g(\Xi^*)| \geq \delta\right\}\right) \leq P\left(\bigcup_{k \geq \ell} \left\{|\Delta_k - \lambda E g(\Xi^*)| \geq \delta\right\} \cap A_k \cap B_k\right) + P\left(\bigcup_{k \geq \ell} \left\{|\Delta_k - \lambda E g(\Xi^*)| \geq \delta\right\} \cap (A_k \cap B_k)^c\right)
$$

$$
\leq P\left(\bigcup_{k \geq \ell} \left\{|\Delta_k - \lambda E g(\Xi^*) + \delta| \cap A_k\right\} \right) + P\left(\bigcup_{k \geq \ell} A_k^c\right) + P\left(\bigcup_{k \geq \ell} \left\{|\Delta_k - \lambda E g(\Xi^*) - \delta| \cap B_k\right\}\right) + P\left(\bigcup_{k \geq \ell} B_k^c\right),
$$

where the events $A_k$ and $B_k$ have been defined in (4.7) and (4.8), respectively. Taking into account the properties of the functional $g : \mathcal{C} \to [0, \infty)$, on $A_k$ we can verify the inequality

$$
\Delta_k \leq \frac{1}{|W_k|} \sum_{n \geq 1} \mathbf{1}_{W_{\kappa+g(q)}}(\alpha(\Xi_n)) g(\Xi_n^0)
$$

for each $k \geq \ell$ with $\ell$ sufficiently large. Likewise, for $k \geq \ell$, we have on $B_k$ that

$$
\Delta_k \geq \frac{1}{|W_k|} \sum_{n \geq 1} \mathbf{1}_{W_{\kappa-g(q)}}(\alpha(\Xi_n)) g(\Xi_n^0).
$$

Hence,

$$
P\left(\bigcup_{k \geq \ell} \left\{|\Delta_k \geq \lambda_0 E g(\Xi^*) + \delta| \cap A_k\right\}\right) + P\left(\bigcup_{k \geq \ell} \left\{|\Delta_k \leq \lambda E g(\Xi^*) - \delta| \cap B_k\right\}\right)
$$

$$
\leq \sum_{\kappa \in \{-1, +1\}} P\left(\sup_{k \geq \ell} \frac{1}{|W_k|} \sum_{n \geq 1} \mathbf{1}_{W_{\kappa+g(q)}}(\alpha(\Xi_n)) g(\Xi_n^0) - \lambda E g(\Xi^*) \geq \delta\right),
$$

However, the latter sum tends to zero as $\ell \to \infty$, since the spatial ergodic theorem (4.1) yields

$$
\frac{1}{|W_{\kappa+g(q)}|} \sum_{n \geq 1} \mathbf{1}_{W_{\kappa+g(q)}}(\alpha(\Xi_n)) g(\Xi_n^0) \xrightarrow{a.s.} \lambda E g(\Xi^*)
$$
and the asymptotic behaviour of $q(\ell)$ as $\ell \to \infty$ implies that
\[ \frac{|W_{\ell} + \kappa q(\ell)|}{|W_{\ell}|} = \left( 1 + \kappa \frac{q(\ell)}{\ell} \right)^d \xrightarrow{\ell \to \infty} 1 \]
for $\kappa = \pm 1$. Finally, Lemma 4.1 shows the validity of (4.12).

4.3. Proof of Theorem 4.2

By Lemma 4.1 and the arguments of the foregoing proof of Theorem 4.1, one can show that (4.6) is equivalent to
\[ \frac{1}{|W_{\ell}|} \sum_{n \geq 1} \mathbb{I}_{W_n} (\alpha(\Xi_n)) J_n^{(i)}(\Xi_n) \xrightarrow{\ell \to \infty} \lambda^{(i)} \text{ for } i = 1, \ldots, m. \]

Furthermore, a simple application of (4.1) to the function $h(C) = |C|$ for $C \in \mathcal{T}^0$ yields
\[ \frac{1}{|W_{\ell}|} \sum_{n \geq 1} \mathbb{I}_{W_n} (\alpha(\Xi_n)) |\Xi_n| \xrightarrow{N \to \infty} \lambda |\Xi^*| = 1, \]
where the latter equality follows from (2.4). Thus, (4.6) is equivalent to $S_N/|W_N| \xrightarrow{N \to \infty} 0$, where $S_N$ can be written as a partial sum $S_N = U_1 + \cdots + U_N$ with
\[ U_k = \sum_{n \geq 1} \mathbb{I}_{W_k \setminus W_{k-1}} (\alpha(\Xi_n)) \left( J_n^{(i)}(\Xi_n) - \lambda^{(i)} |\Xi_n| \right), \quad k = 1, 2, \ldots. \]

Here and below the index $i = 1, \ldots, m$ is fixed. In order to prove that $S_N/|W_N| \xrightarrow{N \to \infty} 0$ holds, it is necessary and sufficient to show that for any given $\delta, \eta > 0$, there exists an integer $\ell_0 = \ell_0(\delta, \eta)$ such that
\[ \mathbb{P} \left( \sup_{k \geq \ell} \left| \frac{S_k}{W_k} \right| \geq \delta \right) \leq \eta \quad \text{for any } \ell \geq \ell_0. \quad (4.13) \]

For $\varepsilon > 0$ (below chosen as a function of $\delta$ and $\eta$) and $a_k = |W_k|$, we introduce the truncated random variables
\[ U_k^{(\varepsilon)} = \sum_{n \geq 1} \mathbb{I}_{W_k \setminus W_{k-1}} (\alpha(\Xi_n)) \left( J_n^{(i)}(\Xi_n) - \lambda^{(i)} |\Xi_n| \right) \mathbb{I}_{\{|J_n^{(i)}(\Xi_n) - \lambda^{(i)} |\Xi_n|\} \leq \varepsilon (a_k \lor a_k^*)} \]
and their partial sums $S_k^{(\varepsilon)} = U_1^{(\varepsilon)} + \cdots + U_k^{(\varepsilon)}$ for $k \geq 1$. Note that the random variables $U_k^{(\varepsilon)}$ are conditionally independent given the tessellation $X = \{\Xi_n\}_{n \geq 1}$. Since $S_k^{(\varepsilon)}(\omega) = S_k(\omega)$ for any $k \geq 1$, whenever
\[ \omega \in A_{\varepsilon, \ell}(X) = \bigcap_{k \geq 1} \bigcap_{n:a(\Xi_n) \in W_k \setminus W_{k-1}} \{ \{J_n^{(i)}(\Xi_n) - \lambda^{(i)} |\Xi_n|\} \leq \varepsilon (a_k \lor a_k^*) \}, \]
it follows by a standard estimate that, for any fixed $\delta > 0$,
\[ \mathbb{P} \left( \sup_{k \geq \ell} \frac{|S_k|}{a_k} \geq \delta \right) - \mathbb{P} \left( \sup_{k \geq \ell} \frac{|S_k^{(\varepsilon)}|}{a_k} \geq \delta \right) \leq \mathbb{P} (A_{\varepsilon, \ell}(X)). \quad (4.14) \]
Using Campbell’s theorem, a straightforward computation reveals that

\[
\mathbb{P}(A_n^c, \ell(X)) \leq \sum_{k \geq 1} \mathbb{P}\left( \sum_{n \geq 1} I_{W_n \setminus W_{n-1}}(\alpha(\Xi_n)) I_{\{J_n(\Xi_n) - \lambda \Xi_n \geq \varepsilon(a_k \vee a_\ell)\}} \geq 1 \right)
\]

\[
\leq \lambda a_\ell \mathbb{P}(V > \varepsilon a_\ell) + \lambda \sum_{k > \ell} (a_k - a_{k-1}) \mathbb{P}(V > \varepsilon a_k) \leq \frac{\lambda}{\varepsilon} \mathbb{E}(V I_{\{V > \varepsilon a_\ell\}}),
\]

where, for fixed i, V denotes the non-negative random variable \(|J_0(\Xi^*) - \lambda^*| \Xi^*|\) with \(\mathbb{E}V < \infty\). Thus, \(\mathbb{P}(A_n^c, \ell(X)) \to 0 \) as \(\ell \to \infty\). Furthermore, we have

\[
\mathbb{P} \left( \sup_{k \geq \ell} \frac{|S_k^{(c)}|}{a_k} \geq \delta \right) \leq \mathbb{E} \left( \mathbb{P} \left( \sup_{k \geq \ell} \frac{|S_k^{(c)} - \mathbb{E}X S_k^{(c)}|}{a_k} \geq \frac{\delta}{2} \right) \right) + \mathbb{E} \left( \mathbb{P} \left( \sup_{k \geq \ell} \frac{|\mathbb{E}X S_k^{(c)}|}{a_k} \geq \frac{\delta}{2} \right) \right), \tag{4.15}
\]

where \(\mathbb{P}_X\) denotes the conditional probability given the tessellation \(X\). To estimate the first term on the right-hand side of (4.15), we make use of the well-known Hájek–Rényi inequality, which reads as follows; see e.g. Theorem 2.5 in [20]. For (conditionally) independent mean zero random variables \(V_1, V_2, \ldots\) with finite variances and positive constants \(c_1, c_2, \ldots\) satisfying \(c_1 \geq c_2 \geq \cdots\), the inequality

\[
\mathbb{P}_X \left( \max_{1 \leq k \leq L} c_k |V_1 + \cdots + V_k| \geq x \right) \leq \frac{1}{x^2} \left( c_1^2 \sum_{k=1}^\ell \mathbb{E}X V_k^2 + \sum_{k=\ell+1}^L c_k^2 \mathbb{E}X V_k^2 \right)
\]

holds for any \(x > 0\) and \(1 \leq \ell \leq L\). Applying this inequality to the conditionally independent random variables \(V_k = U_k^{(c)} - \mathbb{E}X U_k^{(c)}\) (having conditional mean zero) with \(c_k = 1/a_k\) for \(k \geq \ell\), gives

\[
\mathbb{P} \left( \sup_{k \geq \ell} \frac{|S_k^{(c)} - \mathbb{E}X S_k^{(c)}|}{a_k} \geq \frac{\delta}{2} \right) \leq 4 \frac{\text{Var}_X(S_k^{(c)})}{a_k^2} + 4 \frac{\text{Var}_X(U_k^{(c)})}{a_k^2}, \tag{4.16}
\]

Having in mind that, given the tessellation \(X = \{\Xi_n\}_{n \geq 1}\), the random variables \(J_n^{(i)}(\Xi_n) - \lambda(\Xi_n)\), \(n \geq 1\), are mutually independent with mean zero, it is clear that

\[
\text{Var}_X(S_k^{(c)}) \leq \sum_{n \geq 1} I_{W_n}(\alpha(\Xi_n)) \mathbb{E}_X \left( (J_n^{(i)}(\Xi_n) - \lambda(\Xi_n))^2 I_{\{J_n^{(i)}(\Xi_n) - \lambda(\Xi_n) \leq \varepsilon(a_k \vee a_\ell)\}} \right),
\]

and, for \(k > \ell\),

\[
\text{Var}_X(U_k^{(c)}) \leq \sum_{n \geq 1} I_{W_n \setminus W_{n-1}}(\alpha(\Xi_n)) \mathbb{E}_X \left( (J_n^{(i)}(\Xi_n) - \lambda(\Xi_n))^2 I_{\{J_n^{(i)}(\Xi_n) - \lambda(\Xi_n) \leq \varepsilon(a_k \vee a_\ell)\}} \right).
\]
Applying Campbell’s theorem again and using that \( a_k \leq a_{k+1} \) and \( a_k^2 \geq a_k \) for \( k \geq 1 \), we find (after a series of elementary rearrangements) that

\[
\frac{\mathbb{E} \text{Var}_X (S_{\ell}^{(\varepsilon)})}{a_\ell^2} + \sum_{k > \ell} \frac{\mathbb{E} \text{Var}_X (U_k^{(\varepsilon)})}{a_k^2} \\
\leq \frac{\lambda}{a_\ell} \mathbb{E} (V^2 I_{\{V \leq \varepsilon a_\ell\}}) + \lambda \sum_{k > \ell} \frac{a_k - a_{k-1}}{a_{k-1} a_k} \mathbb{E} (V^2 I_{\{V \leq \varepsilon a_k\}}) \\
\leq \lambda \varepsilon \mathbb{E} V + \lambda \varepsilon \frac{a_{k+1}}{a_\ell} \mathbb{E} (V I_{\{V \leq \varepsilon a_{k+1}\}}) + \lambda \varepsilon \sum_{k > \ell} \frac{a_k + 1}{a_k} \mathbb{E} (V I_{\{\varepsilon a_k < V \leq \varepsilon a_{k+1}\}}) \\
\leq \lambda \varepsilon \left( 1 + \frac{a_{\ell+1}}{a_\ell} \right) \mathbb{E} V \leq \frac{\eta \delta^2}{12}
\]

for any \( \ell \geq 1 \) provided that we put \( \varepsilon = \eta \delta^2 / (1 + 2^d) \lambda \mathbb{E} V \). Thus, the right-hand side of (4.16) does not exceed \( \eta / 3 \) for \( \ell \geq 1 \). Further, since \( \mathbb{E} X \left( J_{n}^\beta (\Xi_n) - \lambda (\beta) |\Xi_n| \right) = 0 \), one can easily show that \( |\mathbb{E} X S_k^{(\varepsilon)}| \leq Z_{k, \ell}^{(\varepsilon)} \) for any \( k \geq \ell \), where

\[
Z_{k, \ell}^{(\varepsilon)} = \sum_{n \geq 1} \mathbb{I}_{w_\alpha (\Xi_n)} \left| \mathbb{E} X \left( J_{n}^\beta (\Xi_n) - \lambda (\beta) |\Xi_n| \right) \right| \mathbb{I}_{\{J_{n}^\beta (\Xi_n) < \lambda (\beta) |\Xi_n| \geq \varepsilon a_k\}} .
\]

With the above choice of \( \varepsilon > 0 \), take \( \ell_1 = \ell_1 (\delta, \eta) \) to be the smallest integer such that \( \lambda \mathbb{E} V \mathbb{I}_{\{V > \varepsilon a_{\ell_1}\}} \leq (\delta \wedge \eta) / 3 \). This implies that \( \max_{\ell \geq \ell_1} \mathbb{P} \left( A_{\ell, \varepsilon} (X) \right) \leq \eta / 3 \) and, for \( \ell \geq \ell_1 \),

\[
\mathbb{P} \left( \sup_{k \geq \ell} \left| \frac{\mathbb{E} X S_k^{(\varepsilon)}}{a_k} \right| \geq \frac{\delta}{2} \right) \leq \mathbb{P} \left( \sup_{k \geq \ell} \frac{|Z_{k, \ell}^{(\varepsilon)}|}{a_k} \geq \frac{\delta}{2} \right) \\
\leq \mathbb{P} \left( \sup_{k \geq \ell} \left| \frac{|Z_{k, \ell}^{(\varepsilon)}|}{a_k} - a_k \lambda \mathbb{E} V \mathbb{I}_{\{V > \varepsilon a_{\ell_1}\}} \right| \geq \frac{\delta}{6} \right). \tag{4.17}
\]

However, since the ergodic theorem (4.1) yields \( (Z_{N, \ell}^{(\varepsilon)} / a_N) \overset{\text{a.s.}}{\to} \lambda \mathbb{E} V \mathbb{I}_{\{V > \varepsilon a_{\ell_1}\}} \) as \( N \to \infty \), there exists an integer \( \ell_0 = \ell_0 (\delta, \eta) \) (larger than \( \ell_1 \)) such that the right-hand side of (4.17) becomes smaller than \( \eta / 3 \) for any \( \ell \geq \ell_0 \). Together with the other estimates above and combined with (4.14)–(4.16), this yields (4.13).

5. Multivariate central limit theorem

In this section we prove a central limit theorem, which states asymptotic normality of the normalized random vector \( \tilde{Z}_q = (\tilde{Z}_q^{(1)}, \ldots, \tilde{Z}_q^{(m)}) \) defined in (1.2) as \( q \to \infty \).

**Theorem 5.1.** Suppose that the conditions (3.2) (or (3.10)) and \( \mathbb{E} D^d (\Xi^*) < \infty \) are satisfied. Further, assume that the asymptotic covariance matrix \( K = (\sigma_{ij}^2)_{i,j=1}^m \) is distinct from the null matrix, i.e. \( \max_{1 \leq i \leq m} \sigma_{ii}^2 > 0 \). Then,

\[
\lim_{q \to \infty} \sup_{x \in \mathbb{R}^m} \left| \mathbb{P} \left( \tilde{Z}_q \leq x \right) - \Phi_K (x) \right| = 0, \tag{5.1}
\]

where \( \Phi_K \) denotes the distribution function of the \( m \)-dimensional mean zero Gaussian vector \( N(0, K) \) with covariance matrix \( K \).
To make the proof of Theorem 5.1 more transparent, we first collect some preliminary
results in Section 5.1 and postpone the main part of the proof to Section 5.2 below.

In the particular case $d = 2$, the isodiametric inequality (2.2) implies that the second
moment of the perimeter $\mathcal{H}_1(\Xi^*)$ of $\Xi^*$ exists if and only if the second moment of the
diameter of $\Xi^*$ exists. Therefore, in this case, $\mathbb{E}|\Xi^*|^2 < \infty$ and $\mathbb{E}|\Xi^*| < \infty$ are the
only conditions on the typical cell $\Xi^*$ of $X$, which are needed to show (5.1).

5.1. Some auxiliary results

For any fixed vector $t = (t_1, \ldots, t_m)^\top \in \mathbb{R}^m$ and any $\varepsilon > 0$, define $\sigma^2(t) = t^\top K t$ ($\geq 0$) and the event
\[
\mathcal{E}_\varepsilon(t, \varepsilon) = \left\{ \left| B^2_\varepsilon(t, X) - \sigma^2(t) \right| |W_\varepsilon| < \varepsilon |W_\varepsilon| \right\},
\]
where
\[
B^2_\varepsilon(t, X) = \sum_{n \geq 1} \mathbb{E}_X \left( \sum_{i=1}^m t_i \left( J^{(i)}_0(\Xi_n \cap W_\varepsilon) - \lambda^{(i)} |\Xi_n \cap W_\varepsilon| \right) \right)^2.
\]

Lemma 5.1. Under the conditions of Theorem 5.1, for any $t \in \mathbb{R}^m$ and $\varepsilon > 0$,
\[
\lim_{\varepsilon \to \infty} \mathbb{P}(\mathcal{E}_\varepsilon(t, \varepsilon)) = 0. \tag{5.2}
\]

Proof. Using the fact that $\mathbb{E}_X \left( J^{(i)}_0(\Xi_n \cap W_\varepsilon) \right) = \lambda^{(i)} |\Xi_n \cap W_\varepsilon|$ for $i = 1, \ldots, m$, it
is easy to see that the identity
\[
\mathbb{E}_X \left( \sum_{i=1}^m t_i \left( J^{(i)}_0(\Xi_n) - \lambda^{(i)} |\Xi_n| \right) \right)^2 = \mathbb{E}_X \left( \sum_{i=1}^m t_i \lambda^{(i)} \right)^2 - |\Xi_n^*|^2 \sum_{i=1}^m t_i^2 \sigma^2_\varepsilon
\]
holds for any cell $\Xi_n \in \mathcal{X}$, where $\Xi_n \subset W_\varepsilon$. Thus, by the relations (4.4) and
(4.5) of Corollary 4.1,
\[
\frac{B^2_\varepsilon(t, X)}{|W_\varepsilon|} \xrightarrow[\varepsilon \to \infty]{} \lambda \mathbb{E} \left( \sum_{i=1}^m t_i \lambda^{(i)} \right)^2 - |\Xi^*|^2 \sum_{i=1}^m t_i^2 \sigma^2_\varepsilon, \tag{5.3}
\]
where $\sigma^2_\varepsilon = \lambda \left( \mathbb{E}_X J^{(i)}_0(\Xi^*) J^{(j)}_0(\Xi^*) - \lambda^{(i)} \lambda^{(j)} |\Xi^*|^2 \right)$ for $i, j = 1, \ldots, m$. But these
quantities coincide with the entries of the matrix $K$ as defined in (3.4). In other
words, the ratio $B^2_\varepsilon(t, X)/|W_\varepsilon|$ converges a.s. to $\sigma^2(t)$ as $\varepsilon \to \infty$. This implies the
convergence in probability which is just the assertion (5.2).

In this and the subsequent section we use the abbreviation
\[
g(t, J_0, C) = t^\top J_0(C) - t^\top \lambda_0 |C| = \sum_{i=1}^m t_i \left( J^{(i)}_0(C) - \lambda^{(i)} |C| \right) \tag{5.4}
\]
for any $t \in \mathbb{R}^m$ and $C \in \mathcal{C}$, where $\lambda_0 = (\lambda^{(1)}, \ldots, \lambda^{(m)})^\top$.

Lemma 5.2. Under the condition (3.2) of Theorem 3.1, for any $\delta > 0$,
\[
\frac{1}{|W_\varepsilon|} \mathbb{E} \left( \sum_{n \geq 1} \mathbb{E}_X \left( g^2(t, J_0, \Xi_n \cap W_\varepsilon) \mathbb{1}_{|g(t, J_0, \Xi_n \cap W_\varepsilon)| \geq \delta |W_\varepsilon|} \right) \right) \xrightarrow[\varepsilon \to \infty]{} 0. \tag{5.5}
\]
Proof. Since \( g(t, J_0, C) \overset{a}{=} g(t, J_0, C + x) \) for any \( C \in \mathcal{C} \) and \( x \in \mathbb{R}^d \), by means of the theorems of Campbell and Fubini we get

\[
\mathbb{E} \left( \sum_{n \geq 1} \mathbb{E}_X \left( g^2(t, J_0, Z_n \cap W_\theta) \mathbb{I} \{ g(t, J_0, Z_n \cap W_\theta) \geq \sqrt{n W_\theta} \} \right) \right) \\
= \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E} \left( g^2(t, J_0, C \cap (W_\theta - x)) \mathbb{I} \{ g(t, J_0, C \cap (W_\theta - x)) \geq \sqrt{n W_\theta} \} \right) dx f^0(dC). 
\]

By the definition of \( g(t, J_0, C) \) introduced in (5.4), it is easy to see that

\[
| g(t, J_0, C \cap (W_\theta - x)) | \leq ||t|| \left( \sum_{i=1}^{m} J_0^{(i)}(C) + ||\lambda_0|| |C| \right)
\]

and that

\[
g^2(t, J_0, C \cap (W_\theta - x)) \leq 2 ||t||^2 \left( \sum_{i=1}^{m} (J_0^{(i)}(C \cap (W_\theta - x)))^2 + ||\lambda_0||^2 |C \cap (W_\theta - x)|^2 \right).
\]

Together with (3.7), this yields the inequality

\[
\int_{\mathbb{R}^d} g^2(t, J_0, C \cap (W_\theta - x)) dx \leq 2 ||t||^2 |W_\theta| \left( \sum_{i=1}^{m} (J_0^{(i)}(C))^2 + ||\lambda_0||^2 |C|^2 \right).
\]

Thus, summarizing the above estimates shows that the left-hand side of (5.5) is bounded by the product of \( 2 \lambda \|t\|^2 \) and

\[
\int_{\mathbb{R}^d} \mathbb{E} \left( \sum_{i=1}^{m} \left( (J_0^{(i)}(C))^2 + ||\lambda_0||^2 |C|^2 \right) \mathbb{I} \{ ||t|| \left( \sum_{i=1}^{m} J_0^{(i)}(C) + ||\lambda_0|| |C| \right) \geq \sqrt{n W_\theta} \} \right) f^0(dC).
\]

By the integrability conditions in (3.2), the latter expression converges to zero as \( \theta \to \infty \).

Finally, in order to prove Theorem 5.1, we need the following generalization of the well-known Berry–Essen inequality for independent random variables.

**Lemma 5.3.** For independent mean zero random variables \( U_1, U_2, \ldots \) with finite variances, there exist absolute constants \( a_1, a_2 > 0 \) such that, for any \( \varepsilon \in (0, 1) \) and \( n \in \mathbb{N} \),

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \sum_{i=0}^{n} U_i \leq x B_n \right) - \Phi(x) \right| \leq a_1 \varepsilon + \frac{a_2}{B_n^2} \sum_{i=1}^{n} \mathbb{E} U_i^2 \mathbb{I}_{\{ |U_i| \geq B_n \}},
\]

where \( B_n^2 = \sum_{i=1}^{n} \mathbb{E} U_i^2 \) and \( \Phi \) denotes the standard normal distribution function on \( \mathbb{R} \).

Notice that Lemma 5.3 can be easily obtained from Theorem 5.6 in [20] if in the latter theorem we put \( X_i = U_i / B_n \) and consider the function \( g : \mathbb{R} \to [0, \infty) \) with

\[
g(x) = \begin{cases} 
\varepsilon & \text{if } |x| < \varepsilon, \\
|x| & \text{if } \varepsilon \leq |x| < 1, \\
1 & \text{if } |x| \geq 1.
\end{cases}
\]
5.2. Proof of Theorem 5.1

Recall that \( t^\top \tilde{Z}_\varrho = \sum_{i=1}^{m} t_i \tilde{Z}_{\varrho}^i \) and \( \sigma^2(t) = t^\top K t \) for \( t = (t_1, \ldots, t_m) \in \mathbb{R}^m \). The well-known Cramér-Wold device states that the \( m \)-variate central limit theorem (5.1) is equivalent to

\[
\lim_{\varrho \to \infty} \sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( t^\top \tilde{Z}_\varrho \leq x \right) - \Phi \left( \frac{x}{\sigma(t)} \right) \right| = 0, \tag{5.6}
\]

for all \( t \in \mathbb{R}^m \) with \( \sigma^2(t) > 0 \), and \( t^\top \tilde{Z}_\varrho \xrightarrow{\varrho \to \infty} 0 \) in probability if \( \sigma^2(t) = 0 \). The latter holds since \( E(t^\top \tilde{Z}_\varrho)^2 \xrightarrow{\varrho \to \infty} \sigma^2(t) \) by (3.8). Thus, let \( t \in \mathbb{R}^m \) be fixed such that \( \sigma^2(t) > 0 \). Since the random vector measures \( J_n = (j_n^{(1)}, \ldots, j_n^{(m)})^\top \) are mutually independent and independent of the tessellation \( X = \{ \Xi_n \}_{n \geq 1} \), we may write

\[
\mathbb{P} \left( t^\top \tilde{Z}_\varrho \leq x \right) = \mathbb{E} \left( \mathbb{P}_X \left( \sum_{n \geq 1} t^\top (J_n(\Xi_n \cap W_\varrho) - \lambda_0 |\Xi_n \cap W_\varrho|) \leq x \sqrt{|W_\varrho|} \right) \right).
\]

Lemma 5.3, with the notation introduced in Section 5.1, yields the estimate

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}_X \left( \sum_{n \geq 1} t^\top (J_n(\Xi_n \cap W_\varrho) - \lambda_0 |\Xi_n \cap W_\varrho|) \leq x \sqrt{|W_\varrho|} \right) - \Phi \left( \frac{x \sqrt{|W_\varrho|}}{B_\varrho(t,X)} \right) \right| \leq a_1 \varepsilon + \frac{a_2}{B_\varrho^2(t,X)} \mathbb{E}_X \left( g^2(t, J_0, \Xi_n \cap W_\varrho) \mathbb{I}_{\{|g(t,J_0,\Xi_n \cap W_\varrho)| \geq \varepsilon B_\varrho(t,X)\}} \right),
\]

where \( \varepsilon \in (0, 1) \) can be chosen arbitrarily small such that \( \varepsilon \leq \frac{3}{4} \sigma^2(t) \). Furthermore, (5.3) and the inequality

\[
\left| \frac{B_\varrho^2(t,X)}{|W_\varrho|} - \sigma^2(t) \right| \leq \varepsilon \leq \frac{3}{4} \sigma^2(t)
\]

imply that

\[
\frac{\sqrt{|W_\varrho|}}{B_\varrho(t,X)} \leq \frac{2}{\sigma(t)} \quad \text{and} \quad \left| \frac{\sqrt{|W_\varrho|}}{B_\varrho(t,X)} - \frac{1}{\sigma(t)} \right| \leq \frac{2 \varepsilon}{\sigma^3(t)}
\]

for all \( \varrho > 0 \) sufficiently large. The mean value theorem, together with \( \max_{x \in \mathbb{R}} \Phi'(x) = 1/\sqrt{2\pi} \), yields

\[
\left| \Phi \left( \frac{x \sqrt{|W_\varrho|}}{B_\varrho(t,X)} \right) - \Phi \left( \frac{x}{\sigma(t)} \right) \right| \leq \frac{|x|}{\sqrt{2\pi}} \left| \frac{\sqrt{|W_\varrho|}}{B_\varrho(t,X)} - \frac{1}{\sigma(t)} \right|.
\]

By the estimates derived above, it is easily seen that

\[
\left| \mathbb{P}(t^\top \tilde{Z}_\varrho \leq x) - \Phi \left( \frac{x}{\sigma(t)} \right) \right| \leq \mathbb{P}(E^c_\varrho(t, \varepsilon)) + \mathbb{E}(\mathbb{I}_{E_\varrho(t, \varepsilon)} | \Phi(x \sqrt{|W_\varrho|} / B_\varrho(t,X)) - \Phi(x / \sigma(t))) |
\]

\[+ \mathbb{E}(\mathbb{I}_{E_\varrho(t, \varepsilon)} | \mathbb{P}_X \left( \sum_{n \geq 1} t^\top (J_n(\Xi_n \cap W_\varrho) - \lambda_0 |\Xi_n \cap W_\varrho|) \leq x \sqrt{|W_\varrho|} \right) - \Phi(x \sqrt{|W_\varrho|} / B_\varrho(t,X)) |)
\]

\[\leq \mathbb{P}(E^c_\varrho(t, \varepsilon)) + \frac{2 \varepsilon |x|}{\sqrt{2\pi} \sigma^3(t)} + a_1 \varepsilon
\]

\[+ \frac{4 a_2}{\sigma^2(t) |W_\varrho|} \mathbb{E} \left( \sum_{n \geq 1} \mathbb{E}_X \left( g^2(t, J_0, \Xi_n \cap W_\varrho) \mathbb{I}_{\{|g(t,J_0,\Xi_n \cap W_\varrho)| \geq \varepsilon \sigma(t) \sqrt{|W_\varrho|}/2\}} \right) \right).
\]
Since, by Čebychev’s inequality, we have

$$\left| P\left( t^T \tilde{Z}_q \leq x \right) - \Phi\left( \frac{x}{\sigma(t)} \right) \right| \leq \varepsilon \max \left\{ \mathbb{E}(t^T \tilde{Z}_q)^2, \sigma^2(t) \right\}$$

for $|x| \geq 1/\sqrt{\varepsilon}$, we conclude from (3.8), Lemma 5.1 and Lemma 5.2 that

$$\lim_{\theta \to \infty} \sup_{x \in \mathbb{R}} \left| P\left( t^T \tilde{Z}_q \leq x \right) - \Phi\left( \frac{x}{\sigma(t)} \right) \right| \leq a_1 \varepsilon + \frac{\sqrt{2} \varepsilon}{\sqrt{\pi} \sigma^3(t)} + \varepsilon \sigma^2(t)$$

for any sufficiently small $\varepsilon > 0$. This proves (5.6).

6. Numerical examples

In this section, we assume that $d = 2$ and present some numerical results regarding the asymptotic covariance matrix $K$ as well as the asymptotic distribution of certain functionals. Related numerical results for superpositions of Poisson–Voronoi tessellations can be found, e.g., in [2].

6.1. Poisson nestings

The (initial) tessellation $X$ is chosen to be either a Poisson line tessellation (PLT) or a Poisson-Voronoi tessellation (PVT), where in both cases $\lambda = 0.01$. The inner structure of the cells $\Xi_n$ of $X$ is assumed to be induced by (component) tessellations $X_n$, where either one of four possible Poisson nesting types $X_n^{(1)}, \ldots, X_n^{(4)}$ is chosen; see Table 1. In each case, $X_n$ can be described by two parameters $\gamma_1 > 0$ and $\gamma_2 > 0$, where

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>Type of $X_n^{(j)}$</td>
<td>PLT/PVT</td>
<td>PLT/PVT</td>
<td>PVT/PLT</td>
<td>PVT/PVT</td>
</tr>
</tbody>
</table>

**Table 1: Choices of Poisson nesting types for $X_n$**

$\gamma_2$ is the intensity of the nested tessellation. We concentrate on the case $m = 2$, i.e. $J_n = (J_n^{(1)}, J_n^{(2)})^\top$. Particularly, $J_n^{(1)}$ counts the nodes of the component tessellation $X_n$, and $J_n^{(2)}$ measures the length of the edges of $X_n$. In order to calculate $\gamma_1$ and $\gamma_2$, we assume that the vector $(\lambda^{(1)}, \lambda^{(2)})^\top$ is the same for all four choices of $X_n$. This means that we obtain the system of equations

$$\lambda^{(1)} = \lambda_1^{(1)}(\gamma_1^{(1)}, \gamma_2^{(1)}) \quad \text{and} \quad \lambda^{(2)} = \lambda_2^{(2)}(\gamma_1^{(2)}, \gamma_2^{(2)})$$

(6.1)

for $j = 1, \ldots, 4$. Particularly, the following formulae (6.2) to (6.5) show the system of equations for $X_n^{(1)}, \ldots, X_n^{(4)}$, respectively (see, e.g. [14]):

$$\lambda^{(1)} = 2\gamma_1^{(1)} + \frac{\gamma_1^{(1)}}{\pi} + \frac{8}{\pi} \gamma_1^{(1)} \sqrt{\gamma_1^{(1)}}, \quad \lambda^{(2)} = \gamma_1^{(1)} + 2\sqrt{\gamma_1^{(1)}}$$

(6.2)

$$\lambda^{(1)} = \frac{\gamma_1^{(2)}}{\pi} + \frac{\gamma_2^{(2)}}{\pi} + \frac{4}{\pi} \gamma_1^{(2)} \gamma_2^{(2)} \gamma_1^{(2)}$$

(6.3)
\[ \lambda^{(1)} = 2\gamma_1^{(3)} + \frac{(\gamma_2^{(3)})^2}{\pi} + \frac{8}{\pi} \gamma_2^{(3)} \sqrt{\gamma_1^{(3)}}, \quad \lambda^{(2)} = \gamma_2^{(3)} + 2\sqrt{\gamma_1^{(3)}} \] (6.4)

\[ \lambda^{(1)} = 2\gamma_1^{(4)} + 2\gamma_2^{(4)} + \frac{16}{\pi} \sqrt{\gamma_1^{(4)} \gamma_2^{(4)}}, \quad \lambda^{(2)} = 2\sqrt{\gamma_1^{(4)}} + 2\sqrt{\gamma_2^{(4)}} \] (6.5)

Note that it is difficult, however not impossible, to find values for the intensities \( \lambda^{(1)} \) and \( \lambda^{(2)} \) such that (6.2) to (6.5) have positive solutions \( \gamma_1^{(j)}, \gamma_2^{(j)} \) simultaneously for each \( j = 1, \ldots, 4 \). As Table 2 shows, we confine ourselves to the case where (6.2) through (6.4) are regarded.

<table>
<thead>
<tr>
<th>Type of ( X_n )</th>
<th>( \gamma_1 )</th>
<th>( \gamma_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PLT/PVT (( X_n^{(1)} ))</td>
<td>0.0857086</td>
<td>0.0000511</td>
</tr>
<tr>
<td>PLT/PLT (( X_n^{(2)} ))</td>
<td>0.084829</td>
<td>0.0151171</td>
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<tr>
<td>PVT/PLT (( X_n^{(3)} ))</td>
<td>0.0000511</td>
<td>0.0857086</td>
</tr>
</tbody>
</table>

Table 2: Intensities of \( X_n^{(1)}, X_n^{(2)} \) and \( X_n^{(3)} \) for \( (\lambda^{(1)}, \lambda^{(2)})^\top = (0.004, 0.1)^\top \)

In the above examples, the integrability conditions of (3.10) (see also the remark after Theorem 5.1) are obviously satisfied whenever

\[ \mathbb{E}D^4(\Xi^*) < \infty \quad \text{and} \quad \mathbb{E}\left( J_0^{(i)}([0, 1]^2) \right)^2 < \infty, \quad i = 1, 2. \] (6.6)

The first part of (6.6) is true because the diameter of the typical cell \( \Xi^* \) has an exponentially bounded tail, both for Poisson–Voronoi tessellations and for Poisson line tessellations; see, e.g., [13]. Having in mind that the random measures \( J_0^{(i)} \) are induced by the Poisson–type nestings mentioned in Table 1, the second–order properties of Poisson line and Poisson–Voronoi tessellations provide that the second part of (6.6) is satisfied as well.

### 6.2. Computation of asymptotic covariance matrices

For each of \( n_1 \) realizations \( \xi^* \) of the typical cell \( \Xi^* \) of \( X \), we consider \( n_2 \) realizations of \( X_n^{(1)}, X_n^{(2)}, X_n^{(3)} \), where the respective intensities are chosen according to Table 2. Hence, a covariance estimator based on the \( n_2 \) measured values of \( J_n^{(1)}(\xi^*) \) and \( J_n^{(2)}(\xi^*) \) can be calculated by using the natural approach \( S_{UV} = \frac{1}{n_2} \left( \sum_{i=1}^{n_2} U_i V_i - n_2 \bar{U} \bar{V} \right) \), where \( (U_1, \ldots, U_{n_2})^\top \) and \( (V_1, \ldots, V_{n_2})^\top \) denote two vectors of sample variables. Finally, the sample mean of the \( n_1 \) estimates of covariances is multiplied by \( \lambda \) in order to get an estimate of \( K \). Table 3 shows \( K \) for \( n_1 = n_2 = 100000 \) in the case where \( X \) is a PLT. In Table 4, an estimate of \( K \) is shown with \( X \) being a PVT.

Clearly, the estimated covariance between \( J_n^{(1)}(\Xi^*) \) and \( J_n^{(2)}(\Xi^*) \) seems to depend strongly on the type of \( X \), while the choice of \( X_n \) given a certain \( X \) does not yield much difference between the values.

The simulations have been performed using packages of the GeoStoch library [8]; see also [15]. Further simulations evaluating our central limit theorem given in Theorem 5.1 showed that the asymptotic normality of the distribution of \( (Z_k^{(1)}, Z_k^{(2)})^\top \) is justified quite well, even in the case of a relatively small quadratic sampling window \( W_\theta \) of area \( |W_\theta| = 200^2 \).
\[
\begin{pmatrix}
0.0122837 & 0.1171656 \\
0.1171656 & 1.6270537
\end{pmatrix}
\begin{pmatrix}
0.0125047 & 0.1137313 \\
0.1137313 & 1.6069373
\end{pmatrix}
\begin{pmatrix}
0.0128533 & 0.1220669 \\
0.1220669 & 1.6841176
\end{pmatrix}
\]
(a) Nesting type \(X_n^{(1)}\)  
(b) Nesting type \(X_n^{(2)}\)  
(c) Nesting type \(X_n^{(3)}\)

Table 3: Asymptotic covariance matrix \(K\) if \(X\) is a PLT with \(\lambda = 0.01\)

\[
\begin{pmatrix}
0.0090550 & 0.0711104 \\
0.0711104 & 0.9873148
\end{pmatrix}
\begin{pmatrix}
0.000969 & 0.071946 \\
0.071946 & 0.9913954
\end{pmatrix}
\begin{pmatrix}
0.0093042 & 0.0701055 \\
0.0701055 & 0.9832765
\end{pmatrix}
\]
(a) Nesting type \(X_n^{(1)}\)  
(b) Nesting type \(X_n^{(2)}\)  
(c) Nesting type \(X_n^{(3)}\)

Table 4: Asymptotic covariance matrix \(K\) if \(X\) is a PVT with \(\lambda = 0.01\)

7. Conclusion

In this paper, a (normalized) vector of functionals \(\tilde{Z}_q\) has been considered, where its components describe the inner structure of the cells of a stationary and ergodic tessellation \(X\). We have shown that under certain conditions the distribution function of \(\tilde{Z}_q\) converges uniformly to the (m-dimensional) multivariate normal distribution \(N(\mu, \Sigma)\) if the sampling window \(W_q\) grows unboundedly as \(q \to \infty\). The asymptotic covariance matrix \(K\) has been determined and laws of large numbers are shown, which provide unbiased and consistent estimators for the intensities \(\lambda^{(1)}, \ldots, \lambda^{(m)}\) of the stationary random measures \(J_0^{(1)}, \ldots, J_0^{(m)}\).

There are several interesting perspectives for further research. In particular, the vector of functionals \(J_0^{(1)}, \ldots, J_0^{(m)}\) can be generalized such that each component \(J_n^{(k)}\) of this vector is a functional defined on the \(k\)-facets \(0 \leq k \leq d\) of the cell \(\Xi_n\) of \(X\). For example, if \(d = 2\) and \(J_n^{(k)}\) is induced by the stationary tessellation \(X_n\), then \(k\)-crossings can be analyzed on the boundary of \(\Xi_n\), induced by the edges of a nested tessellation \(X_n\).

Another interesting problem is the derivation of an unbiased and consistent estimator for the asymptotic covariance matrix \(K = \lim_{q \to \infty} \text{Cov} \tilde{Z}_q\). Such an estimator is needed if one wants to construct asymptotic hypothesis tests for the intensities \(\lambda^{(1)}, \ldots, \lambda^{(m)}\) of \(J_0^{(1)}, \ldots, J_0^{(m)}\), based on the central limit theorem stated in Theorem 5.1.

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References


