Asymptotic properties of Euclidean shortest-path trees in random geometric graphs

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Abstract

We consider asymptotic properties of two functionals on Euclidean shortest-path trees appearing in random geometric graphs in $\mathbb{R}^2$ which can be used, for example, as models for fixed-access telecommunication networks. First, we determine the asymptotic bivariate distribution of the two backbone lengths inside a certain class of typical Cox-Voronoi cells as the size of this cell grows unboundedly. The corresponding Voronoi tessellation is generated by a stationary Cox process which is concentrated on the edges of the random geometric graph and whose intensity tends to 0. The limiting random vector can be represented as a simple geometric functional of a decomposition of a typical Poisson-Voronoi cell induced by an independent random sector. Using similar methods, we consider the asymptotic bivariate distribution of the total lengths of the two subtrees inside the Cox-Voronoi cell.

Keywords: random geometric graph, Cox process, Voronoi tessellation, shortest-path tree, backbone length, total length

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1. Introduction

The topology and geometry of fixed-access telecommunication networks is strongly influenced by the respective properties of the underlying road system. Typically, inside a so-called serving zone (the supply area of higher-level components) all lower-level components (e.g. end users) are connected to a fixed higher-level component along the shortest Euclidean path in the road system, which is always assumed below. Tracing the path from the lower-level components to the higher-level component induces a natural tree structure, the so-called shortest-path tree, see [7] and [11]. Therefore, a detailed understanding of distributional properties of shortest-path trees on random networks can allow telecommunication companies to compute elements for accurate cost-estimates for large-scale operations such as upgrading copper networks to optical fiber networks.

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Deriving a simple description for the distribution of the entire shortest-path tree, which is applicable to various network models for underlying road systems (such as Poisson-Delaunay, Poisson-Voronoi and Poisson line tessellation), is considered to be a daunting task; see [10] for the setting of sparse trees. A feasible approach to tackle this problem is to develop a good description of several important characteristics of the shortest-path tree in order to gradually improve the understanding of the complex tree structure. If the intensity of higher-level components is not too small, i.e., the shortest-path tree is not too large, a parametric copula approach has been proposed in [11, 12] for directly simulating various functionals of shortest-path trees. In the present paper, we consider the asymptotic bivariate distribution of the backbone lengths (Theorem 1) as well as the asymptotic bivariate distribution of the total lengths (Theorem 2) of the two subtrees at the network root as the intensity of higher-level components tends to zero. Here, the backbone of a subtree is understood to be its longest branch.

The rest of this paper is organized as follows. In Section 2, we state our main results (Theorems 1 and 2). Section 3 provides some preliminary tools, which are used in the proofs of these results. Finally, in Sections 4 and 5, we use these tools to prove Theorems 1 and 2, respectively.

2. Main results

In the present section, we introduce the principal objects of investigation in this paper, namely *shortest-path trees*. We also state our main results. Loosely speaking, a shortest-path tree can be thought of as the union of all shortest Euclidean paths (on the road system) emanating from some given root component in a spatial network structure. For instance, in the context of fixed-access telecommunication networks, this root component could be an access point to which subscribers in the network are connected along their shortest Euclidean path.

To make this more precise, we first explain the notion of a random geometric graph that is used in this paper. We write $\mathbb{G}$ for the family of all locally finite sets of line segments in $\mathbb{R}^2$. Endowing $\mathbb{G}$ with the smallest σ-algebra $\mathcal{G}$ that contains all open sets of the Fell topology on $\mathbb{G}$, see [14, p. 563], we can consider random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{G}$, which will be called *random geometric graphs* in the following.

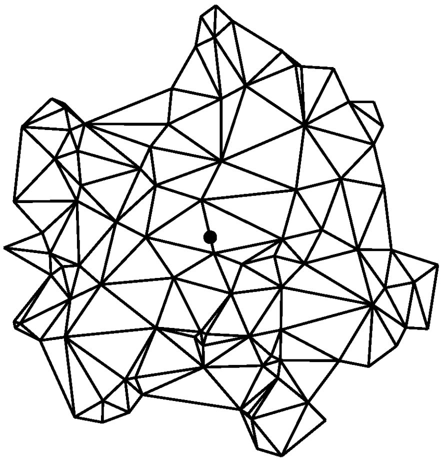
As mentioned above, when considering applications to telecommunication networks, it is conceptually useful to consider a random geometric graph from the perspective of a typical access point, which can be thought of as a point that is chosen at random from the edge set of the graph. This idea can be made more precise by using Palm calculus see, e.g. [5, p. 127] and [14, p. 73]. Indeed, let G be a stationary random geometric graph whose *length intensity* $\gamma = \mathbb{E}\nu_1(G \cap [0, 1]^2)$ is assumed to be positive and finite, where $\nu_1(\cdot)$ denotes the one-dimensional Hausdorff measure in $\mathbb{R}^2$. The *Palm version* G^* of G is a random geometric graph whose distribution is determined by

$$\mathbb{E}h(G^*) = \frac{1}{\gamma}\,\mathbb{E}\int_{G\cap[0,1]^2} h(G - x)\nu_1(\mathrm{d}x),$$

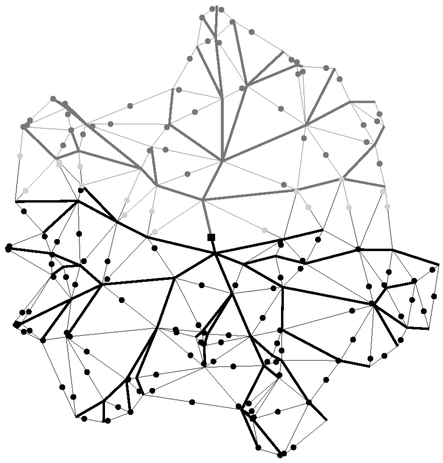
where $h : \mathbb{G} \to [0, \infty)$ is any $\mathcal{G}$-measurable function.

We investigate a scenario where low-level components can be thought of as being distributed arbitrarily densely on the road network. Considering the Palm version G^* of G, the shortest-path tree is defined as the union of all shortest

Euclidean paths emanating from the origin $o \in \mathbb{R}^2$, see [7, 11]. To be more precise, for $x, y \in G^*$ let $\rho(x, y)$ denote a shortest path in the graph G^* connecting x and y. Possible issues of non-uniqueness of shortest paths can be overcome by first endowing the nodes with i.i.d. uniform weights and then selecting the shortest path that additionally minimizes the sum of weights. Then, define the *shortest-path tree* $\mathsf{spt}(G^*)$ of G^* rooted at o as a directed graph whose vertex set is given by the union of $\{o\}$, the vertex set of G^* and the set of distance peaks. Here, a point z on an edge $[x, y]$ of G^* is called *distance peak* if $\ell(o, x) + \nu_1([x, z]) = \ell(o, y) + \nu_1([y, z])$, where $\ell(\cdot, \cdot) = \nu_1(\rho(\cdot, \cdot))$ denotes the length of the shortest Euclidean path along the edges of G^*. Furthermore, $\mathsf{spt}(G^*)$ contains three types of edges. First, if $[x_o, y_o]$ denotes the edge of G^* containing o, then in $\mathsf{spt}(G^*)$ an edge is drawn from o to x_o and from o to y_o. Second, if $[x, y]$ is an edge of G^* such that $x \in \rho(o, y)$, then in $\mathsf{spt}(G^*)$ we add an edge from x to y. Third, if z is a distance peak located on an edge $[x, y]$ of G^*, then in $\mathsf{spt}(G^*)$ an edge is drawn from x to z and from y to z. In particular, $\mathsf{spt}(G^*)$ forms a tree with root o and with probability 1, this tree can be represented as the union of two subtrees $\mathsf{spt}(G^*) = T_1 \cup T_2$, both rooted at o. An illustration of this decomposition is shown in Figure 1, where, in order to make the tree structure clearly visible, edges incident to a distance peak are drawn thinner. The subtrees T_1 and T_2 are drawn in black and dark gray, respectively. The interface between the subtrees T_1 and T_2, i.e., distance peaks adjacent both to an edge of T_1 and T_2, is drawn in light gray. The remaining distance peaks are drawn in dark gray.



(a) Cutout of the Palm version of a Poisson-Delaunay graph



(b) Cutout of two subtrees of the shortest-path tree (black and dark gray) and their interface (points in light gray)

Figure 1: Construction of the shortest-path tree on the Palm version of a Poisson-Delaunay graph

For the purpose of modeling the underlying infrastructure in fixed-access telecommunication networks, different choices for G are thinkable. Nevertheless, the classical planar random tessellations, namely Poisson-Delaunay tessellation, Poisson-Voronoi tessellation and isotropic Poisson line tessellation are the most thoroughly investigated models, see [5, 14]. In order to simplify the presentation, we will therefore always assume that G is either one of these classical tessellations or is given by a Poisson relative neighborhood graph, see [1].

When considering telecommunication networks at a large scale, typically there is not only a single access point located at the origin, but the access points (called *higher-level components* in the following) are spread over the entire edge set of G^*. More precisely, these additional higher-level components form a Cox process X_λ on G^* whose random intensity measure is given by $\lambda\nu_1(\cdot \cap G^*)$ for some linear intensity $\lambda > 0$. Considering the Voronoi tessellation on the higher-level components, the plane is divided into serving zones, and we write $\Xi_{o,\lambda}$ for the serving zone of the origin, i.e., the zero-cell of the Voronoi tessellation on $X_\lambda \cup \{o\}$. In the present paper, we consider the asymptotic behavior of certain characteristics of the restrictions $T_1 \cap \Xi_{o,\lambda}$ and $T_2 \cap \Xi_{o,\lambda}$ of the subtrees T_1 and T_2 to the serving zone $\Xi_{o,\lambda}$ as $\lambda \to 0$, i.e., as the size of the serving zone $\Xi_{o,\lambda}$ tends to infinity.

For $i \in \{1,2\}$ and $\lambda \in (0,1)$ let $Z_\lambda^{(i)} = \sup_{x\in T_i\cap\Xi_{o,\lambda}} \ell(o,x)$ denote the length of the longest branch of the subtree T_i inside $\Xi_{o,\lambda}$. These two longest branches can be seen as the backbone of $\mathsf{spt}(G^*)$. An illustration is shown in Figure 2.

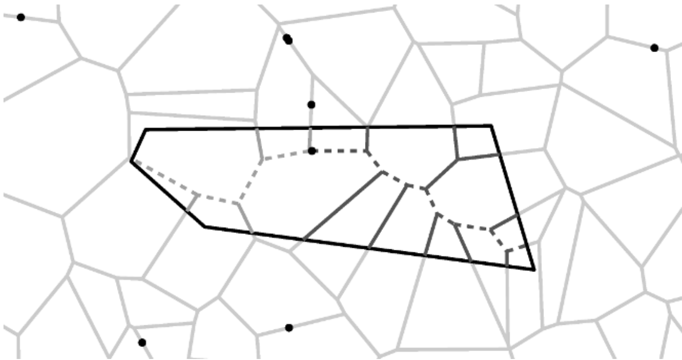


Figure 2: Subdivision of shortest-path tree in $\Xi_{o,\lambda}$ into two subtrees (dark and light gray); the longest branches are dashed

In order to describe the asymptotic behavior of the distribution of the random vector $(Z_\lambda^{(1)}, Z_\lambda^{(2)})$ as $\lambda \to 0$, we first state some properties of the subtrees T_1 and T_2. Let $I \subset [0,2\pi]$ be an arbitrary interval and $C_I = \{y \in \mathbb{R}^2 : \angle(e_1,[o,y]) \in I\}$ be the planar sector of points whose angle with the x-axis is contained in I. Using a methodology developed by Howard and Newman [9], it is shown in [8] that there exist random intervals $I_1, I_2 \subset [0,2\pi]$ that have disjoint interiors and satisfy the following properties. We have $I_1 \cup I_2 = [0,2\pi]$ and for all $\delta > 0$ there exists an a.s. finite random number $R_0 > 0$ such that $G^* \cap C_{I_i\ominus[-\delta,\delta]} \setminus [-R_0,R_0]^2 \subset T_i$ and $T_i \setminus [-R_0,R_0]^2 \subset C_{I_i\oplus[-\delta,\delta]}$ for all $i \in \{1,2\}$, where we put $I_i \oplus [-\delta,\delta] = \{a+b : a \in I_i, b \in [-\delta,\delta]\}$ and $I_i \ominus [-\delta,\delta] = \{a \in I_i : \{a\} \oplus [-\delta,\delta] \subset I_i\}$. Here, subintervals of $[0,2\pi]$ are identified with subsets of the unit circle in $\mathbb{R}^2$ at the corresponding angles. If one of the subtrees, say T_1, is bounded, then the decomposition of $[0,2\pi]$ is trivial in the sense that $I_1 = \emptyset$ and $I_2 = [0,2\pi]$, where we formally put $C_\emptyset = \{o\}$. In [3], a related result is discussed for the radial spanning tree, where the number of sectors varies between 1 and 5.

Next, we introduce the limiting random vector $(Z^{(1)}, Z^{(2)})$ of the vector of appropriately rescaled main-branch lengths $(Z_\lambda^{(1)}, Z_\lambda^{(2)})$. For $i \in \{1,2\}$ consider the random variables $Z^{(i)} = \mu \max_{x\in C_{I_i}\cap\Xi_o} |x|$, where $\mu \geq 1$ is the so-called time constant (see, e.g. [4, 8], and also [2, 13] for related models) and Ξ_o denotes a typical Poisson-Voronoi cell that is independent of (I_1, I_2) and whose underlying Poisson point process has intensity γ. In other words, Ξ_o is the zero-cell of the Voronoi tessellation associated with the union of this Poisson point process and the origin. An illustration of this setup is shown in Figure 3. Loosely speaking, asymptotically, the main branches are close to the dashed lines in Figure 3.

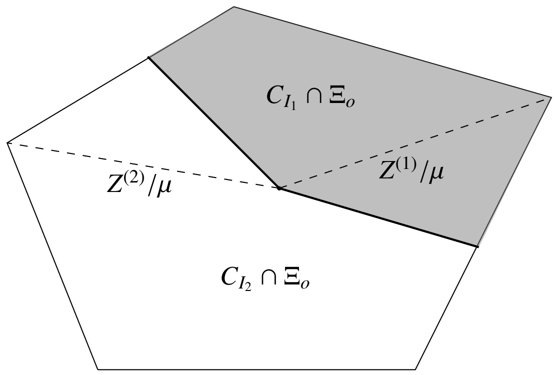


Figure 3: Sectors $C_{I_1} \cap \Xi_o$ (gray) and $C_{I_2} \cap \Xi_o$ (white)

The random vector $(Z_\lambda^{(1)}, Z_\lambda^{(2)})$ exhibits the following asymptotic behavior.

Theorem 1. *As $\lambda \to 0$, the random vector $\left(\sqrt{\lambda} Z_\lambda^{(1)}, \sqrt{\lambda} Z_\lambda^{(2)}\right)$ converges to $\left(Z^{(1)}, Z^{(2)}\right)$ in distribution.*

Theorem 1 is proven in Section 4. The explicit description of the random vector $(Z^{(1)}, Z^{(2)})$ stated above allows us to devise an asymptotically accurate simulation algorithm for the vector $(Z_\lambda^{(1)}, Z_\lambda^{(2)})$ of main-branch lengths, which for small values of λ is far less time-consuming than simulating $(Z_\lambda^{(1)}, Z_\lambda^{(2)})$ directly [11]. Indeed, the size of the serving zones decreases in λ, so that for small values of λ the random geometric graph G^* needs to be constructed in a large environment of the origin in order to determine $(Z_\lambda^{(1)}, Z_\lambda^{(2)})$ by the direct simulation algorithm proposed in [11].

Apart from the length of the two main branches, also the total edge lengths $Y_\lambda^{(i)} = \nu_1(T_i \cap \Xi_{o,\lambda})$ with $i \in \{1,2\}$ in each of the two subtrees rooted at the origin are important structural characteristics of the underlying network. In fact, our methods allow us to consider not only the convergence of an appropriately scaled version of the bivariate random vector $(Y_\lambda^{(1)}, Y_\lambda^{(2)})$, but also of the bivariate vector of random measures $(H_\lambda^{(1)}, H_\lambda^{(2)})$, where for any $i \in \{1,2\}$ and Borel set $B \subset \mathbb{R}^2$, we put $H_\lambda^{(i)}(B) = \nu_1(B \cap \sqrt{\lambda}(T_i \cap \Xi_{o,\lambda}))$. Similar to the main-branch lengths considered in Theorem 1, we can provide an explicit description of the limiting distribution of $(\sqrt{\lambda} H_\lambda^{(1)}, \sqrt{\lambda} H_\lambda^{(2)})$ as $\lambda \to 0$. For $i \in \{1,2\}$ consider the random measure $H^{(i)}(\cdot) = \gamma\nu_2(\cdot \cap C_{I_i} \cap \Xi_o)$, where ν_2 denotes the Lebesgue measure in $\mathbb{R}^2$.

Theorem 2. *As $\lambda \to 0$, the random vector $\left(\sqrt{\lambda} H_\lambda^{(1)}, \sqrt{\lambda} H_\lambda^{(2)}\right)$ converges to $\left(H^{(1)}, H^{(2)}\right)$ in distribution.*

Theorem 2 is proven in Section 5. Furthermore, we show in Section 5 that Theorem 2 can be used to determine the asymptotic behavior of the random vector $(Y_\lambda^{(1)}, Y_\lambda^{(2)})$ of the total subtree-lengths. For $i \in \{1,2\}$ consider the random variables $Y^{(i)} = \gamma\nu_2(C_{I_i} \cap \Xi_o)$.

Corollary 3. *As $\lambda \to 0$, the random vector $\left(\lambda Y_\lambda^{(1)}, \lambda Y_\lambda^{(2)}\right)$ converges to $\left(Y^{(1)}, Y^{(2)}\right)$ in distribution.*

As for the branch lengths considered in Theorem 1, our results stated in Theorem 2 and Corollary 3 can be used to derive an efficient approximate simulation algorithm for small values of λ, see [12].

3. Some preliminary results

Before providing rigorous proofs of the main results stated in Section 2, we begin by discussing some heuristics. To fix ideas, we consider the case of $(Z_\lambda^{(1)}, Z_\lambda^{(2)})$ noting that similar heuristics can be provided for $(H_\lambda^{(1)}, H_\lambda^{(2)})$. Assume that the indexing of the subtrees is chosen such that T_1 is a.s. unbounded. If the subtree T_2 is bounded, then using the results of [7], it is not difficult to see that the random vector of scaled main-branch lengths $(\sqrt{\lambda}Z_\lambda^{(1)}, \sqrt{\lambda}Z_\lambda^{(2)})$ converges in distribution to $(\mu R, 0)$ as $\lambda \to 0$, where μ is the time constant in the respective first-passage model and R denotes the smallest radius such that the disk $B_R(o)$ with radius R centered at the origin contains the typical Poisson-Voronoi cell Ξ_o. On the other hand, if T_2 is unbounded, then it is shown in [8] that the interface between T_1 and T_2 asymptotically approaches ∂C_{I_1}. Furthermore, we also see that the scaled Voronoi cell $\sqrt{\lambda}\Xi_{o,\lambda}$ converges (in distribution) to a typical Poisson-Voronoi cell and that, in a sense, asymptotically this cell becomes independent of the underlying graph G^*, see Lemma 6 below. Finally, by the shape theorem derived in [7, Corollary 1.2], shortest-path lengths behave asymptotically as a constant multiple of the Euclidean distance.

Recall that we assume that G is a Poisson-Delaunay tessellation, Poisson-Voronoi tessellation, Poisson line tessellation or a Poisson relative neighborhood graph. First, we need the following stretched-exponential large-deviation estimate from [10, Lemma A.3] for the total length $\nu_1(G^* \cap Q_s(x))$, where $Q_s(x) = [-s/2, s/2]^2 + x$ denotes the square of side length $s \geq 0$ centered at $x \in \mathbb{R}^2$.

Lemma 4. *For every $\varepsilon > 0$,*

$$\liminf_{s\to\infty} \frac{\log\big(-\log\big(\sup_{x\in\mathbb{R}^2} \mathbb{P}(|\nu_1\big(G^* \cap Q_s(x)) - \gamma s^2| \geq \varepsilon s^2)\big)\big)}{\log s} > 0.$$

A further key step in the derivation of the asymptotic behavior of $\big(\sqrt{\lambda}Z_\lambda^{(1)}, \sqrt{\lambda}Z_\lambda^{(2)}\big)$ is the asymptotic independence of G^* and the zero-cell in the Voronoi tessellation based on $X_\lambda \cup \{o\}$. To begin with, we note that there is a good chance that the Cox-Voronoi cell at the origin contains a given small square and is contained in a given large square. Recall that X_λ denotes a Cox process on G^* with linear intensity $\lambda > 0$ and $\Xi_{o,\lambda}$ is the zero-cell of the Voronoi tessellation based on $X_\lambda \cup \{o\}$. We restate two auxiliary results from [10], see Lemmas A.4 and A.5 in [10].

Lemma 5. *It holds that* $\lim_{r\to\infty} \limsup_{\lambda\to 0} \mathbb{P}\big(Q_{\lambda^{-1/2}/r}(o) \subset \Xi_{o,\lambda} \subset Q_{r\lambda^{-1/2}}(o)\big) = 1.$

Remark. In the proof of Theorem 2, see Section 5 below, we use the fact that Lemma 5 remains true if $\Xi_{o,\lambda}$ is replaced by the typical Voronoi cell associated with a Poisson point process with intensity $\gamma\lambda > 0$.

We write $d_{\mathsf{Haus}}(\cdot,\cdot)$ for the Hausdorff distance between non-empty compact subsets of $\mathbb{R}^2$. Loosely speaking, the following coupling result formalizes a certain asymptotic independence of G^* and the Cox-Voronoi cell $\Xi_{o,\lambda}$.

Lemma 6. *For $\lambda \in (0,1]$ there exists a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which G^*, X_λ, and a homogeneous Poisson point process X with intensity γ are given such that the following two properties hold. If Ξ_o denotes the zero-cell of the Voronoi tessellation on $X \cup \{o\}$, then for every $\varepsilon > 0$,*

$$\lim_{\lambda \to 0} \mathbb{P}\big(d_{\mathsf{Haus}}\big(\sqrt{\lambda}\Xi_{o,\lambda}, \Xi_o\big) \geq \varepsilon\big) = 0. \tag{1}$$

Moreover, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ can be chosen so that X is independent of G^*.

We conclude this subsection by noting a useful elementary result concerning the erosion operation on convex sets.

Lemma 7. *Let $r > 0$ and $K \subset \mathbb{R}^2$ an arbitrary convex set. Then, $(K \oplus B_r(o)) \ominus B_r(o) = K$.*

Proof. Let $x \in \mathbb{R}^2 \setminus K$ be arbitrary. Since K is convex, there exists $y \in B_r(o)$ such that $d(x + y, K) > r$. Hence, $x + y \notin K \oplus B_r(o)$, so that $x \notin \big(K \oplus B_r(o)\big) \ominus B_r(o)$. The inclusion $K \subset (K \oplus B_r(o)) \ominus B_r(o)$ is immediate. □

4. Proof of Theorem 1

In this section, we show how the results of Section 3 can be used in order to prove Theorem 1. We first recall some preliminary results from previous work. Lemma 8 below is a consequence of [7, Theorem 2, Proposition 1], whereas Lemma 9 restates [7, Lemma 28]. For $r > 0$ let $B_r^{G^*}(o) = \{x \in \mathbb{R}^2 : \ell(o, q(x)) \leq r\}$ denote the ball of radius r and center o in the pseudometric induced by ℓ, where $q(x)$ is the closest element on G^* seen from the point x. If the set of closest elements contains more than one point, $q(x)$ is chosen as the lexicographical minimum in this set.

Lemma 8. *Let $\varepsilon > 0$. Then, $\mathbb{P}\big(B_{(1-\varepsilon)t}(o) \subset B_{\mu t}^{G^*}(o) \subset B_{(1+\varepsilon)t}(o)$ for all sufficiently large $t > 0\big) = 1$.*

Lemma 9. *Let $\alpha' \in (0,1)$ be arbitrary. Then, $\lim_{s\to\infty} \mathbb{P}\big(\sup_{x \in Q_s(o)} |x - q(x)| > s^{\alpha'}\big) = 0$.*

Using the results of Section 3, we now show that the family of random vectors $\big(\sqrt{\lambda}Z_\lambda^{(1)}, \sqrt{\lambda}Z_\lambda^{(2)}\big)$ converges in distribution to $(Z^{(1)}, Z^{(2)})$ as $\lambda \to 0$. To simplify notation, we write $I_i^{\delta,+} = I_i \oplus [-\delta, \delta]$ and $I_i^{\delta,-} = I_i \ominus [-\delta, \delta]$ from now on.

Proof of Theorem 1. Let $\delta, \varepsilon > 0$ be arbitrary. Then, by Lemma 9, for all $x_1, x_2 > 0$ and all sufficiently small $\lambda > 0$,

$$\begin{aligned}
\mathbb{P}\big(\sqrt{\lambda}Z_\lambda^{(1)} \leq x_1,\, \sqrt{\lambda}Z_\lambda^{(2)} \leq x_2\big) &= \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{\max_{P^{(i)} \in \Xi_{o,\lambda} \cap T_i} \ell(o, P^{(i)}) \leq x_i/\sqrt{\lambda}\big\}\Big) \\
&\leq \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{\max_{P^{(i)} \in \Xi_{o,\lambda} \cap \big(C_{I_i^{\delta,-}} \ominus B_{\delta/\sqrt{\lambda}}(o)\big)} \ell(o, q(P^{(i)})) \leq x_i/\sqrt{\lambda}\big\}\Big) + \varepsilon \\
&= \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{\Xi_{o,\lambda} \cap \big(C_{I_i^{\delta,-}} \ominus B_{\delta/\sqrt{\lambda}}(o)\big) \subset B^{G^*}_{x_i/\sqrt{\lambda}}(o)\big\}\Big) + \varepsilon.
\end{aligned}$$

Furthermore, Lemmas 6, 7 and 8 yield

$$\begin{aligned}
\mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{\Xi_{o,\lambda} \cap \big(C_{I_i^{\delta,-}} \ominus B_{\delta/\sqrt{\lambda}}(o)\big) \subset B^{G^*}_{x_i/\sqrt{\lambda}}(o)\big\}\Big) &\leq \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{\Xi_{o,\lambda} \cap \big(C_{I_i^{\delta,-}} \ominus B_{\delta/\sqrt{\lambda}}(o)\big) \subset B_{(x_i+\delta)/(\mu\sqrt{\lambda})}(o)\big\}\Big) + \varepsilon \\
&\leq \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{\big(\Xi_o \ominus B_\delta(o)\big) \cap \big(C_{I_i^{\delta,-}} \ominus B_\delta(o)\big) \subset B_{(x_i+\delta)/\mu}(o)\big\}\Big) + 2\varepsilon
\end{aligned}$$

for all sufficiently small $\lambda > 0$. Letting $\delta \to 0$ and $\varepsilon \to 0$ yields

$$\limsup_{\lambda\to 0} \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{ \sqrt{\lambda} Z_\lambda^{(i)} \le x_i \big\} \Big) \le \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{ \operatorname{int}\Xi_o \cap \operatorname{int} C_{I_i} \subset B_{x_i/\mu}(o) \big\} \Big) = \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \{ Z^{(i)} \le x_i \} \Big),$$

where $\operatorname{int}\Xi_o$ denotes the topological interior of Ξ_o. The reverse inequality is similar, but we provide some details for the convenience of the reader. Again, let $\delta, \varepsilon > 0$ be arbitrary. Then, for all $x_1, x_2 > 0$ and all sufficiently small $\lambda > 0$,

$$\begin{aligned}
\mathbb{P}\big(\sqrt{\lambda} Z_\lambda^{(1)} \le x_1,\ \sqrt{\lambda} Z_\lambda^{(2)} \le x_2 \big) &= \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{ \max_{P^{(i)} \in \Xi_{o,\lambda} \cap T_i} \ell(o, P^{(i)}) \le x_i / \sqrt{\lambda} \big\} \Big) \\
&\ge \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{ \max_{P^{(i)} \in \Xi_{o,\lambda} \cap (C_{I_i^{\delta,+}} \oplus B_{\delta/\sqrt{\lambda}}(o))} \ell(o, q(P^{(i)})) \le x_i / \sqrt{\lambda} \big\} \Big) - \varepsilon \\
&= \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{ \Xi_{o,\lambda} \cap \big(C_{I_i^{\delta,+}} \oplus B_{\delta/\sqrt{\lambda}}(o) \big) \subset B^{G^*}_{x_i/\sqrt{\lambda}}(o) \big\} \Big) - \varepsilon.
\end{aligned}$$

Furthermore, Lemmas 6 and 8 yield

$$\begin{aligned}
\mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{ \Xi_{o,\lambda} \cap \big(C_{I_i^{\delta,+}} \oplus B_{\delta/\sqrt{\lambda}}(o) \big) \subset B^{G^*}_{x_i/\sqrt{\lambda}}(o) \big\} \Big) &\ge \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{ \Xi_{o,\lambda} \cap \big(C_{I_i^{\delta,+}} \oplus B_{\delta/\sqrt{\lambda}}(o) \big) \subset B_{(x_i-\delta)/(\mu\sqrt{\lambda})}(o) \big\} \Big) - \varepsilon \\
&\ge \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{ \big(\Xi_o \oplus B_\delta(o) \big) \cap \big(C_{I_i^{\delta,+}} \oplus B_\delta(o) \big) \subset B_{(x_i-\delta)/\mu}(o) \big\} \Big) - 2\varepsilon
\end{aligned}$$

for all sufficiently small $\lambda > 0$. Letting $\delta \to 0$ and $\varepsilon \to 0$ therefore implies

$$\liminf_{\lambda\to 0} \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \big\{ \sqrt{\lambda} Z_\lambda^{(i)} \le x_i \big\} \Big) \ge \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \{ \Xi_o \cap C_{I_i} \subset \operatorname{int} B_{x_i/\mu}(o) \} \Big). \qquad \square$$

5. Proof of Theorem 2

Next, we prove Theorem 2 and Corollary 3. In order to achieve this goal, we need two additional auxiliary results. First, we show that asymptotically the total edge length of G^* in large convex sampling windows does not change substantially, when enlarging this window slightly. In the following, $\mathcal{K}$ denotes the family of all convex subsets of $\mathbb{R}^2$.

Lemma 10. *For every $b \ge 1$ and $\varepsilon > 0$, there exists $\delta > 0$ satisfying*

$$\lim_{s\to\infty} \sup_{\substack{K\in\mathcal{K} \\ K\subset Q_b(o)}} \mathbb{P}\big(\nu_1\big(G^* \cap (sK \oplus B_{\delta s}(o)) \big) - \nu_1(G^* \cap sK) \ge \varepsilon s^2 \big) = 0.$$

Proof. Let $K \subset Q_b(o)$ be fixed. Let $\delta \in (0, 1/16)$ and $K' = \{z \in \mathbb{Z}^2 : Q_\delta(\delta z) \subset (K \oplus B_{3\delta}(o)) \setminus K \ne \emptyset\}$ denote the set of δ-lattice squares contained in $(K \oplus B_{3\delta}(o)) \setminus K$. In particular, putting $n' = \#K'$, convexity of K implies

$$n' \le \delta^{-2}(\nu_2(K \oplus B_{3\delta}(o)) - \nu_2(K)) \le 24b/\delta.$$

Note that for every $z' \in \mathbb{Z}^2$ with $Q_\delta(\delta z') \cap ((K \oplus B_\delta(o)) \setminus K) \ne \emptyset$ there exists $z \in K'$ with $|z - z'|_\infty \le 1$. Thus,

$$\mathbb{P}\big(\nu_1\big(G^* \cap (sK \oplus B_{\delta s}(o)) \big) - \nu_1(G^* \cap sK) \ge \varepsilon s^2 \big) \le \sum_{z\in K'} \mathbb{P}\big(\nu_1(G^* \cap Q_{3\delta s}(\delta s z)) \ge \varepsilon s^2 / n' \big).$$

Hence, choosing $\delta > 0$ sufficiently small and applying Lemma 4 completes the proof. $\square$

The second auxiliary result concerns concentration properties of integrals with respect to $\nu_1(\cdot \cap G^*)$.

Lemma 11. *Let $f : \mathbb{R}^2 \to [0,\infty)$ be a continuous function with compact support. Then, for every $b \ge 1$ and $\varepsilon > 0$,*

$$\lim_{s\to\infty} \sup_{\substack{K\in\mathcal{K}\\ K\subset Q_b(o)}} \mathbb{P}\Big(\big|s^{-1}\int_{G^*\cap\sqrt{s}K} f(s^{-1/2}x)\nu_1(\mathrm{d}x) - \gamma\int_K f(x)\mathrm{d}x\big| \ge \varepsilon\Big) = 0.$$

Proof. The family of $s^{-1/4}$-squares contained in K and the family of $s^{-1/4}$-squares intersecting the topological boundary ∂K of K play important roles. Hence, we put $K_z = K \cap Q_{s^{-1/4}}(s^{-1/4}z)$, $z \in \mathbb{Z}^2$, $S_{\mathsf{int}} = \{z \in \mathbb{Z}^2 : Q_{s^{-1/4}}(s^{-1/4}z) \subset K\}$ and $S_\partial = \{z \in \mathbb{Z}^2 : Q_{s^{-1/4}}(s^{-1/4}z) \cap \partial K \ne \emptyset\}$. We also write $n_{\mathsf{int}} = \#S_{\mathsf{int}}$ and $n_\partial = \#S_\partial$. First, note that

$$\begin{aligned}
&\mathbb{P}\Big(\big|s^{-1}\int_{G^*\cap\sqrt{s}K} f(s^{-1/2}x)\nu_1(\mathrm{d}x) - \gamma\int_K f(x)\mathrm{d}x\big| \ge \varepsilon\Big)\\
&\le \mathbb{P}\Big(\sum_{z\in S_{\mathsf{int}}\cup S_\partial} s^{-1}\int_{G^*\cap\sqrt{s}K_z} |f(s^{-1/2}x) - f(s^{-1/4}z)|\nu_1(\mathrm{d}x) \ge \varepsilon/4\Big) && (2)\\
&\quad + \mathbb{P}\Big(\sum_{z\in S_{\mathsf{int}}\cup S_\partial} \big|s^{-1/2}\int_{s^{-1/2}G^*\cap K_z} f(s^{-1/4}z)\nu_1(\mathrm{d}x) - \gamma\int_{K_z} f(s^{-1/4}z)\mathrm{d}x\big| \ge \varepsilon/4\Big) && (3)\\
&\quad + \mathbb{P}\Big(\sum_{z\in S_{\mathsf{int}}\cup S_\partial} \gamma\int_{K_z} |f(s^{-1/4}z) - f(x)|\mathrm{d}x \ge \varepsilon/4\Big). && (4)
\end{aligned}$$

We analyze the three summands separately and begin with (2). Put $\varepsilon' = \varepsilon/(16b\gamma)$. Then, continuity of f implies that $|f(s^{-1/4}z) - f(x)| \le \varepsilon'$ for all $z \in \mathbb{Z}^2$ and $x \in Q_{s^{-1/4}}(s^{-1/4}z)$, provided that $s > 0$ is sufficiently large. Moreover, $n_{\mathsf{int}} + n_\partial = \#\big(S_{\mathsf{int}} \cup S_\partial\big) \le 2b^2\sqrt{s}$. Hence,

$$\mathbb{P}\Big(\sum_{z\in S_{\mathsf{int}}\cup S_\partial} s^{-1/2}\int_{s^{-1/2}G^*\cap K_z} |f(x) - f(s^{-1/4}z)|\nu_1(\mathrm{d}x) \ge \varepsilon/4\Big) \le \sum_{z\in S_{\mathsf{int}}\cup S_\partial} \mathbb{P}(2b^2\varepsilon' s^{-1/2}\nu_1(G^*\cap Q_{s^{1/4}}(s^{1/4}z)) \ge \varepsilon/4),$$

and by Lemma 4, the latter sum exhibits stretched exponential decay in s. Next, we consider (4) and obtain that

$$\sum_{z\in S_{\mathsf{int}}\cup S_\partial} \gamma\int_{K_z} |f(s^{-1/4}z) - f(x)|\mathrm{d}x \le \sum_{z\in S_{\mathsf{int}}\cup S_\partial} \gamma\varepsilon' s^{-1/2} < \varepsilon/4.$$

Finally, we deal with (3) and put $c = \sup_{x\in\mathbb{R}^2} f(x)$. Observe that

$$\big|s^{-1/2}\int_{s^{-1/2}G^*\cap K_z} f(s^{-1/4}z)\nu_1(\mathrm{d}x) - \gamma\int_{K_z} f(s^{-1/4}z)\mathrm{d}x\big| \le c|s^{-1/2}\nu_1(s^{-1/2}G^*\cap K_z) - \gamma\nu_2(K_z)|.$$

Furthermore, putting $\varepsilon'' = \varepsilon/(4c)$ yields

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{z\in S_{\mathsf{int}}\cup S_\partial} \big|s^{-1/2}\int_{s^{-1/2}G^*\cap K_z} f(s^{-1/4}z)\nu_1(\mathrm{d}x) - \gamma\int_{K_z} f\big(s^{-1/4}z\big)\mathrm{d}x\big| \ge \varepsilon/4\Big)\\
&\le \mathbb{P}\Big(\sum_{z\in S_{\mathsf{int}}} |s^{-1}\nu_1(G^*\cap Q_{s^{1/4}}(s^{1/4}z)) - \gamma| \ge \varepsilon''/2\Big) + \mathbb{P}\Big(\sum_{z\in S_\partial} |s^{-1/2}\nu_1(s^{-1/2}G^*\cap K_z) - \gamma\nu_2(K_z)| \ge \varepsilon''/2\Big)\\
&\le \sum_{z\in S_{\mathsf{int}}} \mathbb{P}(|s^{-1}\nu_1(G^*\cap Q_{s^{1/4}}(s^{1/4}z)) - \gamma| \ge \varepsilon''/(2n_{\mathsf{int}})) + \sum_{z\in S_\partial} \mathbb{P}\big(|s^{-1/2}\nu_1(s^{-1/2}G^*\cap K_z) - \gamma\nu_2(K_z)| \ge \varepsilon''/(2n_\partial)\big).
\end{aligned}$$

As mentioned above $n_{\mathsf{int}} \le 2b^2\sqrt{s}$, so that Lemma 4 shows stretched exponential decay of the first sum. The second sum can be bounded from above by

$$n_\partial 1_{\gamma s^{-1/2}\ge\varepsilon''/(4n_\partial)} + \sum_{z\in S_\partial} \mathbb{P}\big(s^{-1/2}\nu_1(s^{-1/2}G^*\cap Q_{s^{-1/4}}(s^{-1/4}z)) \ge \varepsilon''/(4n_\partial)\big).$$

In order to analyze this expression, we first note that for every $z \in \mathbb{Z}^2$ with $Q_{s^{-1/4}}(s^{-1/4}z) \cap \partial K \neq \emptyset$ there exists $z' \in \mathbb{Z}^2$ such that $s^{-1/4}z' \notin K$ and $|z - z'|_\infty \leq 1$. Hence, by convexity of K,

$$n_\partial \leq 9s^{1/2}(\nu_2(K \oplus B_{3s^{-1/4}}(o)) - \nu_2(K)) \leq 9s^{1/2}(\nu_2(Q_{b+6s^{-1/4}}(o)) - \nu_2(Q_b(o))),$$

which is at most $9s^{1/2} \cdot 24bs^{-1/4}$, so that $n_\partial s^{-1/2} \leq 216bs^{-1/4}$. In particular, for all sufficiently large $s > 0$, we have $\varepsilon''/(4n_\partial) > 2\gamma s^{-1/2}$, so that $1_{\gamma s^{-1/2} \geq \varepsilon''/(4n_\partial)} = 0$. Finally, we conclude from Lemma 4 that the expression

$$\sum_{z \in S_\partial} \mathbb{P}(s^{-1/2}\nu_1(s^{-1/2}G^* \cap Q_{s^{-1/4}}(s^{-1/4}z)) \geq \varepsilon''/(4n_\partial)) = \sum_{z \in S_\partial} \mathbb{P}(\nu_1(G^* \cap Q_{s^{1/4}}(s^{1/4}z)) \geq \varepsilon'' s/(4n_\partial))$$

tends to 0 as $s \to \infty$, which completes the proof of Lemma 11. □

In order to show that the bivariate vector of scaled random measures $\big(\sqrt{\lambda}H_\lambda^{(1)}, \sqrt{\lambda}H_\lambda^{(2)}\big)$ converges in distribution to $(H^{(1)}, H^{(2)})$, it suffices to verify convergence in distribution of integrals with respect to continuous functions with compact support, see [6, Proposition 11.1.VIII]. For $i \in \{1, 2\}$ and $f : \mathbb{R}^2 \to [0, \infty)$ a continuous function with compact support, we put $Y_\lambda^{(f,i)} = \int_{\mathbb{R}^2} f(x)H_\lambda^{(i)}(\mathrm{d}x)$. Then, it suffices to show that for every such f the random vector $\big(\sqrt{\lambda}Y_\lambda^{(f,1)}, \sqrt{\lambda}Y_\lambda^{(f,2)}\big)$ converges in distribution to $\big(Y^{(f,1)}, Y^{(f,2)}\big)$, where $Y^{(f,i)} = \int_{\Xi_o \cap C_{I_i}} f(x)\nu_2(\mathrm{d}x)$.

Proof of Theorem 2. Let $\delta, \varepsilon > 0$ be arbitrary. Then, for every $x_1, x_2 > 0$ and every sufficiently small $\lambda > 0$,

$$\begin{aligned}\mathbb{P}\big(\sqrt{\lambda}Y^{(f,1)}(\lambda) \leq x_1,\ \sqrt{\lambda}Y^{(f,2)}(\lambda) \leq x_2\big) &\leq \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \Big\{\lambda \int_{T_i \cap \Xi_{o,\lambda} \setminus Q_{\lambda^{-1/4}}(o)} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x) \leq x_i\Big\}\Big) \\ &\leq \mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \Big\{\lambda \int_{G^* \cap \Xi_{o,\lambda} \cap C_{I_i^{\delta,-}} \setminus Q_{\lambda^{-1/4}}(o)} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x) \leq x_i\Big\}\Big) + \varepsilon.\end{aligned}$$

Note that this probability can be bounded from above by

$$\mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \Big\{\lambda \int_{G^* \cap \Xi_{o,\lambda} \cap C_{I_i^{\delta,-}}} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x) \leq x_i + \delta\Big\}\Big) + \mathbb{P}\Big(\lambda \int_{G^* \cap Q_{\lambda^{-1/4}}(o)} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x) \geq \delta\Big).$$

Putting $c = \sup_{x \in \mathbb{R}^2} f(x)$, Lemma 4 implies that the second probability is at most $\mathbb{P}\big(\lambda c \nu_1(G^* \cap Q_{\lambda^{-1/4}}(o)) \geq \delta\big) \leq \varepsilon$ provided that $\lambda > 0$ is sufficiently small. Next, we consider the expression

$$\mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \Big\{\lambda \int_{G^* \cap \Xi_{o,\lambda} \cap C_{I_i^{\delta,-}}} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x) \leq x_i + \delta\Big\}\Big),$$

and note that it can be bounded from above by

$$\begin{aligned}\mathbb{P}\Big(\bigcap_{i\in\{1,2\}} \Big\{\lambda \int_{G^* \cap \lambda^{-1/2}\Xi_o \cap C_{I_i^{\delta,-}}} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x) \leq x_i + 2\delta\Big\}\Big) + \mathbb{P}\Big(\lambda \int_{G^* \cap (\Xi_{o,\lambda} \oplus B_{\delta_1 \lambda^{-1/2}}(o) \setminus \Xi_{o,\lambda})} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x) \geq \delta\Big) \\ + \mathbb{P}\big(\Xi_o \not\subset \sqrt{\lambda}\Xi_{o,\lambda} \oplus B_{\delta_1}(o)\big),\end{aligned}$$

where $\delta_1 > 0$ is arbitrary. Lemma 6 implies that the third summand is smaller than ε for sufficiently small $\lambda > 0$. To bound the second summand, we conclude from Lemmas 5 and 10 that for all sufficiently small $\delta_1 = \delta_1(\delta) > 0$,

$$\mathbb{P}\Big(\lambda \int_{G^* \cap (\Xi_{o,\lambda} \oplus B_{\delta_1 \lambda^{-1/2}}(o) \setminus \Xi_{o,\lambda})} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x) \geq \delta\Big) \leq \mathbb{P}\Big(\lambda c \nu_1\big(\big(G^* \cap (\Xi_{o,\lambda} \oplus B_{\delta_1 \lambda^{-1/2}}(o)) \setminus \Xi_{o,\lambda}\big)\big) \geq \delta\Big) \leq \varepsilon.$$

Finally, we use Lemma 11 and the remark after Lemma 5 to obtain a bound for the first summand:

$$\mathbb{P}\Big(\bigcap_{i\in\{1,2\}}\Big\{\lambda\int_{G^*\cap\lambda^{-1/2}\Xi_o\cap C_{I_i^{\delta,-}}} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x)\le x_i+2\delta\Big\}\Big)\le\mathbb{P}\Big(\bigcap_{i\in\{1,2\}}\Big\{\gamma\int_{\Xi_o\cap C_{I_i^{\delta,-}}} f(x)\mathrm{d}x\le x_i+3\delta\Big\}\Big)+\varepsilon.$$

Thus,

$$\mathbb{P}\big(\sqrt{\lambda}Y_\lambda^{(f,1)}\le x_1,\ \sqrt{\lambda}Y_\lambda^{(f,2)}\le x_2\big)\le\mathbb{P}\Big(\bigcap_{i\in\{1,2\}}\Big\{\gamma\int_{\Xi_o\cap C_{I_i^{\delta,-}}} f(x)\mathrm{d}x\le x_i+3\delta\Big\}\Big)+5\varepsilon,$$

provided that $\lambda>0$ is sufficiently small. Passing to the limits $\varepsilon\to 0$ and $\delta\to 0$ yields

$$\limsup_{\lambda\to 0}\mathbb{P}\big(\sqrt{\lambda}Y_\lambda^{(f,1)}\le x_1,\ \sqrt{\lambda}Y_\lambda^{(f,2)}\le x_2\big)\le\mathbb{P}\Big(\bigcap_{i\in\{1,2\}}\Big\{\gamma\int_{\Xi_o\cap\mathrm{int}\,C_{I_i}} f(x)\mathrm{d}x\le x_i\Big\}\Big).$$

As in the proof of Theorem 1 it remains to complement the above arguments with a suitable lower bound. For the convenience of the reader, we present some details. Again, let $\delta,\varepsilon>0$ be arbitrary. Then, for every $x_1,x_2\ge 0$ and every sufficiently small $\lambda>0$,

$$\begin{aligned}\mathbb{P}\big(\sqrt{\lambda}Y_\lambda^{(f,1)}\le x_1,\ \sqrt{\lambda}Y_\lambda^{(f,2)}\le x_2\big)\ge{}&\mathbb{P}\Big(\bigcap_{i\in\{1,2\}}\Big\{\lambda\int_{G^*\cap C_{I_i^{\delta,+}}\setminus Q_{\lambda^{-1/4}}(o)} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x)\le x_i-\delta\Big\}\Big)\\&-\mathbb{P}\Big(\lambda\int_{G^*\cap Q_{\lambda^{-1/4}}(o)} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x)\ge\delta\Big)-\sum_{i=1}^{2}\mathbb{P}\big(T_i\setminus Q_{\lambda^{-1/4}}(o)\not\subset C_{I_i^{\delta,+}}\big),\end{aligned}$$

and we have already seen that the probabilities in the last line tend to 0 as $\lambda\to 0$. Next, Lemmas 5, 6 and 10 yield

$$\mathbb{P}\Big(\bigcap_{i\in\{1,2\}}\Big\{\lambda\int_{G^*\cap\Xi_{o,\lambda}\cap C_{I_i^{+,\delta}}\setminus Q_{\lambda^{-1/4}}(o)} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x)\le x_i-\delta\Big\}\Big)\ge\mathbb{P}\Big(\bigcap_{i\in\{1,2\}}\Big\{\lambda\int_{G^*\cap\lambda^{-1/2}\Xi_o\cap C_{I_i^{\delta,+}}} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x)\le x_i-2\delta\Big\}\Big)-\varepsilon.$$

Finally, Lemma 11 and the remark after Lemma 5 imply that

$$\mathbb{P}\Big(\bigcap_{i\in\{1,2\}}\Big\{\lambda\int_{G^*\cap\lambda^{-1/2}\Xi_o\cap C_{I_i^{\delta,+}}} f\big(\sqrt{\lambda}x\big)\nu_1(\mathrm{d}x)\le x_i-2\delta\Big\}\Big)\ge\mathbb{P}\Big(\bigcap_{i\in\{1,2\}}\Big\{\gamma\int_{\Xi_o\cap C_{I_i^{\delta,+}}} f(x)\mathrm{d}x\le x_i-3\delta\Big\}\Big)-\varepsilon,$$

provided that $\lambda>0$ is sufficiently small, and we can now conclude as in the first case. □

In general, weak convergence of random measures only yields convergence of integrals for continuous functions with compact support. However, an additional argument shows that in the present setting, we also obtain distributional convergence when integrating the function which is constant and equal to one on the entire Euclidean plane $\mathbb{R}^2$.

Proof of Corollary 3. In order to establish the convergence in distribution of the scaled random vector $\big(\lambda Y_\lambda^{(1)},\lambda Y_\lambda^{(2)}\big)$ as $\lambda\to 0$, we investigate the convergence of the characteristic function. Let $\varepsilon>0$ be arbitrary and let $r>0$ be a large real number whose precise value is fixed below. Furthermore, let $f:\mathbb{R}^2\to[0,1]$ be a continuous function with compact support and such that $f(x)=1$ for all $x\in Q_r(o)$. Then, we consider the following decomposition, where $\mathbf{t}=i(t_1,t_2)$, where $t_1,t_2>0$ and i is a complex number satisfying $i^2=-1$.

$$\begin{aligned}\Big|\mathbb{E}\exp\big(\mathbf{t}\big(\lambda Y_\lambda^{(1)},\lambda Y_\lambda^{(2)}\big)^\top\big)-\exp\big(\mathbf{t}\big(Y^{(1)},Y^{(2)}\big)^\top\big)\Big|\le{}&\Big|\mathbb{E}\exp\big(\mathbf{t}\big(\sqrt{\lambda}Y_\lambda^{(f,1)},\ \sqrt{\lambda}Y_\lambda^{(f,2)}\big)^\top\big)-\exp\big(\mathbf{t}\big(Y^{(f,1)},Y^{(f,2)}\big)^\top\big)\Big|\\&+\Big|\mathbb{E}\exp\big(\mathbf{t}\big(\sqrt{\lambda}Y_\lambda^{(f,1)},\ \sqrt{\lambda}Y_\lambda^{(f,2)}\big)^\top\big)-\exp\big(\mathbf{t}\big(\lambda Y_\lambda^{(1)},\lambda Y_\lambda^{(2)}\big)^\top\big)\Big|\\&+\Big|\mathbb{E}\exp\big(\mathbf{t}\big(Y^{(f,1)},Y^{(f,2)}\big)^\top\big)-\exp\big(\mathbf{t}\big(Y^{(1)},Y^{(2)}\big)^\top\big)\Big|.\end{aligned}$$

As $\lambda \to 0$, the first expression converges to 0 by Theorem 2. The second and third expressions are bounded from above by $2\mathbb{P}\big(\sqrt{\lambda}\Xi_{o,\lambda} \not\subset Q_r(o)\big)$ and $2\mathbb{P}\big(\Xi_o \not\subset Q_r(o)\big)$, respectively. Finally, an application of Lemma 5 shows that the latter two are bounded from above by ε, provided that $r > 0$ is sufficiently large. $\square$

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