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# MODERATE DEVIATIONS FOR SHORTEST-PATH LENGTHS ON RANDOM SEGMENT PROCESSES

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**Abstract**. We consider first-passage percolation on segment processes and provide concentration results concerning moderate deviations of shortest-path lengths from a linear function in the distance of their endpoints. The proofs are based on a martingale technique developed by H. Kesten for an analogous problem on the lattice. Our results are applicable to graph models from stochastic geometry. For example, they imply that the time constant in Poisson-Voronoi and Poisson-Delaunay tessellations is strictly greater than 1. Furthermore, applying the framework of Howard and Newman, our results can be used to study the geometry of geodesics in planar shortest-path trees.

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## 1. Introduction

In recent years, several results from classical first-passage percolation on lattices have been generalized to models involving random segment processes embedded in the $d$-dimensional Euclidean space $\mathbb{R}^d$ for some $d \geq 2$. For instance, consider the following scenario. Starting from a stationary point process $X$ in $\mathbb{R}^d$, construct a random segment process $G$ with endpoints of segments given by $X$ and assign i.i.d. passage times to the segments. This framework is considered e.g. in [25, 27]. In contrast to its lattice analogon, first-passage percolation on random segment processes allows to investigate a further natural model, where the segment-passage times are given by the Euclidean lengths of the corresponding segments, see e.g. [1, 15, 16].

Apart from their inherent mathematical appeal, Euclidean first-passage models play an important role in fixed-access telecommunication networks. For instance, for the so-called *stochastic subscriber line model* in $\mathbb{R}^2$, we showed in [14] how a Euclidean analogon of Kesten's shape theorem may be used to derive an asymptotic formula for the distribution of the longest shortest-path length of the segment system in the typical serving zone of the network. For further background information on this model, the reader may consult the introductory article [11].

In the present paper, building on the foundational work of [16, 18], we consider concentration results concerning moderate deviations of a) shortest-path lengths from a linear function in the Euclidean distance of their endpoints (Theorem 1) and b) geodesics from the line segment connecting their endpoints (Theorem 2) for a class of random segment processes including Voronoi and Delaunay tessellations, as well as the relative

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neighborhood graph generated from suitable stationary Poisson-type point processes. In order to apply the martingale approach of [18] to first-passage percolation models on graphs with random topology and independent edge weights, a geometric regularization technique has been developed in [25]. In the present paper, we extend this technique to the scenario of weights given by the actual Euclidean distance. Using different methods, these concentration results are also verified for the isotropic Poisson line tessellation in $\mathbb{R}^2$.

In order to illustrate the strength of our main results, we present two corollaries. First, in Theorem 26, we explain how the concentration property of shortest-path lengths can be used to show that the so-called time constants in Poisson-Voronoi and Poisson-Delaunay tessellations are strictly greater than 1. In other words, the ratio of the shortest-path length and the Euclidean distance of the endpoints tends to a value strictly greater than 1 as the Euclidean distance of the endpoints tends to infinity. So far, there has mainly been empirical evidence for this property, see e.g. [28] for the planar case. Second, by the general framework developed in [16], our main result has important implications for the geometry of planar shortest-path trees, such as the existence of competition interfaces. In [13], we illustrate how our results can be applied to fixed-access telecommunication networks. To be more precise, the concentration results of the present paper are used to describe the asymptotic bivariate distribution of the lengths of the two main branches inside the typical serving zone of the network, a quantity that has already been analyzed empirically in [22].

The rest of this paper is organized as follows. First, in Section 2, we state our main results for shortest-path lengths in Delaunay, Voronoi and relative neighborhood graphs on suitable stationary point processes. Specific examples of point processes that are covered by our framework are given in 3. Then, in Section 4, we explain how a martingale approach is used to obtain the aforementioned concentration results for a general class of connected random segment processes. In Section 5, we verify that this class includes Voronoi and Delaunay tessellations, as well as the relative neighborhood graph on suitable stationary Poisson-type point processes; we also show directly that the isotropic Poisson line tessellation in $\mathbb{R}^2$ exhibits the same concentration property. In Section 6, we show how concentration properties for geodesics can be deduced from concentration properties of shortest-path lengths, and we derive lower bounds on the time constant in Voronoi and Delaunay tessellations. Finally, in Section 7, we restrict to planar case and investigate competition interfaces defined by Euclidean first-passage models.

## 2. Main results

### 2.1. Random segment processes based on point processes

In the present paper, we consider random segment processes that are constructed from point processes by a deterministic rule. Note that this includes many classical examples from stochastic geometry. In particular, the edge set of the Delaunay tessellation and the relative neighborhood graph form segment processes in $\mathbb{R}^d$ that are constructed by applying a deterministic connection rule to a given set of vertices. Similarly, a deterministic rule is used to construct the edge set of the Voronoi tessellation from a given point process of cell centers. In our main result, we show that when constructing these random segment processes from a class of suitable stationary Poisson-type point processes, then the shortest-path lengths exhibit concentration with respect to moderate deviations from a linear function in the Euclidean distance of their endpoints.

Assuming that $d \geq 2$, let $\mathbb{N}$ denote the family of all locally finite sets in $\mathbb{R}^d$ endowed with the $\sigma$-algebra $\mathcal{N}$ such that the function which maps $\varphi$ to $\#(\varphi \cap B)$, the cardinality of the set $\varphi \cap B$, is measurable for each Borel set $B \subset \mathbb{R}^d$. Let $Q_r(x) = x + [-r/2, r/2]^d$ be the cube of side length $r > 0$ centered at $x \in \mathbb{R}^d$. Moreover, let $\mathbb{M}$ be a Polish space. We consider the family $\mathbb{N}_{\mathbb{M}}$ of $\mathbb{M}$-marked locally finite sets in $\mathbb{R}^d$ and write $\mathcal{N}_{\mathbb{M}}$ for the $\sigma$-algebra of subsets of $\mathbb{N}_{\mathbb{M}}$ that is generated by the evaluation maps $\varphi \to \#(\varphi \cap (B \times M))$, where $B$ and $M$ are Borel subsets of $\mathbb{R}^d$ and $\mathbb{M}$, respectively. The products $\mathbb{R}^d \times \mathbb{M}$ and $Q_r(x) \times \mathbb{M}$ of $\mathbb{R}^d$ and $Q_r(x)$ with the mark space $\mathbb{M}$ are denoted by $\mathbb{R}^{d,\mathbb{M}}$ and $Q_r^{\mathbb{M}}(x)$, respectively. Additionally, we assume that every rotation $\zeta \in \mathsf{SO}_d$ of $\mathbb{R}^d$ corresponds to a measurable map $\zeta : \mathbb{M} \to \mathbb{M}$ of the mark space onto itself.

In this paper, we always assume that $X$ forms an independently $\mathbb{M}$-marked homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda > 0$, which is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We also assume that

$X$ is *isotropically marked*, i.e., the distribution of the marks is invariant with respect to rotations. Furthermore, for $x \in \mathbb{R}^d$, $\zeta \in \mathsf{SO}_d$ and $\varphi \in \mathbb{N}_{\mathbb{M}}$ we put $\varphi + x = \{(y+x, m) : (y, m) \in \varphi\}$ and $\zeta(\varphi) = \{(\zeta(y), \zeta(m)) : (y, m) \in \varphi\}$.

In the following, we consider point processes of the form $f_{\mathsf{geom}}(X)$, where $f_{\mathsf{geom}} : \mathbb{N}_{\mathbb{M}} \to \mathbb{N}$ is motion-covariant, i.e., for every $x \in \mathbb{R}^d$, $\zeta \in \mathsf{SO}_d$ and $\varphi \in \mathbb{N}_{\mathbb{M}}$ we have $f_{\mathsf{geom}}(\varphi + x) = f_{\mathsf{geom}}(\varphi) + x$ and $f_{\mathsf{geom}}(\zeta(\varphi)) = \zeta(f_{\mathsf{geom}}(\varphi))$. We also say that $f_{\mathsf{geom}}(\varphi)$ defines the *geometric realization* of $\varphi$. For instance, Poisson-cluster processes [7] can be represented as geometric realizations of independently marked homogeneous Poisson point processes. Indeed, we begin with a (primary) point process $X_{\mathsf{pr}}$, which in our case forms a homogeneous Poisson point process in $\mathbb{R}^d$, and as marks we attach i.i.d. (secondary) point processes to the points of $X_{\mathsf{pr}}$. In this setting, $f_{\mathsf{geom}}$ constructs the Poisson-cluster process from $X$ by forming the union of all secondary point processes shifted by their corresponding primary points. Besides Poisson-cluster processes, many Matérn-type hard-core models and modulated Poisson point processes [7] can also be generated from independently marked homogeneous Poisson point processes by applying a deterministic, motion-covariant construction rule. In Section 3, we provide a more detailed discussion of these examples.

Additionally, we require that $f_{\mathsf{geom}}$ exhibits further useful properties, which intuitively speaking, can be described as follows. First, $f_{\mathsf{geom}}$ should be capable of creating lattice configurations. Second, it should satisfy a sub-additivity property in the sense that the geometric realization of the union of two locally finite sets should be contained in the union of their geometric realizations. Third, the geometric realization inside a bounded sampling window with non-empty interior should not depend on the configuration of $X$ far away from the window. We also need appropriate growth conditions allowing us to control the total number of points of $f_{\mathsf{geom}}(X)$ in rectangular sampling windows. The total number of points in a bounded sampling window with non-empty interior should be positive with positive probability and should also admit a finite exponential moment. Finally, the second factorial moment density should be bounded.

To be more precise, we make the following assumptions, where $o \in \mathbb{R}^d$ denotes the origin and for $B_1, B_2 \subset \mathbb{R}^d$ we put $B_1 \ominus B_2 = \{x \in \mathbb{R}^d : x + b_2 \in B_1 \text{ for all } b_2 \in B_2\}$. We suppose that there exists $h_0 > 0$, $r_0, \tau > 0$ and $\varphi^{(0)} \in \mathbb{N}_{\mathbb{M}}$ such that

- (F1) $f_{\mathsf{geom}}(\emptyset) = \emptyset$ and $f_{\mathsf{geom}}(\varphi^{(0)}) = r_0\mathbb{Z}^d$ (lattice condition),
- (F2) $f_{\mathsf{geom}}(\varphi \cup \psi) \subset f_{\mathsf{geom}}(\varphi) \cup f_{\mathsf{geom}}(\psi)$ for all $\varphi, \psi \in \mathbb{N}_{\mathbb{M}}$ (sub-additivity condition),
- (F3) $f_{\mathsf{geom}}(\varphi \cup \psi) \cap \big(B \ominus Q_\tau(o)\big) = f_{\mathsf{geom}}(\varphi) \cap \big(B \ominus Q_\tau(o)\big)$ for all $\varphi, \psi \in \mathbb{N}_{\mathbb{M}}$ and bounded Borel sets $B \subset \mathbb{R}^d$ such that $\varphi \subset B \times \mathbb{M}$ and $\psi \subset \mathbb{R}^{d,\mathbb{M}} \setminus (B \times \mathbb{M})$ (stability condition),
- (F4) $\mathbb{P}(f_{\mathsf{geom}}(X) = \emptyset) = 0$ (non-emptiness condition),
- (F5) $\mathbb{E} \exp\big(h_0 \# f_{\mathsf{geom}}(X \cap Q^{\mathbb{M}}_\tau(o))\big) < \infty$ (exponential moment condition),
- (F6) the second factorial moment measure of $f_{\mathsf{geom}}(X)$ has a bounded density with respect to the $2d$-dimensional Lebesgue measure (second moment condition).

In order to have a specific example satisfying these abstract conditions, the reader may think of the special case where $X$ is an unmarked homogeneous Poisson point process, and $f : \mathbb{N} \to \mathbb{N}$ is the identity. Then, conditions (F1)–(F4), (F6) are obviously satisfied and condition (F5) is a consequence of the existence of exponential moments for Poisson random variables. Further examples of stationary point processes covered by our framework are presented in Section 3.

We now consider marked locally finite sets in $\mathbb{R}^d$, where the marks are line segments. Conceptually, this marking is different from the one of the marked Poisson point process $X$. Let $\mathbb{L}$ denote the family of all line segments in $\mathbb{R}^d$. This family forms a Polish space in the Fell topology [26]. We write $\mathbb{G}$ for the family of all locally finite sets on $\mathbb{R}^d \times \mathbb{L}$ such that $\#(\varphi \cap (B \times \mathbb{L}))$ is finite for every bounded Borel set $B \subset \mathbb{R}^d$. Furthermore, let $\mathcal{G}$ denote the $\sigma$-algebra on $\mathbb{G}$ that is generated by the evaluation maps $\varphi \mapsto \#(\varphi \cap (B \times L))$, where $B$ and $L$ are Borel sets in $\mathbb{R}^d$ and $\mathbb{L}$, respectively. Random variables with values in $\mathbb{G}$ are called *random segment processes*. It will be convenient to think of an element $\{(x_n, u_n)\}_{n \ge 1}$ of $\mathbb{G}$ as the subset of $\mathbb{R}^d$ formed by the union $\cup_{n \ge 1}(u_n + x_n)$. In the following, we consider random segment processes in $\mathbb{R}^d$ of the form $G = g(f_{\mathsf{geom}}(X))$, where $g : \mathbb{N} \to \mathbb{G}$ is a suitable measurable mapping. For instance, in the Poisson cluster setting, $f_{\mathsf{geom}}(X)$ would first construct the Poisson cluster process $f_{\mathsf{geom}}(X)$ from the marked Poisson point process $X$. The random segment process $G$ is then obtained by applying $g$ to the unmarked point process $f_{\mathsf{geom}}(X)$. This framework includes many of the classical random segment processes considered in stochastic geometry.

For instance, for any locally finite $\varphi \subset \mathbb{R}^d$, we let $\mathsf{Rng}(\varphi)$ and $\mathsf{Del}(\varphi)$ denote the relative neighborhood graph and the Delaunay graph on the vertex set $\varphi$, respectively. Here, two vertices $x, y \in \varphi$ are connected by a segment in $\mathsf{Rng}(\varphi)$ if there does not exist a vertex $z \in \varphi$ such that $\max\{|x-z|, |z-y|\} < |x-y|$. The vertices $x, y \in \varphi$ are connected by a segment in $\mathsf{Del}(\varphi)$ if and only if there exists a ball $B \subset \mathbb{R}^d$ such that $x, y \in B$ and $\#(\varphi \cap B) \le d+1$. For further properties of relative neighborhood and Delaunay graphs, the reader is referred to [1, 26], respectively. We stress that in general, $\varphi$ need not coincide with the set of segment endpoints. For instance, for $x \in \varphi$ consider the Voronoi cell $Z(x, \varphi) = \{y \in \mathbb{R}^d : |y-x| \le \inf_{x' \in \varphi} |y - x'|\}$ with cell center $x$. It is shown in [26] that if the convex hull of $\varphi$ equals $\mathbb{R}^d$, then the family $\{Z(x,\varphi)\}_{x \in \varphi}$ consists of bounded, convex polytopes. We let $\mathsf{Vor}(\varphi)$ denote the union of edge sets of these polytopes.

For the remainder of this section, we assume that $g = \mathsf{Rng}$, $g = \mathsf{Del}$ or $g = \mathsf{Vor}$. One of the main results of the present paper deals with concentration properties of shortest-path lengths along the edges of $G$. To be more precise, for $x \in \mathbb{R}^d$ we write $q(x) = q(x, G) = \mathrm{argmin}_{y \in G}|x - y|$ for the point on the random segment process $G$ which has the smallest distance to $x$; if this choice is not unique we pick the lexicographically smallest point on $G$ with this property. For $x, y \in \mathbb{R}^d$, let $\ell(x, y)$ denote the length of the shortest Euclidean path from $q(x)$ to $q(y)$ on $G$ and put $\ell_r = \ell(o, re_1)$, where $e_1 = (1, 0, \dots, 0)$ and $r > 1$. Note that we put $\ell_r = \infty$ if $q(o)$ and $q(re_1)$ belong to different connected components of $G$, an event which occurs with probability 0. This is not entirely obvious, but at the end of Section 4.1, we note that $\ell_r$ has even stretched exponential tails.

We say that a family of events $\{A_r\}_{r>1}$ in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ occurs *with high probability (short: whp)* if

$$\liminf_{r \to \infty} \frac{\log(-\log(1 - \mathbb{P}(A_r)))}{\log r} > 0. \tag{1}$$

Note that a family of events $\{A_r\}_{r>1}$ occurs whp if and only if there exist constants $c_1, c_2 > 0$ such that $1 - \mathbb{P}(A_r) \le c_1 \exp(-r^{c_2})$ for all $r > 1$.

**Theorem 1.** *Let $\beta > 1/2$ be an arbitrary fixed number, assume that $g = \mathsf{Del}$, $g = \mathsf{Rng}$ or $g = \mathsf{Vor}$, and that conditions* (F1)–(F6) *are satisfied. Then, for $r > 1$ the events*

$$|\ell_r - \mu r| \le r^\beta \tag{2}$$

*occur whp, where $\mu = \inf_{n \ge 1} \mathbb{E}\ell_n / n$ is a finite number.*

We expect that for Poisson-Delaunay and Poisson-Voronoi tessellations our first-passage model should be in the same universality class as the lattice analogue, so that for $d = 2$ we conjecture that the precise fluctuation exponent should be given by $1/3$ rather than $1/2$. An important consequence of Theorem 1 is that for both Delaunay and Voronoi tessellations constructed on suitable stationary Poisson-type point processes, the time constant $\mu$ is strictly greater than 1, see Theorem 26. So far, there has mainly been empirical evidence for this property which, moreover, was restricted to the two-dimensional homogeneous Poisson point process, see [28]. Very recently a lower bound has been established by different methods for the specific example of the planar Poisson-Delaunay tessellation [6].

### 2.2. A general class of random segment processes

In addition to the random segment processes discussed in Section 2.1, the concentration property (2) is also satisfied with high probability for the Poisson line tessellation in $\mathbb{R}^2$, see Section 5.1. Therefore, in the present section, we assume that $G$ denotes an arbitrary stationary, ergodic and isotropic random segment process in $\mathbb{R}^d$ for which (2) holds whp and for which the events stated below in (G1) and (G2) occur whp. In the following, $\nu_1(\cdot)$ denotes the one-dimensional Hausdorff measure in $\mathbb{R}^d$. Then, as in [14], for $r > 1$ we consider the events

(G1) $\{\nu_1(G \cap Q_r(o)) > 0\} \cap \{\nu_1(G \cap Q_1(o)) \le r\}$ (growth condition), (G2) $\{G \cap Q_{r/2}(o)$ is contained in a connected component of $G \cap Q_r(o)\}$ (connectivity condition),

and assume that both the events in (G1) and in (G2) occur whp. In Section 6.2, we elaborate on how the concentration property (2), which deals with shortest-path lengths, implies a concentration property for deviations of shortest paths from the line segment connecting their endpoints. See Figure 1 for an illustration, where for $a \in \mathbb{R}^d$ and $B \subset \mathbb{R}^d$ we put $\mathsf{dist}(a, B) = \inf_{b \in B} |a - b|$. In order to state this result precisely, for $x, y \in G$ let $R(x, y)$ denote the family of all paths $\gamma$ in $G$ from $x$ to $y$ satisfying $\nu_1(\gamma) = \ell(x, y)$. The elements of $R(x, y)$ are also called *geodesics* between $x$ and $y$.

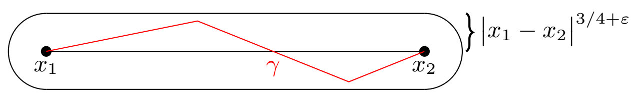



FIGURE 1. Every element $\gamma \in R(x_1, x_2)$ (red) is contained in the tube $\{y \in \mathbb{R}^d : \mathsf{dist}(y, [x_1, x_2]) \le |x_1 - x_2|^{3/4+\varepsilon}\}$

**Theorem 2.** *Let $\varepsilon > 0$ be arbitrary. Then, there exists a $(1, \infty)$-valued random variable $V_0$ such that almost surely,* $\sup_{\substack{\gamma \in R(x_1, x_2) \\ y \in \gamma}} \mathsf{dist}(y, [x_1, x_2]) \le |x_1 - x_2|^{3/4+\varepsilon}$ *for all $x_1 \in G \cap Q_1(o)$ and all $x_2 \in G$ with $|x_2| \ge V_0$.*

Apart from generalizing known results from classical lattice first-passage percolation (see, e.g. [23]) to models involving random segment processes, Theorem 2 also has important consequences for the geometry of so-called *shortest-path trees*, such as the existence of competition interfaces in dimension $d = 2$, see Section 7 for details.

## 3. Examples

In this section, we present examples of Poisson-based cluster, Poisson-based hard-core and modulated Poisson point processes satisfying conditions (F1)–(F6).

### 3.1. Poisson cluster processes

To begin with, we consider the Poisson cluster model. For $\tau > 0$ let $\mathbb{N}_\tau$ denote the subfamily of $\mathbb{N}$ formed by all locally finite sets $\varphi \subset \mathbb{R}^d$ satisfying $\varphi \subset B_{\tau/2}(o)$. Note that every rotation $\zeta \in \mathsf{SO}_d$ induces a canonical map $\zeta : \mathbb{N}_\tau \to \mathbb{N}_\tau$. Furthermore, we define the motion-covariant geometric realization $f_{\mathsf{geom}} : \mathbb{N}_{\mathbb{N}_\tau} \to \mathbb{N}$ by $\varphi \mapsto \bigcup_{(x,\psi) \in \varphi} (\psi + x)$. Then $f_{\mathsf{geom}}$ obviously satisfies (F1)–(F3). To show (F4) assume that the primary point process forms a homogeneous Poisson point process with intensity $\lambda > 0$. Let $Y$ denote the typical secondary process of the cluster process $X$, where we assume that $Y$ is isotropic. Clearly, for $X$ to satisfy (F4), it is necessary and sufficient that the second-order process $Y$ is non-empty with positive probability. Next, we verify condition (F5), imposing the existence of some exponential moment of $\#Y$ as an additional assumption.

**Lemma 3.** *Assume that there exists $h > 0$ such that $\log \mathbb{E} \exp(h\#Y) < \infty$. Then $X$ satisfies* (F5).

*Proof.* Indeed, putting $c_1 = \log \mathbb{E} \exp(h\#Y)$,

$$\mathbb{E} \exp(h\# f_{\mathsf{geom}}(X \cap Q_\tau^{\mathbb{M}}(o))) = \mathbb{E} \prod_{(x,\psi) \in X \cap Q_\tau^{\mathbb{M}}(o)} \exp(h\#\psi) = \mathbb{E} \exp(c_1 \#(X \cap Q_\tau^{\mathbb{M}}(o))) = \exp\big(\lambda \tau^d (\exp(c_1) - 1)\big). \quad \square$$

Finally, we note that if the second factorial moment measure of $Y$ has a bounded density with respect to the $2d$-dimensional Lebesgue measure, then the same is true for $f_{\mathsf{geom}}(X)$, see [17, Section 6.2.2].

### 3.2. Matérn-type hard-core point processes

Next, we consider Matérn-type hard-core point processes. In this example, the mark space $\mathbb{M}$ is chosen as the product $\mathbb{M}_\tau = [0,1] \times \mathbb{C}_\tau$, where $\mathbb{C}_\tau$ denotes the family of compact subsets of $B_{\tau/2}(o)$. Any $\zeta \in \mathsf{SO}_d$ defines a measurable map $\mathbb{M}_\tau \to \mathbb{M}_\tau$ by leaving the first component invariant and acting on the second in the canonical way. For $\varphi \in \mathbb{N}_{\mathbb{M}_\tau}$ the geometric realization $f_{\mathsf{geom}}(\varphi)$ consists of all $x \in \mathbb{R}^d$ for which there exists $(x, u, m) \in \varphi \subset \mathbb{R}^d \times [0,1] \times \mathbb{C}$ such that $u > v$ for all $(y, v, n) \in \varphi$ with $(x + m) \cap \big(y + n\big) \neq \emptyset$. As before, $f_{\mathsf{geom}}$ clearly satisfies (F1)–(F4). Moreover, since $f_{\mathsf{geom}}(X)$ is a thinning of the Poisson point process with intensity $\lambda$, conditions (F5) and (F6) are also obvious.

### 3.3. Modulated Poisson point processes

Finally, we discuss modulated Poisson point processes. Here, the mark space $\mathbb{M}$ is chosen as $\mathbb{M}_\tau^{\mathsf{mod}} = \mathbb{M}_\tau^{(0)} \cup \mathbb{M}_\tau^{(1)}$, where $\mathbb{M}_\tau^{(i)} = \{i\} \times [0,1] \times \mathbb{C}_\tau$ for any $i \in \{0,1\}$. A mark from $\mathbb{M}_\tau^{(0)}$ is interpreted as a germ in the germ-grain model used to define the modulated Poisson point process, whereas points of the dominating point process from which the modulated Poisson point process is obtained as thinning are endowed with a mark from $\mathbb{M}_\tau^{(1)}$. We also fix $\lambda_1, \lambda_2 \geq 0$ as the intensities in the two phases of the modulated Poisson point process. In order to establish sub-additivity easily, we assume that $\lambda_2 \geq \lambda_1$. Any $\zeta \in \mathsf{SO}_d$ defines a measurable map $\mathbb{M}_\tau^{\mathsf{mod}} \to \mathbb{M}_\tau^{\mathsf{mod}}$ by leaving the first two components invariant and acting canonically on the last component. To define the geometric realization we make use of the observation that a modulated Poisson point process can be regarded as a suitable thinning of a Poisson point process. The geometric realization $f_{\mathsf{geom}} : \mathbb{N}_{\mathbb{M}_\tau^{\mathsf{mod}}} \to \mathbb{N}$ is defined by $\varphi \mapsto S_1(\varphi) \cup S_2(\varphi)$, where

$$S_1(\varphi) = \big\{x \in \cup_{(y,0,v,n) \in \varphi}(y + n) : (x, 1, u, m) \in \varphi \text{ for some } (u, m) \in [0, \lambda_1/\lambda_2] \times \mathbb{C}_\tau\big\},$$
$$S_2(\varphi) = \big\{x \in \mathbb{R}^d \setminus \cup_{(y,0,v,n) \in \varphi}(y + n) : (x, 1, u, m) \in \varphi \text{ for some } (u, m) \in [0,1] \times \mathbb{C}_\tau\big\}.$$

Then, properties (F1)–(F3) are immediate. Next, we introduce our precise assumptions on the random input for the geometric realization. Note that any independently $\mathbb{M}_\tau^{\mathsf{mod}}$-marked Poisson point process $X$ can be decomposed as an independent superposition $X = X^{(0)} \cup X^{(1)}$, where $X^{(i)} = X \cap \mathbb{M}_\tau^{(i)}$, $i \in \{0,1\}$. Furthermore, the last two components of the typical marks of both $X^{(0)}$ and $X^{(1)}$ are assumed to be of the form $Y = (U, M)$, where $U, M$ are independent, $U$ is uniform on $[0,1]$ and $M$ forms an isotropic random element of $\mathbb{C}_\tau$. Then, (F4) is obvious provided that $\lambda_2 > 0$. Finally, as in the case of Matérn-type hard-core point processes, $f_{\mathsf{geom}}(X)$ can be considered as thinning of a homogeneous Poisson point process, so that (F5) and (F6) follow immediately.

## 4. Moderate deviations of shortest-path lengths

To prove Theorem 1, we follow the approach of [25], where first-passage percolation using i.i.d. weights on the edges of the planar Poisson-Delaunay tessellation is considered. The key idea of this approach is based on a martingale concentration technique appearing originally in [18]. In the proof of Theorem 1 we have to overcome several difficulties that do not appear in the i.i.d. Delaunay setting. Since we consider Euclidean first-passage percolation, the edge weights are no longer independent of the underlying random segment process. This gives rise to more complex dependencies that have to be dealt with accordingly. Additionally, the Delaunay tessellation is a random segment process with a very high degree of connectivity, whereas our techniques also apply to sparser random segment processes such as the relative neighborhood graph.

We provide a rough outline of the proof, before going into the technical details. First, using conditions (H3) and (H5) stated below, it can be seen that it suffices to prove the assertion for the case where $r = n$ is an integer. Next, in Section 4.1, we introduce a general class of random segment processes that encompasses Delaunay tessellations, Voronoi tessellations as well as relative neighborhood graphs, and for which Theorem 1 is shown. In Section 4.2, we explain how to apply [18, Theorem 3] in order to obtain a concentration result for moderate deviations of $\big|\overline{\ell}_n - \mathbb{E}\overline{\ell}_n\big|$, where $\overline{\ell}_n$ describes the length of the shortest path from $q(o)$ to $q(ne_1)$, when the considered random segment process is constructed from a regularized variant of $f_{\mathsf{geom}}(X)$. This regularization is discussed in detail in Section 4.2 (see also Figure 2 below). In Section 4.3, we prove that $\overline{\ell}_n$ is sufficiently close to $\ell_n$, so that a concentration result for $|\ell_n - \mathbb{E}\ell_n|$ follows from the corresponding concentration result for $|\overline{\ell}_n - \mathbb{E}\overline{\ell}_n|$. Finally, in Section 4.4 it is shown that in the expression $|\ell_n - \mathbb{E}\ell_n|$ it is possible to replace $\mathbb{E}\ell_n$ by $\mu n$.

## 4.1. **A general class of random segment processes**

In order to prove Theorem 1 for Voronoi tessellations, Delaunay tessellations and relative neighborhood graphs, it is convenient to introduce a special class of random segment processes that encompasses all three models. To be more precise, we consider random segment processes of the form $G = g_{\mathsf{geom}}(X)$, where $g_{\mathsf{geom}}(X) = g(f_{\mathsf{geom}}(X))$ and $g : \mathbb{N} \to \mathbb{G}$ is a measurable and motion-covariant function. Similar to [14], we need to impose additional constraints on the random segment process $G = g_{\mathsf{geom}}(X)$, which are described in the following.

First, the random segment process $G$ should satisfy two suitable stability conditions with respect to $X$. On the one hand, the configuration of $X$ far away from a bounded sampling window $W \subset \mathbb{R}^d$ does not influence the configuration of the random segment process inside the set $W$. Conversely, changing $X$ inside $W$ does not affect the random segment process far away from the set $W$. Furthermore, we require a strong connectivity condition. In an appropriately chosen environment of the sampling window $W$, any two points on $G \cap W$ can be connected by a path on $G$. Finally, we also need two growth conditions in order to bound the total length of the random segment process $G$ inside cubic sampling windows. On the one hand, the segment process should be sufficiently dense so that the distance from any point in the window to the graph $G$ is not too large. On the other hand, the edge set of $G$ in big windows should still not be too long with high probability.

In order to deal with the relative neighborhood graph, it is more convenient to impose the conditions that we have just described for an auxiliary family of measurable and motion-covariant construction rules $\{\overline{g}_r\}_{r>1}$, $\overline{g}_r : \mathbb{N} \to \mathbb{G}$, which are well-behaved by definition, but still can approximate the original construction rule $g$ arbitrarily closely. This approximation step is not needed for the Delaunay and Voronoi graphs, so that readers mainly interested in these two examples may simply replace $\overline{g}_r$ by $g$ in the following.

For $s > 0$ and $x \in \mathbb{R}^d$ we write $B_s(x) = \{y \in \mathbb{R}^d : |y - x| \le s\}$ for the ball in $\mathbb{R}^d$ with center $x \in \mathbb{R}^d$ and radius $s > 0$. In the general setting, we assume the existence of a family of measurable and translation-covariant construction rules $\{\overline{g}_r\}_{r>1}$, $\overline{g}_r : \mathbb{N} \to \mathbb{G}$ with $g_{\mathsf{geom}}(\varphi) \subset \overline{g}_{r,\,\mathsf{geom}}(\varphi) \overset{\mathsf{def}}{=} \overline{g}_r\big(f_{\mathsf{geom}}(\varphi)\big)$ for all $r > 1$ and $\varphi \in \mathbb{N}_{\mathbb{M}}$ and such that for $r > 1$ the events $g_{\mathsf{geom}}(X) \cap Q_r(o) = \overline{g}_{r,\,\mathsf{geom}}(X) \cap Q_r(o)$ occur whp. Additionally, assume the existence of $\alpha_1, \alpha_2 \ge d$ and a family of events $\{A_r\}_{r>1}$ in $\mathbb{N}_{\mathbb{M}}$ with the following properties

- (A) $(\varphi^{(0)} - rz) \cap Q_r^{\mathbb{M}}(o) \in A_r$ for all sufficiently large $r > 1$ and all $z \in \mathbb{Z}^d$,
- (B) if $\varphi \in \mathbb{N}_{\mathbb{M}}$ is such that $(\varphi - rz) \cap Q_r^{\mathbb{M}}(o) \in A_r$ for all $z \in \mathbb{Z}^d$, then the following properties hold:
  - (H1) $\overline{g}_{r,\,\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o)) \cap Q_r(o) = \overline{g}_{r,\,\mathsf{geom}}\big((\varphi \cap Q_{3r}^{\mathbb{M}}(o)) \cup \psi\big) \cap Q_r(o)$ for all $\psi \in \mathbb{N}_{\mathbb{M}}$ with $\psi \subset \mathbb{R}^{d,\mathbb{M}} \setminus Q_{3r}^{\mathbb{M}}(o)$ (stability condition I),
  - (H1′) $g_{\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o)) \cap Q_r(o) = g_{\mathsf{geom}}\big((\varphi \cap Q_{3r}^{\mathbb{M}}(o)) \cup \psi\big) \cap Q_r(o)$ for all $\psi \in \mathbb{N}_{\mathbb{M}}$ with $\psi \subset \mathbb{R}^{d,\mathbb{M}} \setminus Q_{3r}^{\mathbb{M}}(o)$ (stability condition I′),
  - (H2) $\overline{g}_{r,\,\mathsf{geom}}(\varphi) \setminus Q_{3r}(o) = \overline{g}_{r,\,\mathsf{geom}}\big((\varphi \setminus Q_r^{\mathbb{M}}(o)) \cup \psi\big) \setminus Q_{3r}(o)$ for all $\psi \in \mathbb{N}_{\mathbb{M}}$ with $\psi \subset Q_r^{\mathbb{M}}(o)$ (stability condition II),
  - (H3) $\overline{g}_{r,\,\mathsf{geom}}(\varphi) \cap Q_{5r}(o)$ is contained in a connected component of $\overline{g}_{r,\,\mathsf{geom}}(\varphi) \cap Q_{7r}(o)$ (connectivity condition),
  - (H4′) $q(x, g_{\mathsf{geom}}(\varphi)) \in B_{r/2}(x)$ for all $x \in Q_r(o)$ (growth condition I′),
  - (H5) $\nu_1\big(\overline{g}_{r,\,\mathsf{geom}}(\varphi) \cap Q_{7r}(o)\big) \le r^{\alpha_1}$ (growth condition II),
  - (H6) $q(o, \overline{g}_{r,\,\mathsf{geom}}(\varphi))$ and $q(re_1, \overline{g}_{r,\,\mathsf{geom}}(\varphi))$ can be connected by a path in $\overline{g}_{r,\,\mathsf{geom}}(\varphi)$ of length at most $\alpha_2 r$ (path-length condition).

We assume that for $r > 1$ the events $X \cap Q_r^{\mathbb{M}}(o) \in A_r$ occur whp.

Note that since $g_{\mathsf{geom}}(\varphi) \subset \overline{g}_{r,\,\mathsf{geom}}(\varphi)$ the introduction of a growth condition (H4) with respect to $\overline{g}_{r,\,\mathsf{geom}}$ corresponding to (H4′) would be redundant. Finally, we also assume a linear growth bound for $\ell_n$ in the sense that for $n > 1$ the events $\ell_n \geq n/2$ occur whp and there exists a positive integer $p_1 \in \mathbb{Z} \cap [1, \infty)$ such that

$$\liminf_{\substack{yn \to \infty \\ \min\{y,n\} \geq p_1}} \frac{\log(-\log \mathbb{P}(\ell_n > yn))}{\log(yn)} > 0. \quad (3)$$

In this section, we show how to derive moderate deviations of shortest-path lengths in random segment processes satisfying these conditions.

**Proposition 4.** *Let $\beta > 1/2$ be an arbitrary fixed number, assume that conditions* (A) *and* (B) *above are satisfied and that* (3) *holds. Then, for $r > 1$ the events $|\ell_r - \mu r| \leq r^\beta$ occur whp.*

Next, in Section 5, we prove Theorem 1 by verifying that conditions (A) and (B) above are satisfied for Voronoi tessellations, Delaunay tessellations and relative neighborhood graphs. For these models the validity of condition (3) has been shown in [14, Theorem 1]. Indeed, it follows from Lemmas 21 and 22 below that the events $\{f_{\mathsf{geom}}(X) \cap Q_a(o) \neq \emptyset\}$ and $\{\#(f_{\mathsf{geom}}(X) \cap Q_1(o)) \leq a\}$ occur whp. Finally, condition (D2) of [14, Section 3] coincides with condition (F6) above.

## 4.2. A martingale concentration inequality

A crucial step in the proof of Proposition 4 consists in relating the shortest-path lengths $\ell_n$ for integers $n \geq 1$ to appropriate martingales and to apply the martingale concentration result derived in [18]. To embed our shortest-path problem into a martingale framework, we first define a suitable filtration. As in the lattice model discussed in [18], it turns out that when defining such a filtration surprisingly few properties of the Euclidean space $\mathbb{R}^d$ have to be taken into account. Indeed, we may fix an arbitrary enumeration $\{z_1, z_2, \ldots\}$ of $\mathbb{Z}^d$ and consider the canonical probability space $\big(\mathbb{N}_{\mathbb{M}}, \mathcal{N}_{\mathbb{M}}, \mathbb{P}\big)$ associated with the independently $\mathbb{M}$-marked Poisson point process $X$. Then, we note that for every $n \geq 1$ the $\sigma$-algebras $\mathcal{F}_m^{(n)} = \sigma\big(X \cap \bigcup_{i=1}^m Q^{\mathbb{M}}_{n^\delta}(n^\delta z_i)\big)$, $m \geq 1$ form a filtration of $\mathcal{N}_{\mathbb{M}}$. In particular, it is possible to consider $\ell_n$ as the limit of the martingales $\mathbb{E}\big(\ell_n \mid \mathcal{F}_m^{(n)}\big)$ as $m \to \infty$ and the following concentration result (a special case of [16, Lemma 5.6], which in turn is based on [18, Theorem 3]) explains why this interpretation is worthwhile.

**Lemma 5.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(\mathcal{F}_k)_{k \geq 1}$ be a filtration of $\mathcal{F}$ and let $H$ be an $\mathcal{F}$-measurable random variable in $L^1$. Furthermore, put $H_k = \mathbb{E}\big(H \bigm| \mathcal{F}_k\big)$, $\Delta_k = H_k - H_{k-1}$ and assume the existence of a constant $b > 1$ such that $\mathbb{P}\big(\big|\Delta_k\big| \leq b\big) = 1$. Let $(U_k)_{k \geq 1}$ be a sequence of $\mathcal{F}$-measurable positive random variables such that $\sum_{k=1}^\infty U_k \leq b$ and $\mathbb{E}(\Delta_k^2 \mid \mathcal{F}_{k-1}) \leq \mathbb{E}(U_k \mid \mathcal{F}_{k-1})$ a.s. for all $k \geq 2$. Then $\lim_{k \to \infty} H_k = H$ exists, is finite a.s. and there exist constants $C_1, C_2 > 0$ (independent of $b$), such that*

$$\mathbb{P}\big(|H| \geq x\sqrt{b}\big) \leq C_1 \exp(-C_2 x) \quad \textit{for all } x \leq b.$$

**Remarks.** Apart from Lemma 5 also the concentration inequalities obtained in [5] seem suitable for obtaining concentration results on moderate-deviations in Euclidean first-passage percolation. Indeed, various lattice-based cases have been considered in [4, 10]. However, the conditions of Lemma 5 are easier to verify, since the martingale approach allows a greater degree of averaging in comparison to [5]. Furthermore, it seems unlikely that by using a more advanced concentration inequality one can avoid the necessity of having to introduce regularizations. Finally, we note that simpler martingale concentration inequalities require stronger assumptions, such as the a.s. boundedness of $\sum_{k \geq 1} \mathbb{E}(\Delta_k^2 \mid \mathcal{F}_{k-1})$ (see [21, Theorem 3.15]) and are therefore difficult to apply in the present setting.

Unfortunately though, the a.s. boundedness of the martingale increments $\Delta_k$ is not satisfied when trying to apply Lemma 5 directly to the martingale $\mathbb{E}\big(\ell_n \mid \mathcal{F}_m^{(n)}\big) - \mathbb{E}\ell_n$, $m \geq 1$. As will be explained below, the martingale increments can be interpreted in terms of differences of certain shortest-path lengths, but their absolute value is not almost surely bounded by a fixed threshold $b$. Therefore, we first consider suitable regularizations of the point process $X$ which remove this undesirable property.

Before we provide a precise and formal description of this regularization, let us start by explaining the main idea by considering the case of the Poisson-Delaunay graph. First, every $n^\delta$-cube of the form $Q^{\mathbb{M}}_{n^\delta}(n^\delta z)$, $z \in \mathbb{Z}^d$ is considered, and it is checked whether the configuration of the Poisson point process inside this cube is in a certain sense pathological. For instance, in the Poisson-Delaunay model, this could happen if no points of the Poisson point process are contained inside the cube. Then, for every cube that exhibits a pathological configuration of points, we replace the Poisson point process inside that cube by a regular point pattern.

To make this more precise, recall from condition (F1) that we assumed the existence of a positive number $r_0 > 0$ and a locally finite set $\varphi^{(0)} \in \mathbb{N}_{\mathbb{M}}$ such that $f_{\mathsf{geom}}(\varphi^{(0)}) = r_0\mathbb{Z}^d$. Next, we define the *$r$-coupling of $\varphi$ by $\varphi^{(0)}$* to be the locally finite set $\overline{\varphi} = \overline{\varphi}(r) = \bigcup_{z\in\mathbb{Z}^d} \varphi_z$, where

$$\varphi_z = \begin{cases} \varphi \cap Q^{\mathbb{M}}_r(rz) & \text{if } (\varphi - rz) \cap Q^{\mathbb{M}}_r(o) \in A_r, \\ \varphi^{(0)} \cap Q^{\mathbb{M}}_r(rz) & \text{otherwise.} \end{cases} \tag{4}$$

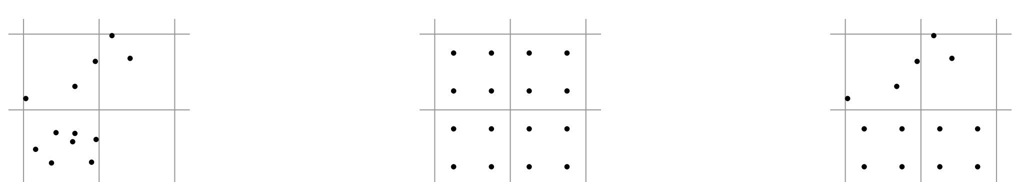
Since we assumed that $(\varphi^{(0)} - rz) \cap Q^{\mathbb{M}}_r(o) \in A_r$ for all $z \in \mathbb{Z}^d$, we see that $\varphi_z - rz = (\overline{\varphi}(r) - rz) \cap Q^{\mathbb{M}}_r(o) \in A_r$ for all $z \in \mathbb{Z}^d$. By property (B) of Section 4.1, this implies that properties (H1)–(H6) hold. If $X$ is an independently and isotropically $\mathbb{M}$-marked homogeneous Poisson point process in $\mathbb{R}^d$, then the expression $X(r)$ is to be understood accordingly. In the remainder of the present subsection, we fix $\delta \in (0, 1/(8\alpha_1))$. For any $\varphi \in \mathbb{N}_{\mathbb{M}}$ let $\overline{\ell}_n(\varphi)$ denote the length of the shortest path from $q(o)$ to $q(re_1)$ in the segment process $\overline{g}(\varphi) = \overline{g}_{n^\delta,\,\mathsf{geom}}\big(\overline{\varphi}(n^\delta)\big)$. Furthermore, we put $\overline{\ell}_n = \overline{\ell}_n(X)$ and

$$H^{(n)}_m = \mathbb{E}\big(\overline{\ell}_n \mid \mathcal{F}^{(n)}_m\big) - \mathbb{E}\overline{\ell}_n. \tag{5}$$

Observe that due to the Poisson assumption, for any $n \geq 1$ the probability space $(\mathbb{N}_{\mathbb{M}}, \mathcal{N}_{\mathbb{M}}, \mathbb{P})$ can be considered as a product space associated with the sequence of random variables $\{X \cap Q_{n^\delta}(n^\delta z_k)\}_{k\geq 1}$. In fact, this observation is the reason for the apparent difficulty to generalize the results to non-Poissonian settings. For $j \geq 1$, let $\sigma_j = \sigma \cap Q^{\mathbb{M}}_{n^\delta}(n^\delta z_j)$ denote the restriction of $\sigma \in \mathbb{N}_{\mathbb{M}}$ to the cube $Q^{\mathbb{M}}_{n^\delta}(n^\delta z_j)$. Formally, we should write $\sigma_{j,n}$ instead of $\sigma_j$, but in the present subsection the value of $n \geq 1$ will always be clear from the context, so that we prefer to use simple notation. For the martingale construction, it will be more intuitive to put $\omega = X$. Hence, the $\sigma$-algebra $\mathcal{F}^{(n)}_m$ is generated by $\{\omega_j\}_{1\leq j\leq m}$. We write $\mathbb{P}_j$ for the distribution of $\omega_j$ and $[\omega, \sigma]^{(j)}$ for the marked point process given by

$$[\omega, \sigma]^{(j)} \cap Q^{\mathbb{M}}_{n^\delta}(n^\delta z_i) = \begin{cases} \omega_i & \text{if } i \leq j, \\ \sigma_i & \text{otherwise,} \end{cases}$$

where $\sigma \in \mathbb{N}_{\mathbb{M}}$. Figure 2 illustrates the construction of $[\omega, \sigma]^{(2)}$.



(a) Configuration $\omega$ (b) Configuration $\sigma$ (c) Configuration $[\omega, \sigma]^{(2)}$

Figure 2. Construction of $[\omega, \sigma]^{(2)}$

Observe that we may compute the conditional expectations $\mathbb{E}\big(\overline{\ell}_n \mid \mathcal{F}_k^{(n)}\big)$ appearing in (5) by integrating over everything except the first $k$ coordinates of the entire product measure $\mathbb{P} = \prod_{j=1}^{\infty} \mathbb{P}_j$, so that

$$\Delta_k^{(n)} = \int \overline{\ell}_n([\omega,\sigma]^{(k)}) - \overline{\ell}_n([\omega,\sigma]^{(k-1)}) \prod_{j=k}^{\infty} \mathbb{P}_j(\mathrm{d}\sigma_j). \tag{6}$$

It is important to note that no regularization is performed in the construction of the point processes $[\omega,\sigma]^{(k)}$ or $[\omega,\sigma]^{(k)}$. The regularization appears only through application of the functional $\overline{\ell}_n$.

Recall from Section 2.2 that for $x, y \in G$ we write $R(x,y)$ for the family of geodesics connecting $x$ and $y$. Throughout Section 4, for any $x, y \in \mathbb{R}^d$ let $\rho(x,y)$ denote the lexicographically smallest element in $R(q(x), q(y))$, where we say that a path $\gamma_1$ is lexicographically smaller than $\gamma_2$ if the list of vertices defining $\gamma_1$ is lexicographically smaller than the list of vertices defining $\gamma_2$. For $n \ge 1$ and $\varphi \in \mathbb{N}_{\mathbb{M}}$ we also write $\overline{\rho}_n(\varphi)$ for the geodesic $\rho(o, ne_1)$ in the segment process $\overline{g}(\varphi)$ and put $\overline{\rho}_n = \overline{\rho}_n(X)$. Using these definitions, we now proceed as in [25, Lemma 2.3] to provide an upper bound for the expression appearing inside the integral (6). For the convenience of the reader, we present a detailed proof. For $k, n \ge 1$, let $I_k^{(n)}$ denote the indicator of the event $\overline{\rho}_n \cap Q_{4n^\delta}(n^\delta z_k) \ne \emptyset$. If the value of $n \ge 1$ is understood, then we also write $I_k$ instead of $I_k^{(n)}$. The event $\{I_k^{(n)} = 1\}$ is illustrated in Figure 3.

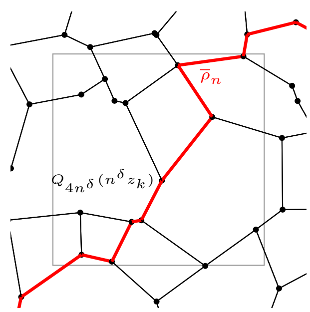



FIGURE 3. Illustration of the event $\{I_k^{(n)} = 1\}$

**Lemma 6.** *Let $k, n \ge 2$ and $\omega, \sigma \in \mathbb{N}_{\mathbb{M}}$. Then,*

$$\big|\overline{\ell}_n([\omega,\sigma]^{(k-1)}) - \overline{\ell}_n([\omega,\sigma]^{(k)})\big| \le n^{\delta\alpha_1} \max\big\{I_k([\omega,\sigma]^{(k-1)}), I_k([\omega,\sigma]^{(k)})\big\}.$$

*Proof.* We begin by considering the case $I_k([\omega,\sigma]^{(k)}) = 0$. Observe that conditions (H2) and (H4′) imply that

$$q\big(x, \overline{g}([\omega,\sigma]^{(k-1)})\big) = q\big(x, \overline{g}([\omega,\sigma]^{(k)})\big)$$

for all $x \in \{o, ne_1\}$. Moreover, from (H2) we conclude that $\overline{\rho}_n([\omega,\sigma]^{(k)})$ also forms a path in the segment process $\overline{g}([\omega,\sigma]^{(k-1)})$, so that

$$\overline{\ell}_n([\omega,\sigma]^{(k-1)}) \le \overline{\ell}_n([\omega,\sigma]^{(k)}). \tag{7}$$

Here, and in the previous statement, we used that $I_k([\omega,\sigma]^{(k)}) = 0$. Next, consider the case $I_k([\omega,\sigma]^{(k)}) = 1$ and assume additionally that $\{o, ne_1\} \subset \mathbb{R}^d \setminus Q_{4n^\delta}(n^\delta z_k)$; the other cases are similar. Then, for every $x \in \{o, ne_1\}$,

$$q\big(x, \overline{g}([\omega,\sigma]^{(k-1)})\big) = q\big(x, \overline{g}([\omega,\sigma]^{(k)})\big).$$

In particular, there is no harm in using the abbreviations $q(o)$ and $q(ne_1)$. Denote by $x_F$ the first point of $\overline{\rho}_n([\omega,\sigma]^{(k)})$ contained in $Q_{5n^\delta}(n^\delta z_k)$ and similarly by $x_L$ the last point of $\overline{\rho}_n([\omega,\sigma]^{(k)})$ contained in $Q_{5n^\delta}(n^\delta z_k)$. Note that we exclude neither $x_F = q(o)$ nor $x_L = q(ne_1)$ and that we used $I_k([\omega,\sigma]^{(k)}) = 1$ to deduce the existence of $x_F$, $x_L$. Observe that (H2) and (H3) imply that $x_F$ and $x_L$ can be connected by a path $\gamma$ in $\overline{g}([\omega,\sigma]^{(k-1)}) \cap Q_{7n^\delta}(n^\delta z_k)$ and we consider the concatenation of the geodesic from $q(o)$ to $x_F$ in the segment process $\overline{g}([\omega,\sigma]^{(k)})$, the path $\gamma$ and the geodesic from $x_L$ to $q(ne_1)$ in the segment process $\overline{g}([\omega,\sigma]^{(k)})$. Due to the definition of $x_F$ and $x_L$, these geodesics are also paths in the segment process $\overline{g}([\omega,\sigma]^{(k-1)})$. Hence, by conditions (H2) and (H5),

$$\overline{\ell}_n([\omega,\sigma]^{(k-1)}) \le \nu_1(\gamma) + \overline{\ell}_n([\omega,\sigma]^{(k)}) \le n^{\delta\alpha_1} + \overline{\ell}_n([\omega,\sigma]^{(k)}). \tag{8}$$

Finally, combining (7) and (8) yields $\overline{\ell}_n([\omega,\sigma]^{(k-1)}) - \overline{\ell}_n([\omega,\sigma]^{(k)}) \le n^{\delta\alpha_1} I_k([\omega,\sigma]^{(k)})$. Since $\overline{\ell}_n([\omega,\sigma]^{(k)}) - \overline{\ell}_n([\omega,\sigma]^{(k-1)}) \le n^{\delta\alpha_1} I_k([\omega,\sigma]^{(k-1)})$ can be shown similarly, this completes the proof of Lemma 6. □

Next, following [25, Lemma 3.4], we put $b^{(n)} = n^{1+8\delta\alpha_1}$, $n \ge 2$. It is also convenient to define $U_k^{(n)} = 2n^{2\delta\alpha_1} I_k(\omega)$, $k \ge 2$.

**Lemma 7.** *Let $k,n \ge 2$. Then, $|\Delta_k^{(n)}| \le b^{(n)}$ and $\mathbb{E}\big((\Delta_k^{(n)})^2 \mid \mathcal{F}_{k-1}^{(n)}\big) \le \mathbb{E}\big(U_k^{(n)} \mid \mathcal{F}_{k-1}^{(n)}\big)$.*

*Proof.* The inequality $|\Delta_k^{(n)}| \le b^{(n)}$ is an immediate consequence of Lemma 6. To deduce the second claim we may apply Lemma 6 and the Cauchy-Schwarz inequality to compute that

$$\begin{aligned}\mathbb{E}\big((\Delta_k^{(n)})^2 \mid \mathcal{F}_{k-1}^{(n)}\big)(\omega) &= \int \Big(\int \overline{\ell}_n([\omega,\sigma]^{(k)}) - \overline{\ell}_n([\omega,\sigma]^{(k-1)}) \prod_{j=k}^{\infty} \mathbb{P}_j(\mathrm{d}\sigma_j)\Big)^2 \mathbb{P}_k(\mathrm{d}\omega_k)\\
&\le n^{2\delta\alpha_1} \int \Big(\int \max\big\{I_k([\omega,\sigma]^{(k)}), I_k([\omega,\sigma]^{(k-1)})\big\} \prod_{j=k}^{\infty} \mathbb{P}_j(\mathrm{d}\sigma_j)\Big)^2 \mathbb{P}_k(\mathrm{d}\omega_k)\\
&\le n^{2\delta\alpha_1} \int\int \max\big\{I_k([\omega,\sigma]^{(k)}), I_k([\omega,\sigma]^{(k-1)})\big\} \prod_{j=k}^{\infty} \mathbb{P}_j(\mathrm{d}\sigma_j)\mathbb{P}_k(\mathrm{d}\omega_k)\\
&\le n^{2\delta\alpha_1} \int\int I_k([\omega,\sigma]^{(k)}) + I_k([\omega,\sigma]^{(k-1)}) \prod_{j=k}^{\infty} \mathbb{P}_j(\mathrm{d}\sigma_j)\mathbb{P}_k(\mathrm{d}\omega_k),\end{aligned}$$

which is equal to $\mathbb{E}\big(U_k^{(n)} \mid \mathcal{F}_{k-1}^{(n)}\big)(\omega)$ a.s., since

$$\mathbb{E}\big(I_k(\omega) \mid \mathcal{F}_{k-1}^{(n)}\big) = \int\int I_k\big([\omega,\sigma]^{(k-1)}\big) \prod_{j=k}^{\infty} \mathbb{P}_j(\mathrm{d}\sigma_j)\mathbb{P}_k(\mathrm{d}\omega_k) = \int\int I_k\big([\omega,\sigma]^{(k)}\big) \prod_{j=k}^{\infty} \mathbb{P}_j(\mathrm{d}\sigma_j)\mathbb{P}_k(\mathrm{d}\omega_k) \text{ a.s.} \qquad \square$$

In order to be able to provide a suitable bound on $\mathbb{P}(\sum_{k=1}^\infty U_k^{(n)} > x)$, we need the following elementary result on the volume of the Minkowski sausage on polygonal curves, where $\kappa_d = \nu_d(B_1(o))$ and $\nu_d$ denotes the Lebesgue measure in $\mathbb{R}^d$. Moreover, for $B_1, B_2 \subset \mathbb{R}^d$ we put $B_1 \oplus B_2 = \{b_1 + b_2 :\ b_1 \in B_1, b_2 \in B_2\}$.

**Lemma 8.** *Let $\gamma \subset \mathbb{R}^d$ be a polygonal curve. Then, $\nu_d\big(\gamma \oplus B_r(o)\big) \le \kappa_d 2^d r^{d-1}\nu_1(\gamma) + \kappa_d 2^d r^d$ for all $r > 0$.*

*Proof.* We prove the claim by induction on $\lfloor \nu_1(\gamma)/r \rfloor$. Let $x$ denote the starting point of the curve $\gamma$. If $\gamma \subset B_r(x)$, then the claim is trivial. Otherwise, $x'$ denotes the first intersection point of $\gamma$ and $\partial B_r(x)$ and $\gamma'$ the sub-path of $\gamma$ starting at $x'$. Then,

$$\nu_d\big(\gamma \oplus B_r(o)\big) \le \nu_d\big(B_{2r}(x)\big) + \nu_d(\gamma' \oplus B_r(o)) \le 2\kappa_d 2^d r^d + \kappa_d 2^d r^{d-1}\nu_1(\gamma') \le \kappa_d 2^d r^d + \kappa_d 2^d r^{d-1}\nu_1(\gamma). \qquad \square$$

As a final preliminary result, we note that $\overline{\ell}_n$ grows linearly in $n$, where $\alpha_2$ is introduced in condition (H6).

**Lemma 9.** *It holds that $\overline{\ell}_n \in [n/2, 3\alpha_2 n]$ for all sufficiently large $n \geq 1$.*

*Proof.* For the lower bound, we note that

$$\overline{\ell}_n \geq |q(ne_1) - q(o)| \geq n - |q(o)| - |q(ne_1) - ne_1| \geq n - n^\delta \geq n/2,$$

for all sufficiently large $n \geq 1$. In order to prove the second claim, we begin by choosing $n' \geq 1$ such that $ne_1 \in Q_{n^\delta}(n'n^\delta e_1)$. Then, using (H6) repeatedly, we see that $q(o)$ and $q(n'n^\delta e_1)$ can be connected by a path of length at most $n'\alpha_2 n^\delta$ and the latter expression is bounded from above by $2\alpha_2 n$ for sufficiently large $n$. Furthermore, combining conditions (H3), (H4′) and (H5) shows that $q(n'n^\delta e_1)$ and $q(ne_1)$ can be connected by a path of length at most $n^{\delta\alpha_1}$, and the latter expression is bounded from above by $n$, since we assumed $\delta \in (0, 1/(8\alpha_1))$. □

Now, we have collected all auxiliary results and proceed to prove the desired almost sure upper bound for $\sum_{k\geq 1} U_k^{(n)}$. In other words, we show that the support of this random variable is bounded, which is a stronger property than the sub-exponential decay considered in [16, 18, 25]. Our improvement is due to the difference in the regularization. While in [16, 18, 25] the regularization is performed on the level of single segment weights, we use a spatial block construction. To be more precise, although [25] uses a construction based on $n^\delta$-blocks, still in the regularization every single weight of a segment in a block is subject to a truncation. On the other hand, we benefit from the observation that it is easier to control spatial averages in large sampling windows than trying to impose a suitable regularization on the microscopic level of single segment weights.

**Lemma 10.** *There exists $n_0 \geq 1$ such that $\mathbb{P}(\sum_{k=1}^\infty U_k^{(n)} > n^{1+8\delta\alpha_1}) = 0$ for all $n \geq n_0$.*

*Proof.* Indeed, applying Lemmas 8 and 9, we see that for all sufficiently large $n \geq 1$,

$$\begin{aligned}\sum_{k=1}^\infty U_k^{(n)} &= 2n^{2\delta\alpha_1}\sum_{k=1}^\infty I_k = 2n^{2\delta\alpha_1}\#\{z \in \mathbb{Z}^d : \overline{\rho}_n \cap Q_{4n^\delta}(n^\delta z) \neq \emptyset\} \leq 2n^{2\delta\alpha_1}\frac{\nu_d\big(\overline{\rho}_n \oplus B_{4\sqrt{d}n^\delta}(o)\big)}{n^{\delta d}}\\ &\leq 2n^{2\delta\alpha_1}\frac{\kappa_d 2^d 4^d n^{\delta d} d^{d/2} + \kappa_d 2^d n^{\delta(d-1)} 4^{d-1} d^{(d-1)/2}\overline{\ell}_n}{n^{\delta d}}\\ &\leq \kappa_d 2^{3d+1} d^{d/2} n^{2\delta\alpha_1} + 3\alpha_2\kappa_d 2^{d+1} d^{(d-1)/2} 4^{d-1} n^{1+\delta(2\alpha_1-1)}.\end{aligned}$$

The proof is completed since the latter expression is at most $n^{1+8\delta\alpha_1}$, provided that $n \geq 1$ is sufficiently large. □

Finally, we can deduce the desired concentration result for $\big|\overline{\ell}_n - \mathbb{E}\overline{\ell}_n\big|$.

**Proposition 11.** *There exist $n_0 \geq 1$ and constants $C_1, C_2 > 0$ such that for all sufficiently large $n \geq n_0$,*

$$\mathbb{P}\big(\big|\overline{\ell}_n - \mathbb{E}\overline{\ell}_n\big| > n^{1/2+8\delta\alpha_1}\big) \leq C_1\exp(-C_2 n^{4\delta\alpha_1}).$$

*Proof.* Indeed, the statement follows by combining Lemmas 5, 7 and 10. □

### 4.3. Comparison of $\ell_n$ and $\overline{\ell}_n$

To show that $\ell_n$ and $\overline{\ell}_n$ are sufficiently close, we follow again the approach of [25]. First, observe that as a corollary to Lemma 9, we obtain an auxiliary result on the diameter of $\overline{\rho}_n$.

**Corollary 12.** *Let $\delta \in (0, 1/(8\alpha_1))$ be arbitrary. Then $\overline{\rho}_n \subset Q_{8\alpha_2 n}(o)$ for all sufficiently large $n \geq 1$.*

*Proof.* By condition (H4′) we have $q(o, \overline{g}(X)) \in Q_{n^\delta}(o)$, so that $\overline{\rho}_n \not\subset Q_{8\alpha_2 n}(o)$ implies $\overline{\ell}_n \geq 4\alpha_2 n - \sqrt{d}n^\delta$. An application of Lemma 9 completes the proof of Corollary 12. □

**Remark.** Applying very similar arguments as in Corollary 12, one may use (3) to show that for $n > 1$ the events $\rho_n \subset Q_{4p_1n}(o)$ occur whp.

Using Corollary 12 we may now deduce the following auxiliary result on the relation between $\ell_n$ and $\overline{\ell}_n$.

**Lemma 13.** *Let $\delta \in (0, 1/(8\alpha_1))$. Then, for $n > 1$ the events $\ell_n = \overline{\ell}_n$ occur whp, and for every $k \geq 1$,*

$$\sup_{n\geq 1} \max\left\{\mathbb{E}\left|\ell_n - \overline{\ell}_n\right|^k, n^{-k}\mathbb{E}\ell_n^k\right\} < \infty.$$

*Proof.* To begin with, we prove the first assertion. Using Corollary 12 in conjunction with its remark implies that $\rho_n \cup \overline{\rho}_n \subset Q_{p_2n}(o)$ whp, where we put $p_2 = \max\{4p_1, 8\alpha_2\}$. Moreover, we claim that for $n \geq 1$ also the events

$$\left\{\overline{g}_{n^\delta,\,\mathsf{geom}}(X) \cap Q_{p_2n}(o) = \overline{g}_{n^\delta,\,\mathsf{geom}}(\overline{X}) \cap Q_{p_2n}(o)\right\} \cap \left\{\rho_n \cup \overline{\rho}_n \subset Q_{p_2n}(o)\right\} \tag{9}$$

occur whp. First, recall from the assumptions stated in Section 4.1 that we have $X \cap Q_{n^\delta}(o) = \overline{X} \cap Q_{n^\delta}(o)$ whp. Since our definition of 'whp' imposes stretched exponential decay for the probabilities of the complements, whereas $Q_{2p_2n}(o)$ is covered by a polynomial number of $n^\delta$-cubes, we conclude that also the events $X \cap Q_{2p_2n}(o) = \overline{X} \cap Q_{2p_2n}(o)$ occur whp. Now, (9) follows from condition (H1). Since, additionally, for $n > 1$ the events $\overline{g}_{n^\delta,\,\mathsf{geom}}(X) \cap Q_{p_2n}(o) = g_{\mathsf{geom}}(X) \cap Q_{p_2n}(o)$ occur whp, the first assertion is proved by noting that the intersection of these events implies $\ell_n = \overline{\ell}_n$. For the second assertion, observe that by (3) and Lemma 9 there exists $\alpha' > 0$ such that

$$\mathbb{P}\big(\left|\ell_n - \overline{\ell}_n\right| > ny\big) \leq \mathbb{P}\big(\ell_n > ny/2\big) + \mathbb{P}\big(\overline{\ell}_n > ny/2\big) \leq \exp\big(-(ny)^{\alpha'}\big).$$

for all $y \geq p_2$ and all sufficiently large $n \geq 1$. In particular,

$$\begin{aligned}\mathbb{E}\big|\ell_n - \overline{\ell}_n\big|^k &= k\int_0^\infty \mathbb{P}\big(\big|\ell_n - \overline{\ell}_n\big| > z\big)z^{k-1}\mathrm{d}z = kn^{k-1}\int_0^\infty \mathbb{P}\big(\big|\ell_n - \overline{\ell}_n\big| > ny\big)y^{k-1}\mathrm{d}y\\ &= kn^{k-1}\Big(p_2\mathbb{P}\big(\ell_n \neq \overline{\ell}_n\big) + \int_{p_2}^\infty \exp\big(-(ny)^{\alpha'}\big)y^{k-1}\mathrm{d}y\Big).\end{aligned}$$

Finally, observe that

$$n^{-k}\mathbb{E}\ell_n^k = n^{-k}k\int_0^\infty \mathbb{P}\big(\ell_n > z\big)z^{k-1}\mathrm{d}z = k\int_0^\infty \mathbb{P}\big(\ell_n > ny\big)y^{k-1}\mathrm{d}y = kp_1^k + k\int_{p_1}^\infty \exp\big(-(ny)^{\alpha'}\big)y^{k-1}\mathrm{d}y,$$

which completes the proof of the second claim. □

As a corollary we obtain the following concentration result on moderate deviations of the shortest-path lengths $\ell_n$ from their expectation.

**Proposition 14.** *Let $\beta > 1/2$ be arbitrary. Then, for $n \geq 1$ the events $|\ell_n - \mathbb{E}\ell_n| \leq n^\beta$ occur whp.*

*Proof.* Put $\delta = (\beta - 1/2)/(8\alpha_1)$ and note that by Lemma 13

$$\mathbb{P}(|\ell_n - \mathbb{E}\ell_n| \geq n^\beta) \leq \mathbb{P}(|\ell_n - \overline{\ell}_n| \geq n^\beta/3) + \mathbb{P}(|\overline{\ell}_n - \mathbb{E}\overline{\ell}_n| \geq n^\beta/3)$$

for all sufficiently large $n \geq 1$. Hence, the claim follows from Corollary 11 and Lemma 13. □

### 4.4. **Controlling** $\mathbb{E}\ell_n$

Next, to deduce Proposition 4 from Proposition 14, we derive suitable bounds for $|\mathbb{E}\ell_n - \mu n|$. In order to achieve this goal, we consider the approach outlined in [16,25]. Using similar arguments as in [16], the following analog of [16, Lemma 4.1] can be derived. For the convenience of the reader, we provide a detailed proof.

**Lemma 15.** *Let $\beta > 1/2$ be arbitrary. Then, $\mathbb{E}\ell_{2n} \ge 2\mathbb{E}\ell_n - n^\beta$ for all sufficiently large $n \ge 1$.*

*Proof.* First, put $\delta = (\beta - 1/2)/(8\alpha_1)$, where we may assume that $\beta \in (1/2, 1)$. Note that since for $r > 1$ the events $X \cap Q_r^{\mathbb{M}}(o) \in A_r$ and $\overline{g}_{r,\,\mathsf{geom}}(X) \cap Q_r(o) = g_{\mathsf{geom}}(X) \cap Q_r(o)$ occur whp, there exists a family of events $\{A'_r\}_{r>1}$ such that for $r > 1$ the events $\{A'_r\}_{r>1}$ occur whp and such that if $X \in A'_r$, then $q(x) \in Q_{3r}(o)$ for all $x \in Q_r(o)$ and $\ell(x, y) \le r^{\alpha_1}$ for all $x, y \in g_{\mathsf{geom}}(X) \cap Q_{3r}(o)$. Finally, put

$$F_n = \big\{X - n^\delta z \in A'_{n^\delta} \text{ for all } z \in \mathbb{Z}^d \cap Q_{4n}(o)\big\}$$

and note that the events $F_n$ occur whp. Next, put $x_1 = ne_1$ and choose $n^d-1$ further points $x_2, \dots, x_{n^d} \in \partial B_n(o)$ such that $\partial B_n(o) \subset \bigcup_{i=1}^{n^d} Q_1(x_i)$. Moreover, define $x'_i = 2ne_1 - x_i$. Then, the first step consists of establishing the following lower bound for $\ell_{2n}$ :

$$\ell_{2n} \ge \big(\min_{1\le i\le n^d} \ell(o, x_i) + \min_{1\le j\le n^d} \ell(x'_j, 2ne_1) - 2n^{\delta\alpha_1}\big)1_{F_n}. \tag{10}$$

Denote by $x_F$ the first intersection point of $\rho_{2n}$ with $\partial B_n(o)$ and by $x_L$ the last intersection point of $\rho_{2n}$ with $\partial B_n(2ne_1)$, so that $\ell_{2n} \ge \ell(o, x_F)+\ell(x_L, 2ne_1)$. By construction, there exists $i_1 \in \{1, \dots, n^d\}$ with $x_F \in Q_1(x_{i_1})$ and we choose $z_i \in \mathbb{Z}^d$ with $x_{i_1} \in Q_{n^\delta}(n^\delta z_i)$. Observe that if $F_n$ occurs, then $q(x_{i_1}) \in Q_{3n^\delta}(n^\delta z_i)$, so that $x_F$ and $q(x_{i_1})$ can be connected by a path whose length is bounded from above by $n^{\delta\alpha_1}$. Therefore,

$$\ell(o, x_F) \ge \big(\ell(o, x_{i_1}) - n^{\delta\alpha_1}\big)1_{F_n} \ge \big(\min_{1\le i\le n^d} \ell(o, x_i) - n^{\delta\alpha_1}\big)1_{F_n},$$

and similarly $\ell(x_L, 2ne_1) \ge \big(\min_{1\le j\le n^d} \ell(x'_j, 2ne_1) - n^{\delta\alpha_1}\big)1_{F_n}$, which shows (10).

Since $x_1 = x'_1 = ne_1$ we can deduce the following refined version of (10)

$$\ell_{2n} + \big(\ell(o, ne_1) + \ell(ne_1, 2ne_1)\big)1_{F_n^c} \ge \min_{1\le i\le n^d} \ell(o, x_i) + \min_{1\le j\le n^d} \ell(x'_j, 2ne_1) - 2n^{\delta\alpha_1}.$$

In particular, by the Cauchy-Schwarz inequality,

$$\mathbb{E}\ell_{2n} + 2n^{\delta\alpha_1} + 2\sqrt{\mathbb{E}\ell_n^2 \mathbb{P}(F_n^c)} \ge 2\mathbb{E} \min_{1\le i\le n^d} \ell(o, x_i).$$

As observed above, the events $F_n$ occur whp so that using Lemma 13, we conclude that

$$\mathbb{E}\ell_{2n} - 2\mathbb{E}\ell_n + 3n^{\delta\alpha_1} \ge -2\mathbb{E}\big(\max_{1\le i\le m(n)} \mathbb{E}\ell(o, x_i) - \ell(o, x_i)\big)$$

for all sufficiently large $n \ge 1$. Finally, put $\varepsilon = \beta - 1/2$ and note that Proposition 14 implies

$$\begin{aligned}\mathbb{E}\big(\max_{1\le i\le n^d} \mathbb{E}\ell(o, x_i) - \ell(o, x_i)\big) &\le n^{1/2+\varepsilon/4} + \sum_{i=1}^{n^d} \mathbb{P}\big(|\ell(o, x_i) - \mathbb{E}\ell(o, x_i)| \ge n^{1/2+\varepsilon/4}\big)\mathbb{E}\ell_n \\ &= n^{1/2+\varepsilon/4} + n^d \mathbb{P}\big(|\ell_n - \mathbb{E}\ell_n| \ge n^{1/2+\varepsilon/4}\big)\mathbb{E}\ell_n,\end{aligned}$$

which is at most $n^{1/2+\varepsilon/2}$ for all sufficiently large $n \ge 1$. □

Next, for the convenience of the reader, we restate [16, Lemma 4.2].

**Lemma 16.** *Let $\mu > 0$ and let $(a_n)_{n\ge1}$, $(b_n)_{n\ge1}$ be sequences in $(0,\infty)$ satisfying $a_n/n \to \mu$, $b_n/n \to 0$, $a_{2n} \ge 2a_n - b_n$ and $\psi = \limsup_{n\ge1} b_{2n}/b_n < 2$. Then, $a_n \le \mu n + cb_n$ for all sufficiently large $n \ge 1$, where $c = 2/(2-\psi)$.*

Finally, using Lemmas 15 and 16 (with $a_n = \mathbb{E}\ell_n$ and $b_n = n^\beta$), we obtain the following corollary which can be used to complete the proof of Proposition 4.

**Corollary 17.** *Let $\beta > 1/2$ be arbitrary. Then, $\mu n \le \mathbb{E}\ell_n \le \mu n + n^\beta$ for all sufficiently large $n \ge 1$.*

*Proof of Proposition* 4. Let $\beta \in (1/2, 1)$ be arbitrary and put $\delta = (\beta - 1/2)/(8\alpha_1)$. Choosing $n_1(r) \ge 1$ such that $re_1 \in Q_1(n_1(r)e_1)$ yields

$$\ell(o, re_1) - \mu r \le |\ell_{n_1(r)} - \mu n_1(r)| + \ell(n_1(r)e_1, re_1) + \mu|n_1(r) - r|.$$

By condition (H4′) we have $q(n_1(r)e_1), q(re_1) \in Q_{3n_1(r)^\delta}(n_1(r)e_1)$ whp. In particular, conditions (H3) and (H5) imply that $\ell(n_1(r)e_1, re_1) \le n_1(r)^{\delta\alpha_1}$ whp. Finally, by Corollary 17, the events $|\ell_{n_1(r)} - \mu n_1(r)| \le n_1(r)^{(2\beta+1)/4}$ occur whp. □

## 5. Proof of Theorem 1

In this section, we complete the proof of Theorem 1. First, we provide a direct proof if the underlying segment process is given by the isotropic Poisson line model in $\mathbb{R}^2$. For Voronoi tessellations, Delaunay tessellations and relative neighborhood graphs, Theorem 1 is proved by verifying conditions (H1)–(H6) from Section 4.1.

### 5.1. **Poisson line tessellation**

First, we discuss the elementary case of the isotropic Poisson line tessellation in $\mathbb{R}^2$. That is, we assume that $G$ is the planar segment process induced by the lines of an isotropic planar Poisson line process with intensity $\lambda > 0$.

**Proposition 18.** *Let $\beta > 1/2$ be arbitrary. Then, for $r > 1$ the events $|\ell_r - r| \le r^\beta$ occur whp.*

*Proof.* We put $\varepsilon = \beta - 1/2$, where we may assume that $\beta \in (1/2, 1)$ and begin by introducing some useful events. Let $E_r^{(1,a)}$ denote the event that there exists a quadrilateral $\Xi_a \subset \mathbb{R}^2$ such that $B_{r^{1/2+\varepsilon/4}}(o) \subset \Xi_a \subset B_{r^{1/2+\varepsilon/2}}(o)$ and whose boundary is defined by four lines of the Poisson line process. Similarly, $E_r^{(1,b)}$ denotes the event that there exists a quadrilateral $\Xi_b \subset \mathbb{R}^2$ such that $B_{r^{1/2+\varepsilon/4}}(re_1) \subset \Xi_b \subset B_{r^{1/2+\varepsilon/2}}(re_1)$ and whose boundary is defined by four lines of the Poisson line process. Furthermore, let $E_r^{(2)}$ denote the event that there exists a line $l_1$ of the line process intersecting $B_{\sqrt{r}}(o)$ and whose angle with the $x$-axis is contained in $[0, r^{-1/2+\varepsilon/8}]$. An illustration for the occurrence of the events $E_r^{(1,a)}$, $E_r^{(1,b)}$ and $E_r^{(2)}$ is shown in Figure 4.

Observe that if $E_r^{(1,a)}$, $E_r^{(1,b)}$ and $E_r^{(2)}$ occur, then the distance of $l_1$ from $re_1$ is at most

$$\sqrt{r} + r \sin r^{-1/2+\varepsilon/8} \le \sqrt{r} + r^{1/2+\varepsilon/8},$$

so that $l_1 \cap B_{r^{1/2+\varepsilon/4}}(re_1) \ne \emptyset$ (provided that $r \ge 1$ is sufficiently large). Hence, if $E_r^{(1,a)}$, $E_r^{(1,b)}$ and $E_r^{(2)}$ occur, then $\ell(o, re_1) \ge |q(o) - q(re_1)| \ge r - 2r^{1/2+\varepsilon/2}$ and choosing suitable $P_1 \in l_1 \cap \Xi_a$ and $P_2 \in l_1 \cap \Xi_b$ yields

$$\ell(o, re_1) \le \ell(q(o), P_1) + \ell(P_1, P_2) + \ell(P_2, q(re_1)) \le 2r^{1/2+\varepsilon/2} + \pi r^{1/2+\varepsilon/2} + r + \pi r^{1/2+\varepsilon/2} + 2r^{1/2+\varepsilon/2}.$$

Therefore, it suffices to show that that $\lim_{r\to\infty} \mathbb{P}(E_r^{(1,a)} \cap E_r^{(1,b)} \cap E_r^{(2)}) = 1$. Note that by definition of the Poisson line process, $1 - \mathbb{P}(E_r^{(2)}) = \exp(-\lambda\sqrt{r} r^{-1/2+\varepsilon/8}/\pi)$, which tends to 0 as $r \to \infty$. Furthermore, by

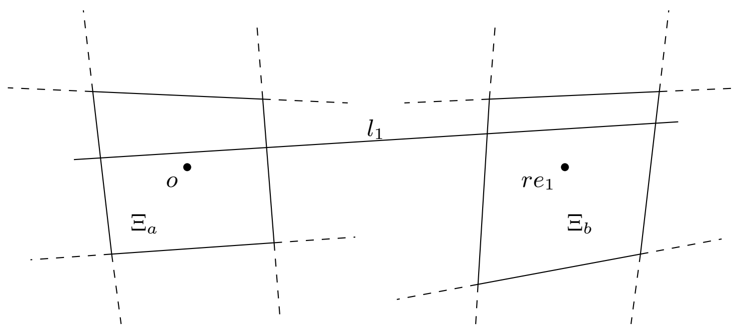



FIGURE 4. Configuration after occurrence of $E_r^{(1,a)} \cap E_r^{(1,b)} \cap E_r^{(2)}$

stationarity $\mathbb{P}(E_r^{(1,a)}) = \mathbb{P}(E_r^{(1,b)})$, so that it suffices to show $\mathbb{P}(E_r^{(1,a)}) \to 1$ as $r \to \infty$. Let $E_r^{(3)}$ denote the event that there exists a line $l$ from the Poisson line process whose angle is contained in $[\pi/2-\pi/6, \pi/2+\pi/6]$ and that intersects the ball $B_{\sqrt{r}}(r^{1/2+3\varepsilon/8}e_1)$. Furthermore, for $i \in \{1, \ldots, 4\}$ let $E_r^{(3,i)}$ denote the event obtained from $E_r^{(3)}$ by applying a rotation of angle $(i-1)\pi/2$ and center $o$. Then, using elementary geometry, we see that $\bigcap_{i=1}^4 E_r^{(3,i)}$ implies the existence of the desired quadrilateral $\Xi_a$, provided that $r > 0$ is sufficiently large. The proof of Proposition 18 is therefore completed upon noting that $1 - \mathbb{P}(E_r^{(3)}) = \exp(-2\lambda r^{1/2}/3)$. □

## 5.2. **Auxiliary results for point-process based models**

In contrast to the Poisson line model considered in Section 5.1, it seems difficult to derive a moderate-deviation estimate for shortest-path lengths on Voronoi and Delaunay tessellations, as well as the relative neighborhood graph directly and our goal is therefore to apply the results of Section 4. In the present subsection, we show that condition (H6) is redundant in the sense that it can be deduced from conditions (H1) to (H5).

Let $g : \mathbb{N}_{\mathbb{M}} \to \mathbb{G}$ be a measurable, covariant construction-rule and let $\{\widetilde{g}_r\}_{r>1}$ be a family of approximations of $g$ such that $g(\varphi) \subset \widetilde{g}_r(\varphi)$ for all $r > 1$ and all $\varphi \in \mathbb{N}_{\mathbb{M}}$. As a first preliminary, we show that if (H1)–(H5) are satisfied at some fine resolution, then these conditions are also satisfied at a coarser resolution.

**Lemma 19.** *Let $\{\widetilde{A}_r\}_{r>1}$ be a family of events that satisfy conditions* (H1)–(H5) *for the construction rules $\{\widetilde{g}_r\}_{r>1}$. For every $r > 1$, put $\overline{g}_r = \widetilde{g}_{r/a'_r}$, where $r' = r/a'_r$ and $a'_r = 2\lceil r^{1-1/\alpha_1}\rceil + 1$, and define*

$$A'_r = \{\varphi \in \mathbb{N}_{\mathbb{M}} : (\varphi - r'z) \cap Q_{r'}^{\mathbb{M}}(o) \in \widetilde{A}_{r'} \text{ for all } z \in \mathbb{Z}^d \cap Q_{a'_r}(o)\}.$$

*Then, for each $r > 1$, the elements of $A'_r$ satisfy* (H1)–(H5) *for the construction rules $\{\overline{g}_r\}_{r>1}$ with $\alpha_1$ replaced by $\alpha_1 + 1$.*

*Proof.* In this proof, we assume that $\varphi \in \mathbb{N}_{\mathbb{M}}$ is such that $(\varphi - rz) \cap Q_r^{\mathbb{M}}(o) \in A'_r$ for all $z \in \mathbb{Z}^d$. Let $\psi \in \mathbb{N}_{\mathbb{M}}$ be such that $\psi \subset \mathbb{R}^d \setminus Q_{3r}^{\mathbb{M}}(o)$. Then, for every $z \in Q_{a'_r}(o) \cap \mathbb{Z}^d$,

$$\begin{aligned}\overline{g}_{r,\,\mathsf{geom}}\big((\varphi \cap Q_{3r}^{\mathbb{M}}(o)) \cup \psi\big) \cap Q_{r'}(r'z) &= \widetilde{g}_{r',\,\mathsf{geom}}\big((\varphi \cap Q_{3r}^{\mathbb{M}}(o)) \cup \psi\big) \cap Q_{r'}(r'z)\\ &= \widetilde{g}_{r',\,\mathsf{geom}}(\varphi \cap Q_{3r'}^{\mathbb{M}}(r'z)) \cap Q_{r'}(r'z),\end{aligned}$$

which equals $\widetilde{g}_{r',\,\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o)) \cap Q_{r'}(r'z)$. For conditions (H1$'$) and (H2) one can argue similarly. To verify condition (H3), let $x, y \in \overline{g}_{r,\,\mathsf{geom}}(\varphi) \cap Q_{5r}(o)$ be arbitrary. Furthermore, choose $z_1, \ldots, z_k \in Q_{5a'_r}(o)$ such that $x \in Q_{r'}(r'z_1)$, $y \in Q_{r'}(r'z_k)$ and $|z_{i+1} - z_i|_\infty \le 1$ for all $i \in \{1, \ldots, k-1\}$. Then, for every $i \in \{1, \ldots, k-1\}$, the points $q(r'z_i)$ and $q(r'z_{i+1})$ can be connected by a path in $\widetilde{g}_{r',\,\mathsf{geom}}(\varphi) \cap Q_{7r'}(r'z_i)$. Additionally, $x$ and $q(r'z_1)$ as well as $q(r'z_k)$ and $y$ can be connected by paths in $\widetilde{g}_{r',\,\mathsf{geom}}(\varphi) \cap Q_{7r'}(r'z_1)$ and $\widetilde{g}_{r',\,\mathsf{geom}}(\varphi) \cap Q_{7r'}(r'z_k)$, respectively. This proves (H3). Moreover, condition (H4′) is clearly satisfied. Finally,

$$\nu_1\big(\overline{g}_{r,\,\mathsf{geom}}(\varphi)\cap Q_{7r}(o)\big)=\sum_{z\in Q_{7a'_r}(o)}\nu_1\big(\widetilde{g}_{r',\,\mathsf{geom}}(\varphi)\cap Q_{r'}(r'z)\big)\le\sum_{z\in Q_{7a'_r}(o)}(r')^{\alpha_1}\le 7^d r^{\alpha_1}(a'_r)^{d-\alpha_1}.$$

Since $\alpha_1\ge d$ and $7^d r^{\alpha_1}\le r^{\alpha_1+1}$ for all sufficiently large $r>1$, this proves the claim. □

In order to verify condition (H6), we consider a refinement of the family of events $\{A'_r\}_{r\ge 1}$ considered in Lemma 19, which ensures that two specific points close to the left and right boundary of $Q_r(o)$ can be connected to $q(o)$ by a short path on $\overline{g}_{r,\,\mathsf{geom}}(\varphi)$. By establishing this short path using a large number of small intermediate steps, we can ensure that this event is not influenced by the configuration of $\varphi$ outside the sampling window $Q_r(o)$. To be more precise, we put $A_r=A'_r\cap B_r$, where $B_r$ denotes the family of all $\varphi\in\mathbb{N}_\mathbb{M}$, such that

$$\ell^{(r)}\big(q(ir'e_1,g_{\mathsf{geom}}(\varphi)),q((i+1)r'e_1,g_{\mathsf{geom}}(\varphi))\big)\le p_1r' \tag{11}$$

for all $i\in\mathbb{Z}$ such that $-(a'_r-1)/2+2p_1+1\le i\le(a'_r-1)/2-2p_1-1$, where $\ell^{(r)}$ denotes the shortest-path length in the segment process $\overline{g}_{r,\,\mathsf{geom}}(\varphi)$, and where $p_1\ge 1$ is chosen as in (3). Furthermore, as in Lemma 19, we put $r'=r/a'_r$.

**Lemma 20.** *For each $r>1$, the elements of $A_r$ satisfy* (H1)–(H6) *for the construction rules $\{\overline{g}_r\}_{r>1}$. Moreover, if the events $\{A'_r\}_{r>1}$ occur whp, then the events $\{A_r\}_{r>1}$ occur whp.*

*Proof.* In this proof, we assume that $(\varphi-rz)\cap Q_r^\mathbb{M}(o)\in A_r$ for all $z\in\mathbb{Z}^d$. It follows from Lemma 19 that conditions (H1)–(H5) are satisfied. Next, we claim that condition (H6) is satisfied with $\alpha_2=6p_1+5$. From condition (H1′) we conclude that for every $z\in Q_{a'_r-2}(o)$, we have

$$g_{\mathsf{geom}}\big(\varphi\cap Q_r^\mathbb{M}(o)\big)\cap Q_{r'}(r'z)=g_{\mathsf{geom}}(\varphi)\cap Q_{r'}(r'z),$$

which implies that

$$g_{\mathsf{geom}}\big(\varphi\cap Q_r^\mathbb{M}(o)\big)\cap Q_{r-2r'}(o)=g_{\mathsf{geom}}(\varphi)\cap Q_{r-2r'}(o).$$

Similarly, condition (H1) shows that these identities remain true when replacing $g_{\mathsf{geom}}$ by $\overline{g}_{r,\,\mathsf{geom}}$. In particular, $\varphi\cap Q_r^\mathbb{M}(o)\in B_r$ yields

$$\ell^{(r)}(q(ir'e_1,g_{\mathsf{geom}}(\varphi)),q((i+1)r'e_1,g_{\mathsf{geom}}(\varphi)))\le p_1r'$$

for all $i\in\mathbb{Z}$ with $-(a'_r-1)/2+2p_1+1\le i\le(a'_r-1)/2-2p_1-1$. Hence, $q(o,g_{\mathsf{geom}}(\varphi))$ and $q\big(((a'_r-1)/2-2p_1-1)r'e_1,g_{\mathsf{geom}}(\varphi)\big)$ can be connected by a path in $\overline{g}_{r,\,\mathsf{geom}}(\varphi)$ of length at most $p_1r$. Next, a repeated application of conditions (H3) and (H5) implies that $q(((a'_r-1)/2-2p_1-1)r'e_1,g_{\mathsf{geom}}(\varphi))$ and $q(((a'_r+1)/2+2p_1+1)r'e_1,g_{\mathsf{geom}}(\varphi))$ can be connected by a path in $\overline{g}_{r,\mathsf{geom}}(\varphi)$ of length at most $(4p_1+3)(r')^{\alpha_1}\le(4p_1+3)r$. We also observe that $q(o,\overline{g}_{\mathsf{geom}}(\varphi))$ and $q(o,\overline{g}_{r,\,\mathsf{geom}}(\varphi))$ as well as $q(re_1,g_{\mathsf{geom}}(\varphi))$ and $q(re_1,\overline{g}_{r,\,\mathsf{geom}}(\varphi))$ can be connected by paths in $\overline{g}_{r,\,\mathsf{geom}}(\varphi)$ of length at most $(r')^{\alpha_1}\le r$. Finally, the same reasoning used to construct a path connecting $q(o,g_{\mathsf{geom}}(\varphi))$ and $q(((a'_r-1)/2-2p_1-1)r'e_1,g_{\mathsf{geom}}(\varphi))$ can be applied to obtain a path connecting $q(re_1,g_{\mathsf{geom}}(\varphi))$ and $q(((a'_r+1)/2+2p_1+1)r'e_1,g_{\mathsf{geom}}(\varphi))$. Concatenation of all constructed paths shows that $q(o,\overline{g}_{r,\,\mathsf{geom}}(\varphi))$ and $q(re_1,\overline{g}_{r,\,\mathsf{geom}}(\varphi))$ can be connected by a path in $\overline{g}_{r,\,\mathsf{geom}}(\varphi)$ of length at most $(r+p_1r+(4p_1+3)r+p_1r+r)$, as desired. It remains to show that for $r>1$ the events $X\cap Q_r^\mathbb{M}(o)\in A_r$ occur whp. First, we observe that if $X\cap Q_r^\mathbb{M}(o)\in A'_r$, then $X\cap Q_r^\mathbb{M}(o)\in B_r$ is equivalent to $X\in B_r$. Moreover, from (3) and $\ell^{(r)}(\cdot,\cdot)\le\ell(\cdot,\cdot)$, we know that for $r>1$ the events $\ell^{(r)}(q(o,g_{\mathsf{geom}}(X)),q(r'e_1,g_{\mathsf{geom}}(X)))\le p_1r'$ occur whp. Hence, for $r>1$ also the events $B_r$ occur whp, which completes the proof of Lemma 20. □

We conclude this subsection with some observations concerning conditions (F1)–(F5) that will be used in the following. First, note that (F2) implies $f_{\mathsf{geom}}(\varphi)\subset f_{\mathsf{geom}}(\psi)$ for all $\varphi,\psi\in\mathbb{N}_\mathbb{M}$ with $\varphi\subset\psi$. Furthermore, recalling the appearance of $\tau$ in condition (F3), conditions (F3) and (F4) have important consequences on the asymptotic behavior of the number of points of $f_{\mathsf{geom}}(X)$ in bounded sampling windows.

**Lemma 21.** *Under conditions* (F3) *and* (F4)*, there exists* $c_0 > 0$ *such that*

$$\mathbb{P}\big(f_{\mathsf{geom}}(X) \cap Q^{\mathbb{M}}_{r-\tau}(o) = \emptyset\big) \le \exp(-c_0 r^d)$$

*for all sufficiently large* $r > 1$.

*Proof.* The cube $Q^{\mathbb{M}}_{r-\tau}(o)$ contains $k = \lfloor (r-2\tau)/2\tau \rfloor^d$ disjoint sub-cubes $Q_{r,1}, \ldots, Q_{r,k}$ of side length $2\tau$. For $i \in \{1, \ldots, k\}$ let $E_i = \big\{f_{\mathsf{geom}}(X) \cap (Q_{r,i} \ominus Q^{\mathbb{M}}_\tau(o)) \ne \emptyset\big\}$ denote the event that the intersection of $f_{\mathsf{geom}}(X)$ with the inner cube $Q_{r,i} \ominus Q^{\mathbb{M}}_\tau(o)$ is non-empty. Note that by (F3) and the Poisson assumption, the events $E_i$, $i \in \{1, \ldots, k\}$ are independent and (F4) implies $p = \mathbb{P}(E_i) > 0$. Hence,

$$\mathbb{P}\big(f_{\mathsf{geom}}(X) \cap Q^{\mathbb{M}}_{r-\tau}(o) = \emptyset\big) \le \prod_{i=1}^{k}(1 - \mathbb{P}(E_i)) = \exp\big(\big(k \log(1-p)/r^d\big) r^d\big). \qquad \square$$

Next, we consider the following result that can be seen as a complement to Lemma 21.

**Lemma 22.** *Under conditions* (F2) *and* (F5)*, there exists* $c > 0$ *such that for all sufficiently large* $r > 1$,

$$\mathbb{P}\big(\#(f_{\mathsf{geom}}(X) \cap Q^{\mathbb{M}}_r(o)) \ge c r^d\big) \le \exp(-r^d).$$

*Proof.* For $r > 1$ sufficiently large we can subdivide $Q^{\mathbb{M}}_r(o)$ into $k \le 2r^d/\tau^d$ not necessarily congruent sub-boxes $Q_1, \ldots, Q_k$ such that the longest side in each of these boxes is bounded from above by $\tau$. By sub-additivity we then obtain $\# f_{\mathsf{geom}}(X \cap Q^{\mathbb{M}}_r(o)) \le \sum_{i=1}^k \# f_{\mathsf{geom}}(X \cap Q_i)$ and note that by the Poisson assumption, the random variables $\# f_{\mathsf{geom}}(X \cap Q_i)$, $i = 1, \ldots, k$ are independent. Put $c_1 = \log \mathbb{E} \exp\big(h_0 \# f_{\mathsf{geom}}(X \cap Q^{\mathbb{M}}_\tau(o))\big)$. Then, for every $c > 0$,

$$\mathbb{P}\big(\# f_{\mathsf{geom}}(X \cap Q^{\mathbb{M}}_r(o)) \ge c r^d\big) \le \exp(-h_0 c r^d) \exp(k c_1) \le \exp\big(-r^d(h_0 c - 2c_1/\tau^d)\big). \qquad \square$$

## 5.3. Voronoi tessellations

In the present subsection, we verify that conditions (F1)–(F6) imply conditions (H1)–(H5) for the Voronoi tessellation. We begin by defining events $\widetilde{A}_r$ for $r > 1$ by putting

$$\widetilde{A}_r = \widetilde{C}_r \cap \widetilde{D}_r, \tag{12}$$

where

$$\widetilde{C}_r = \big\{\varphi \in \mathbb{N}_{\mathbb{M}} : f_{\mathsf{geom}}(\varphi) \cap Q_{r/(4d+1)-\tau}(rz/(4d+1)) \ne \emptyset \text{ for all } z \in \mathbb{Z}^d \cap Q_{4d+1}(o))\big\},$$
$$\widetilde{D}_r = \big\{\varphi \in \mathbb{N}_{\mathbb{M}} : \# f_{\mathsf{geom}}\big(\varphi \cap Q^{\mathbb{M}}_{r/(4d+1)}(rz/(4d+1))\big) \le c(r/(4d+1))^d \text{ for all } z \in \mathbb{Z}^d \cap Q_{4d+1}(o)\big\}.$$

Note that by Lemmas 21 and 22, for $r > 1$ the events $X \cap Q^{\mathbb{M}}_r(o) \in \widetilde{A}_r$ occur whp. Also note that (F3) shows that $\varphi \in \widetilde{C}_r$ implies $f_{\mathsf{geom}}\big(\varphi \cap Q^{\mathbb{M}}_{r/(4d+1)}(rz/(4d+1)) \cup \psi\big) \cap Q_{r/(4d+1)-\tau}(rz/(4d+1)) \ne \emptyset$ for all $z \in \mathbb{Z}^d \cap Q_{4d+1}(o)$ and all $\psi \in \mathbb{N}_{\mathbb{M}}$ with $\psi \subset \mathbb{R}^{d,\mathbb{M}} \setminus Q^{\mathbb{M}}_{r/(4d+1)}(rz/(4d+1))$. For the Voronoi tessellation no additional regularization of the construction rule is needed, so that we may choose $\widetilde{g}_r = \mathsf{Vor}$ for all $r \ge 1$. Next, we verify conditions (H1)–(H5).

**Lemma 23.** *The Voronoi graph satisfies conditions* (H1)–(H5).

*Proof.* Let $(\varphi - rz) \cap Q^{\mathbb{M}}_r(o) \in \widetilde{A}_r$ for all $z \in \mathbb{Z}^d$ and, for readability, put $a = 4d+1$. Moreover, the following arguments are valid for all sufficiently large $r > 1$. Consider the event $\widetilde{A}_r$ given in (12). To prove (H1), let $\psi, \psi' \in \mathbb{N}_{\mathbb{M}}$ with $\psi, \psi' \subset \mathbb{R}^{d,\mathbb{M}} \setminus Q^{\mathbb{M}}_{3r}(o)$ be arbitrary. Then, it suffices to show the following. For every

$x \in f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi)$ whose Voronoi cell $\Theta = Z(x, f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi))$ intersects $Q_r(o)$ it holds that $x \in f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi')$ and $\Theta \cap Q_r(o) = \Theta' \cap Q_r(o)$, where $\Theta' = Z(x, f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi'))$. Let $y \in \Theta \cap Q_r(o)$ be arbitrary and choose $z \in \mathbb{Z}^d \cap Q_a(o)$ such that $y \in Q_{r/a}(rz/a)$. Then, for all

$$x' \in f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi) \cap Q_{r/a-\tau}(rz/a) = f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{r/a}(rz/a)) \cap Q_{r/a-\tau}(rz/a)$$

we have $|x' - y| \le \sqrt{d} r/a < r/2 - \tau$. In particular, the center $x$ of $\Theta$ is contained in

$$f_{\mathsf{geom}}\big(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi\big) \cap Q_{3r-\tau}(o) = f_{\mathsf{geom}}\big(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi'\big) \cap Q_{3r-\tau}(o),$$

and $y \in \Theta'$. A similar argument verifies (H2), but we provide some details for the convenience of the reader. Let $\psi, \psi' \in \mathbb{N}_{\mathbb{M}}$ with $\psi \subset Q^{\mathbb{M}}_r(o)$ be arbitrary. Furthermore, let $\Theta$ be a cell of $\mathsf{Vor}_{\mathsf{geom}}\big((\varphi \setminus Q^{\mathbb{M}}_r(o)) \cup \psi\big)$ with $\Theta \not\subset Q_{3r}(o)$. Let $y \in \Theta \setminus Q_{3r}(o)$ and suppose that the center $x \in f_{\mathsf{geom}}\big((\varphi \setminus Q^{\mathbb{M}}_r(o)) \cup \psi\big)$ of $\Theta$ was not contained $(\mathbb{R}^d \setminus Q_r(o)) \ominus Q_\tau(o)$. Let $\eta \in \partial Q_{3r}(o)$ denote the intersection of $\partial Q_{3r}(o)$ with the line segment connecting $x$ and $y$ and choose $z \in Q_{3a}(o)$ such that $\eta \in Q_{r/a}(rz/a)$. Then, for all

$$x' \in f_{\mathsf{geom}}\big((\varphi \setminus Q^{\mathbb{M}}_r(o)) \cup \psi\big) \cap Q_{r/a-\tau}(rz/a) = f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{r/a}(rz/a)) \cap Q_{r/a-\tau}(rz/a)$$

we have $|\eta - x| \le \sqrt{d} r/a < |\eta - x|$, which contradicts the assumption that $\eta$ lies in the Voronoi cell $\Theta$ associated with $x$. In fact, this argument also shows that $y$ is contained in the Voronoi cell of $x$ with respect to $f_{\mathsf{geom}}\big((\varphi \setminus Q^{\mathbb{M}}_r(o)) \cup \psi'\big)$. In particular, we have verified (H2). For (H3), we observe that the following two statements are satisfied.

(1) the center of every Voronoi cell intersecting $Q_{5r}(o)$ is contained in $Q_{6r}(o)$ and
(2) every Voronoi cell whose center lies in $Q_{6r}(o)$ is contained in $Q_{7r}(o)$.

Similarly, for all $x \in Q_r(o)$ and all $y \in \partial B_{r/2}(x)$ the points $x$ and $y$ lie in different Voronoi cells. This proves (H4′). To check (H5) note that

(1) the center of every Voronoi cell intersecting $Q_{7r}(o)$ is contained in $Q_{8r}(o)$ and
(2) every Voronoi cell whose center is located in $Q_{8r}(o)$ is contained in $Q_{9r-\tau}(o)$.

As any edge in the Voronoi tessellation is determined by $d$ adjacent cells, we obtain that

$$\nu_1(\mathsf{Vor}(\varphi) \cap Q_{7r}(o)) \le 7r\sqrt{d}\big(\#\big(f_{\mathsf{geom}}(\varphi) \cap Q_{9r-\tau}(o)\big)\big)^d.$$

Hence, sub-additivity of $f_{\mathsf{geom}}$ yields that $\nu_1(\mathsf{Vor}(\varphi) \cap Q_{7r}(o))$ can be bounded from above by

$$7r\sqrt{d}\big(\#\big(f_{\mathsf{geom}}(\varphi) \cap Q_{9r-\tau}(o)\big)\big)^d \le 7r\sqrt{d}\big(\# f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{9r}(o))\big)^d \le 7r\sqrt{d}\Big(\sum_{z \in \mathbb{Z}^d \cap Q_{9a}(o)} \# f_{\mathsf{geom}}\big(\varphi \cap Q^{\mathbb{M}}_{r/a}(rz/a)\big)\Big)^d,$$

which is at most $7r\sqrt{d}(c9^d r^d)^d$. □

Finally, we note that when choosing $\varphi^{(0)} \in \mathbb{N}_{\mathbb{M}}$ such that $f_{\mathsf{geom}}(\varphi^{(0)}) = r_0\mathbb{Z}^d$, then $(\varphi^{(0)} - rz) \cap Q_r(o) \in A_r = A'_r \cap B_r$ provided that the discretization parameter $r$ is sufficiently large.

### 5.4. **Delaunay tessellation**

Using very similar arguments, the analogue of Lemma 23 holds for the Delaunay model, too. Also for the Delaunay tessellation no regularization of the construction rule is needed, so that we may choose $\widetilde{g}_r = \mathsf{Del}$ for all $r > 1$.

**Lemma 24.** *The Delaunay graph satisfies conditions* (H1)–(H5).

*Proof.* As in the case of Voronoi tessellations, we suppose that $(\varphi - rz) \cap Q_r^{\mathbb{M}}(o) \in \widetilde{A}_r$ for all $z \in \mathbb{Z}^d$. As in Lemma 23, the following arguments are valid for all sufficiently large $r > 1$ and we put $a = 4d+1$. Consider the events $\widetilde{A}_r$ given in (12). To prove (H1), let $\psi, \psi' \in \mathbb{N}_{\mathbb{M}}$ with $\psi, \psi' \subset \mathbb{R}^{d,\mathbb{M}} \setminus Q_{3r}^{\mathbb{M}}(o)$ be arbitrary. Furthermore, let $e$ be an edge of $\mathsf{Del}_{\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o) \cup \psi)$ with $e \cap Q_r(o) \neq \emptyset$. We show that $e$ also forms an edge in $\mathsf{Del}_{\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o) \cup \psi')$. In order to achieve this goal, we note that there exists a ball $B \subset \mathbb{R}^d$ containing $e$ and satisfying $f_{\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o) \cup \psi) \cap \mathrm{int}(B) = \emptyset$, where $\mathrm{int}(B)$ denotes the topological interior of $B$. If $B$ is not contained in $Q_{3r-\tau}(o)$, then there exists $z \in \mathbb{Z}^d \cap Q_{3a}(o)$ such that $Q_{r/a}(rz/a) \subset B \cap Q_{3r-\tau}(o)$. In particular,

$$\emptyset = Q_{r/a}(rz/a) \cap f_{\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o) \cup \psi) = Q_{r/a}(rz/a) \cap f_{\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o))$$

contradicting the assumption $(\varphi - rz) \cap Q_r^{\mathbb{M}}(o) \in \widetilde{C}_r$. Therefore, $B \subset Q_{3r-\tau}(o)$ which implies that

$$B \cap f_{\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o) \cup \psi) = B \cap f_{\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o)) = B \cap f_{\mathsf{geom}}(\varphi \cap Q_{3r}^{\mathbb{M}}(o) \cup \psi').$$

As (H2) can be shown by analogous arguments, the corresponding proof is omitted. For (H3), we observe that by the same argument as before any Delaunay cell intersecting $Q_{5r}(o)$ is already contained in $Q_{7r}(o)$. Furthermore, (H4′) holds trivially. In order to verify the final condition, observe that any Delaunay simplex intersecting $Q_{7r}(o)$ is already contained in $Q_{9r-\tau}(o)$. In particular, $\nu_1(\mathsf{Del}(\varphi) \cap Q_{7r}(o)) \leq 7r\sqrt{d}\big(\#\big(f_{\mathsf{geom}}(\varphi) \cap Q_{9r-\tau}(o)\big)\big)^2$. This shows that $\nu_1(\mathsf{Del}(\varphi) \cap Q_{7r}(o))$ can be bounded from above by

$$7r\sqrt{d}\big(\#\big(f_{\mathsf{geom}}(\varphi) \cap Q_{9r-\tau}(o)\big)\big)^2 \leq 7r\sqrt{d}\big(\# f_{\mathsf{geom}}(\varphi \cap Q_{9r}^{\mathbb{M}}(o))\big)^2 \leq 7r\sqrt{d}\Big(\sum_{z \in \mathbb{Z}^d \cap Q_{9a}(o)} \# f_{\mathsf{geom}}(\varphi \cap Q_{r/a}^{\mathbb{M}}(rz/a))\Big)^2,$$

which is at most $7r\sqrt{d}\big(c9^d r^d\big)^2$. □

As before, if we choose $\varphi^{(0)} \in \mathbb{N}_{\mathbb{M}}$ such that $f_{\mathsf{geom}}(\varphi^{(0)}) = r_0\mathbb{Z}^d$, then $(\varphi^{(0)} - rz) \cap Q_r^{\mathbb{M}}(o) \in A_r = A_r' \cap B_r$ provided that the discretization parameter $r$ is sufficiently large.

## 5.5. Relative neighborhood graph

When considering moderate-deviation properties, relative neighborhood graphs differ from Voronoi and Delaunay tessellations in two important aspects. First, they tend to be rather unstable in regions, where the underlying point process contains many pairs of atoms that are at (almost) identical distances. Second, it is more difficult to control boundary effects in the sense that even if the configurations of the point process inside two neighboring cubes seem non-pathological, undesirable effects might still occur due to interactions between points close to the interface of the two cubes. In contrast to Voronoi and Delaunay tessellations, we therefore need to regularize not only the underlying point process, but also the construction rule of the relative neighborhood graph.

In the present subsection, we show that the relative neighborhood graph $\mathsf{Rng}_{\mathsf{geom}}(X)$ satisfies conditions (H1)–(H5) if $X$ is an independently marked Poisson point process satisfying conditions (F1)–(F6). First, we introduce a suitable family of regularizations of construction rules $\big\{\widetilde{\mathsf{Rng}}_r\big\}_{r\geq 1}$, where we add some additional edges in order to guarantee good connectivity properties even under rather pathological configurations. To be more precise, for any $\varphi \in \mathbb{N}$, let $\widetilde{\mathsf{Rng}}_r(\varphi)$ denote a segment process on the vertex set $\varphi$ with the following edge set. Two vertices $x, y \in \varphi$ are connected by an edge in $\widetilde{\mathsf{Rng}}_r(\varphi)$ if

(1) $x$ and $y$ are connected by an edge in $\mathsf{Rng}(\varphi)$, or
(2) $|x-y| \leq r^{1/(2d+4)}$ and there exist $x_0, x_1, \ldots, x_m \in \varphi$ such that $\{x_0, x_1\} = \{x, y\}$, $|x_m - x_0| > r/4$ and $|x_0 - x_1| > |x_1 - x_2| > \cdots > |x_{m-1} - x_m|$.

The second possibility can also be rephrased as saying that $|x-y| \leq r^{1/(2d+4)}$ and there exists a descending chain starting from the line segment $[x, y]$ and leaving the $r/4$-ball around the starting point.

Our first goal is the construction of a family of events $\{\widetilde{A}_r\}_{r>1}$ such that conditions (H1)–(H5) are satisfied. To achieve this goal, we put $a_r = (4\lceil r^{1-1/(2d+4)}\rceil + 1)(4d+1)$ and proceed similarly to the Voronoi and the Delaunay model by defining

$$\widetilde{A}_r = \widetilde{C}_r \cap \widetilde{D}_r, \tag{13}$$

where

$$\widetilde{C}_r = \{\varphi \in \mathbb{N}_{\mathbb{M}} : f_{\mathsf{geom}}(\varphi) \cap Q_{r'-\tau}(r'z) \neq \emptyset \text{ for all } z \in \mathbb{Z}^d \cap Q_{a_r}(o)\},$$
$$\widetilde{D}_r = \{\varphi \in \mathbb{N}_{\mathbb{M}} : \# f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{r'}(r'z)) \leq c(r')^d \text{ for all } z \in \mathbb{Z}^d \cap Q_{a_r}(o)\},$$

and $r' = r/a_r$. Again, we observe that by Lemmas 21 and 22, for $r > 1$ the events $X \cap Q^{\mathbb{M}}_r(o) \in \widetilde{A}_r$ occur whp.

**Lemma 25.** *The family of segment processes* $\{\widetilde{\mathsf{Rng}}_r\}_{r>1}$ *satisfies conditions* (H1)–(H5).

*Proof.* As in the case of Voronoi tessellations, we suppose that $(\varphi - rz) \cap Q^{\mathbb{M}}_r(o) \in \widetilde{A}_r$ for all $z \in \mathbb{Z}^d$. The following arguments are valid for all sufficiently large $r > 1$. Consider the events $\widetilde{A}_r$ given in (13). To prove (H1), let $\psi, \psi' \in \mathbb{N}_{\mathbb{M}}$ with $\psi, \psi' \subset \mathbb{R}^{d,\mathbb{M}} \setminus Q^{\mathbb{M}}_{3r}(o)$ be arbitrary. Furthermore, let $e$ be an edge of $\widetilde{\mathsf{Rng}}_{r,\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi_1)$ with $e \cap Q_r(o) \neq \emptyset$. We show that $e$ also forms an edge in $\widetilde{\mathsf{Rng}}_{r,\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi_2)$. First, if $\nu_1(e) \leq r^{1/(2d+4)}$, then the existence of a descending chain starting from the line segment $e$ and leaving a ball of radius $r/4$ only depends on

$$f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi_i) \cap Q_{3r-\tau}(o) = f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o)) \cap Q_{3r-\tau}(o),$$

and in particular does not depend on $\psi_1$ or $\psi_2$. Hence, it remains to consider edges of the relative neighborhood graph $\mathsf{Rng}_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi_i)$, $i \in \{1,2\}$. Note that whether $e$ forms an edge in the graph $\mathsf{Rng}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi_i)$ only depends on $f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi_i) \cap (e \oplus B_{\nu_1(e)}(o))$ and therefore is independent of $\psi_1$ and $\psi_2$ if $\nu_1(e) \leq r/4$. Finally, if $\nu_1(e) \geq r/4$, then there exists a ball $B \subset \mathbb{R}^d$ with diameter $e$ and whose interior does not intersect $f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi_1)$. Moreover, there exists $z \in \mathbb{Z}^d \cap Q_{3a}(o)$ such that $Q_{r'}(r'z) \subset B \cap Q_{3r-\tau}(o)$. In particular,

$$\emptyset = f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o) \cup \psi_1) \cap Q_{r'-\tau}(r'z) = f_{\mathsf{geom}}(\varphi \cap Q^{\mathbb{M}}_{3r}(o)) \cap Q_{r'-\tau}(r'z)$$

contradicting the assumption $(\varphi - rz) \cap Q^{\mathbb{M}}_r(o) \in \widetilde{C}_r$ for all $z \in \mathbb{Z}^d \cap Q_3(o)$. For conditions (H1′) and (H2) very similar arguments can be used and therefore proofs are omitted. To prove (H3), let $x, y \in \widetilde{\mathsf{Rng}}_{r,\,\mathsf{geom}}(\varphi) \cap Q_{5r}(o)$ be arbitrary. As before, we note that $x, y$ are located on edges contained in $\overline{g}_{r,\,\mathsf{geom}}(\varphi) \cap Q_{6r}(o)$ whose endpoints are denoted by $x', x''$ and $y', y''$. Choose $z_1, \ldots, z_k \in Q_{6a_r}(o)$ such that $x' \in Q_{r'}(r'z_1)$, $y' \in Q_{r'}(r'z_k)$ and $z_{i+1} \in Q_3(z_i)$ for all $i \in \{1, \ldots, k-1\}$. Moreover, choose $x_i \in f_{\mathsf{geom}}(\varphi) \cap Q_r(r'z_i)$ for all $i \in \{1, \ldots, k\}$, where we assume that $x_1 = x'$ and $x_k = y'$. Note that $|x_i - x_{i+1}| \leq r^{1/(2d+4)}$ for all $i \in \{1, \ldots, k-1\}$, so that $x_i$ and $x_{i+1}$ are connected by an edge in $\widetilde{\mathsf{Rng}}_{r,\mathsf{geom}}(\varphi) \cap Q_{7r}(o)$ if there exists a descending chain starting at $[x_i, x_{i+1}]$ and leaving $B_{r/4}(x_i)$. On the other hand, if such a chain does not exist, then proceeding as in [1, Lemma 10] or [14, Lemma 6] one can show that $x_i$ and $x_{i+1}$ are connected by a path in $\mathsf{Rng}_{\mathsf{geom}}(\varphi) \cap Q_{7r}(o)$. Condition (H4′) is clearly satisfied, so that it remains to consider condition (H5). Observe that every edge of $\widetilde{\mathsf{Rng}}_{r,\mathsf{geom}}(\varphi)$ intersecting $Q_{7r}(o)$ is already contained in $Q_{9r-\tau}(o)$. In particular, $\nu_1\big(\widetilde{\mathsf{Rng}}_{r,\mathsf{geom}} \cap Q_{7r}(o)\big)$ is bounded from above by $7r\sqrt{d}\big(\#\big(f_{\mathsf{geom}}(\varphi) \cap Q_{9r-\tau}(o)\big)\big)^2$. Hence, $\nu_1\big(\widetilde{\mathsf{Rng}}_{r,\,\mathsf{geom}}(\varphi) \cap Q_{7r}(o)\big)$ can be bounded from above by

$$7r\sqrt{d}\big(\#\big(f_{\mathsf{geom}}(\varphi) \cap Q_{9r-\tau}(o)\big)\big)^2 \leq 7r\sqrt{d}\big(\# f_{\mathsf{geom}}\big(\varphi \cap Q_{9r}(o)\big)\big)^2 \leq 7r\sqrt{d}\Big(\sum_{z \in \mathbb{Z}^d \cap Q_{9a_r}(o)} \# f_{\mathsf{geom}}\big(\varphi \cap Q_{r'}(r'z)\big)\Big)^2,$$

which is at most $7r\sqrt{d}(c9^d r^d)^2$. □

As before, if we choose $\varphi^{(0)} \in \mathbb{N}_{\mathbb{M}}$ such that $f_{\mathsf{geom}}(\varphi^{(0)}) = r_0\mathbb{Z}^d$, then $(\varphi^{(0)} - rz) \cap Q_r^{\mathbb{M}}(o) \in A_r = A'_r \cap B_r$ provided that the discretization parameter $r$ is sufficiently large. We conclude this subsection by noting that for $r > 1$ the events $\widetilde{\mathsf{Rng}}_{r,\,\mathsf{geom}}(X) = \mathsf{Rng}_{\mathsf{geom}}(X)$ occur whp. By definition of the regularization $\widetilde{\mathsf{Rng}}_r$, it suffices to prove that whp there does not exist $x \in X \cap Q_{2r}^{\mathbb{M}}(o)$ for which there exists a descending chain starting from the spatial coordinate of $x$, leaving the $r/4$-ball centered at this point and consisting of segments of length at most $r^{1/(2d+4)}$. But this is shown in [14, Lemma 5]; see also [1, Lemma 11] and [9] for related results.

## 6. Applications

Next, we discuss two applications of Theorem 1 illustrating the usefulness of being able to control the deviation of $\ell_r$ from $\mu r$ up to an error term of order $r^{1/2+\varepsilon}$.

### 6.1. **Lower bounds on time constants**

To begin with, we show that the time constant $\mu$ of the planar Voronoi tessellation, the Delaunay tessellation and the relative neighborhood graph constructed from suitable Poisson-based point processes is strictly greater than 1. In [3] it has been shown rigorously that in the Poisson-Delaunay case $\mu \le 4/\pi$. Similarly, for the Poisson-Voronoi graph numerical results stated in [28] indicate $\mu \approx 1.145$. Using Theorem 1 we provide a rigorous proof that $\mu > 1$. This shows that the asymptotic behavior of shortest-path lengths in these random segment processes is genuinely different from the situation in Poisson line tessellations, where it is known that $\mu = 1$, see Section 5.1 and also [28].

**Theorem 26.** *Let $X$ denote an independently $\mathbb{M}$-marked Poisson point process and assume that conditions* (H1)–(H6) *are satisfied for the construction rules* $(g_{\mathsf{geom}}, \{\overline{g}_r\}_{r\ge 1})$. *Furthermore, assume that there exists $\theta \ge 1$ such that the probability that there exists an edge $e$ in $g_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(o)) \cap Q_{3r}(o)$ with* $\sin\angle(e, e_1) \in (-r^{-\theta}, r^{-\theta})$ *tends to* $0$ *as $r \to \infty$. Then $\mu > 1$.*

*Proof.* Fix $\delta = 1/(8\theta + 12)$. If $\gamma$ is any path in $G$ and $x, y \in \gamma$, then it is convenient to write $\gamma(x, y)$ for the sub-path of $\gamma$ starting at $x$ and ending at $y$. In the following, $\gamma = \gamma(n)$ denotes a shortest path on $g_{\mathsf{geom}}(X)$ connecting $q(o)$ to $q(ne_1)$, $n \ge 1$. We will show that the event $\nu_1(\gamma(n)) - n \ge n^{5/8}$ occurs whp, so that Theorem 1 yields $\mu > 1$. We begin by defining a site-percolation model on $\mathbb{Z}^d$ and put $r = n^\delta$. A site $z \in \mathbb{Z}^d$ is said to be *$n$-good* if there exists an edge $e$ in $g_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(rz)) \cap Q_{3r}(rz)$ with $\sin\angle(e, e_1) \in (-r^{-\theta}, r^{-\theta})$. In particular, if $z$ is $n$-bad, then for every edge $e = [x, y]$ in $g_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(rz)) \cap Q_{3r}(rz)$,

$$1 - \frac{|\langle x - y, e_1\rangle|}{|x - y|} \ge \frac{|x - y|^2 - \langle x - y, e_1\rangle^2}{(|x - y| + |\langle x - y, e_1\rangle|)|x - y|} \ge \big(\sin\angle(x - y, e_1)\big)^2/2 \ge r^{-2\theta}/2,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^d$. Hence, if $\gamma'$ is a path in $g_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(rz)) \cap Q_{3r}(rz)$ consisting of segments $[x_0, x_1], \ldots, [x_{k-1}, x_k]$, then

$$\nu_1(\gamma') - \langle x_k - x_0, e_1\rangle = \sum_{i=1}^{k} |x_i - x_{i-1}| - \langle x_i - x_{i-1}, e_1\rangle \ge \frac{r^{-2\theta}\sum_{i=1}^{k}|x_i - x_{i-1}|}{2} = \frac{r^{-2\theta}\nu_1(\gamma')}{2}. \tag{14}$$

The site-percolation process of $n$-good sites is clearly 5-dependent. Hence, [20, Theorem 0.0] shows that any $*$-connected component of $n$-good sites intersecting $\mathbb{Z}^d \cap Q_n(o)$ consists of at most $n^\delta$ sites whp. Here, we say that two sites are $*$-adjacent if their $|\cdot|_\infty$-distance equals 1. We conclude from Theorem 1 that $\gamma$ is contained in $Q_{nr}(o)$ whp. Furthermore, (H1) implies that whp for every $z \in \mathbb{Z}^d \cap Q_n(o)$ we have $g_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(rz)) \cap Q_{3r}(rz) = g_{\mathsf{geom}}(X) \cap Q_{3r}(rz)$. Therefore, whp there exists a set of $m \ge nr^{-4}$ distinct $n$-bad sites $\{z_1, \ldots, z_m\} \subset \mathbb{Z}^d$ such that

(1) $|z_i - z_j|_\infty \ge 5$ for all $i, j \in \{1, \ldots, m\}$ with $i \ne j$ and
(2) $\gamma \cap Q_r(rz_i) \ne \emptyset$ for all $i \in \{1, \ldots, m\}$.

For each $i \in \{1,\ldots,m\}$ we choose $x_i^{(1)}, x_i^{(2)} \in \gamma \cap \partial Q_{3r}(rz_i)$ such that $\gamma_i = \gamma(x_i^{(1)}, x_i^{(2)}) \subset Q_{3r}(rz_i)$ and $\gamma_i \cap Q_r(rz_i) \neq \emptyset$. Since the site $z_i$ is $n$-bad we can apply (14), to deduce that

$$\nu_1(\gamma_i) - \langle x_i^{(2)} - x_i^{(1)}, e_1\rangle \geq r^{-2\theta}\nu_1(\gamma_i)/2 \geq r^{-2\theta+1}.$$

Hence,

$$\nu_1(\gamma) - \langle q(ne_1) - q(o), e_1\rangle \geq \sum_{i=1}^m \nu_1(\gamma_i) - \langle x_i^{(2)} - x_i^{(1)}, e_1\rangle \geq mr^{-2\theta+1}.$$

Therefore, whp,

$$\nu_1(\gamma) - n \geq mr^{-2\theta+1} + (\langle q(ne_1) - q(o), e_1\rangle - n) \geq n^{1-4\delta}n^{-(2\theta-1)\delta} - n^{1/2} \geq n^{5/8},$$

so that Theorem 1 implies $\mu > 1$. □

We conclude the present subsection by showing that if we consider the planar Voronoi tessellation, the Delaunay tessellation or the relative neighborhood graph, then the second condition in Theorem 26 can be deduced from a simple condition on the underlying marked point process $X$.

**Lemma 27.** *Let $X$ be an independently and isotropically $\mathbb{M}$-marked homogeneous Poisson point process with intensity $\lambda$ and let $g : \mathbb{N} \to \mathbb{G}$ denote the planar Voronoi tessellation, the Delaunay tessellation or the relative neighborhood graph. Furthermore, assume that the probability that there exist distinct $x, y \in f_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(o))$ with $\sin\angle([x,y], e_1) \in (-r^{-\theta}, r^{-\theta})$ tends to $0$ as $r \to \infty$. Then, the probability that there exists an edge $e$ in $g_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(o)) \cap Q_{3r}(o)$ with $\sin\angle(e, e_1) \in (-r^{-\theta}, r^{-\theta})$ tends to $0$ as $r \to \infty$.*

*Proof.* In the Delaunay tessellation (and therefore also in the relative neighborhood graph) every edge of $g_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(o)) \cap Q_{3r}(o)$ is of the form $[x,y]$ for some $x, y \in f_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(o))$, so that the claim is immediate. Similarly, in the Voronoi model every edge of the random segment process $g_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(o)) \cap Q_{3r}(o)$ is perpendicular to a segment of the form $[x,y]$ for some $x, y \in f_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{M}}(o))$, so that isotropy of $X$ allows us to complete the proof of Lemma 27. □

Note that if $f'_{\mathsf{geom}} : \mathbb{N}_{\mathbb{M}} \to \mathbb{N}_{\mathbb{M}}$ describes a thinning of $\mathbb{N}_{\mathbb{M}}$ in the sense that $f'_{\mathsf{geom}}(\varphi) \subset f_{\mathsf{geom}}(\varphi)$ for all $\varphi \in \mathbb{N}_{\mathbb{M}}$, then in order to check the condition of Lemma 27 for $f'_{\mathsf{geom}}$, it suffices to verify it for $f_{\mathsf{geom}}$. Additionally, the following result shows that Lemma 27 can be applied to a large class of finite-range Poisson-cluster processes.

**Lemma 28.** *Let $\tau > 0$ and let $X$ denote an independently $\mathbb{N}_\tau$-marked homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda > 0$. Let $Y \in \mathbb{N}_\tau$ denote the typical mark of $X$. Moreover, assume that $\mathbb{E}\#Y < \infty$ and that there exists $\theta \geq 5$ such that the probability that there exist distinct $x, y \in Y$ with $\sin\angle([x,y], e_1) \in (-r^{-\theta}, r^{-\theta})$ is of order $o(r^{-d})$. Then, the probability that there exist distinct $x, y \in f_{\mathsf{geom}}(X \cap Q_{5r}^{\mathbb{N}_\tau}(o))$ with $\sin\angle([x,y], e_1) \in (-r^{-\theta}, r^{-\theta})$ tends to $0$ as $r \to \infty$.*

*Proof.* We distinguish two cases. First, let $E_{1,r}$ be the event that there exist $(x_0, \psi) \in X \cap Q_{5r}^{\mathbb{N}_\tau}(o)$ and $x, y \in \psi$ with $\sin\angle([x,y], e_1) \in (-r^{-\theta}, r^{-\theta})$. Then,

$$\mathbb{P}(E_{1,r}) \leq \lambda 5^d r^d \mathbb{P}\big(\sin\angle([x,y], e_1) \in (-r^{-\theta}, r^{-\theta}) \text{ for some } x, y \in Y\big),$$

which by our assumption tends to 0 as $r \to \infty$. Next, let $E_{2,r}$ be the event that there exist $(x_0, \psi), (x_0', \psi') \in X \cap Q_{5r}^{\mathbb{N}_\tau}(o)$ and $x \in \psi$, $x' \in \psi'$ with $\sin\angle([x_0 + x, x_0' + x'], e_1) \in (-r^{-\theta}, r^{-\theta})$. Furthermore, let $Y$, $Y'$ be two independent copies of the typical mark of $X$. Then,

$$\begin{aligned}\mathbb{P}(E_{2,r}) &\le \lambda^2\mathbb{E}\sum_{x\in Y}\sum_{x'\in Y'}\int_{Q_{5r}(o)}\int_{Q_{5r}(o)} 1_{(-r^{-\theta},r^{-\theta})}(\sin\angle([x_0+x,x_0'+x'],e_1))\mathrm{d}x_0\mathrm{d}x_0'\\ &\le 2\kappa_{d-1}\lambda^2\mathbb{E}\sum_{x\in Y}\sum_{x'\in Y'}5^dr^d5\sqrt{d}r\big(5\sqrt{d}r\cdot r^{-\theta}\big)^{d-1}\\ &\le 2\kappa_{d-1}\lambda^2 5^{2d}d^{d/2}r^{2d-\theta(d-1)}(\mathbb{E}\#Y)^2,\end{aligned}$$

which tends to 0 as $r\to\infty$. □

## 6.2. Moderate deviations for geodesics

In the present section, we prove Theorem 2, i.e., we provide a result on moderate deviations of geodesics from the straight line segment connecting their endpoints. Let $G$ be a stationary, isotropic and ergodic random segment process in $\mathbb{R}^d$ satisfying (2) as well as (G1) and (G2). Under these assumptions we prove the following variant of [16, Theorem 2.4] (see also [24, Lemma 4.1] for a related result), where we recall that for $\eta,\eta'\in G$ we write $R(\eta,\eta')$ for the set of all paths $\gamma$ in $G$ connecting $\eta$ and $\eta'$ and satisfying $\nu_1(\gamma)=\ell(\eta,\eta')$.

**Proposition 29.** *Let $\beta>3/4$ and $\sigma>0$ be arbitrary. Then, for $r\ge 1$ the events*

$$\sup_{\substack{\eta\in G\cap B_\sigma(o)\\ \eta'\in G\cap B_\sigma(re_1)}}\ \sup_{\substack{\gamma\in R(\eta,\eta')\\ x\in\gamma}}\mathsf{dist}(x,[\eta,\eta'])\le r^\beta$$

*occur whp.*

*Proof.* Fix an arbitrary value $\delta\in(0,1/(8d))$ and assume $\beta<1$ (we may do so without loss of generality). For $r\ge 1$ let $A_r$ denote the event that i) $q(o)\in Q_r(o)$, ii) $G\cap Q_r(o)$ is contained in a connected component of $G\cap Q_{3r}(o)$ and iii) $\nu_1(G\cap Q_{3r}(o))\le r^{d+1}$. For $r\ge 1$ let $E_r^{(1)}$ denote the event that there exist $z\in\mathbb{Z}^d$ with $r^\delta z\in Q_{3r}(o)$ and such that $G-r^\delta z\notin A_{r^\delta}$. Furthermore, let $E_r^{(2)}$ denote the event that there exist $\eta\in G\cap B_\sigma(o),\eta'\in G\cap B_\sigma(re_1)$ with $\mathsf{dist}(x,[\eta,\eta'])\ge r^\beta$ for some $x\in\gamma$ and $\gamma\in R(\eta,\eta')$. Note that if $E_r^{(2)}$ occurs, then there exist $z\in\mathbb{Z}^d$, $\gamma\in R(\eta,\eta')$ and $x\in\gamma\cap Q_{r^\delta}(r^\delta z)$ such that $r^\delta z\in Q_{3r}(o)$ and $\mathsf{dist}(x,[\eta,\eta'])\ge r^\beta$. It is easy to see that this also implies $\mathsf{dist}(x,[o,re_1])\ge r^{3/4+3\varepsilon/4}$, where $\varepsilon=\beta-3/4$. To fix ideas, we assume, additionally, that $\zeta=\langle r^\delta z,e_1\rangle/r\in[-1,2]$ and $|r^\delta z-\langle r^\delta z,e_1\rangle e_1|\ge r^{\beta'}$, where $\beta'=3/4+\varepsilon/2$. The remaining cases may be treated similarly. Finally, let $E_r^{(3,a)}$ and $E_r^{(3,b)}$ denote the events that there exists $z\in\mathbb{Z}^d$ with $r^\delta z\in Q_{3r}(o)$ and such that $\ell(o,r^\delta z)<\mu r^\delta|z|-r^{1/2+\varepsilon/2}$ and $\ell(r^\delta z,re_1)<\mu|r^\delta z-re_1|-r^{1/2+\varepsilon/2}$, respectively. If $E_r^{(2)}$ occurs, then sub-additivity of $\ell$ yields

$$\begin{aligned}\ell(o,re_1)-\mu r &\ge \ell(\eta,\eta')-\ell(\eta,o)-\ell(\eta',re_1)-\mu r=\ell(\eta,x)+\ell(x,\eta')-\ell(\eta,o)-\ell(\eta',re_1)-\mu r\\ &\ge \ell(o,r^\delta z)+\ell(r^\delta z,re_1)-2\ell(x,r^\delta z)-2\ell(\eta,o)-2\ell(\eta',re_1)-\mu r.\end{aligned}$$

Furthermore, if neither of $E_r^{(1)},E_r^{(3,a)}$ and $E_r^{(3,b)}$ occurs, then the last line is at least

$$\mu\big(|r^\delta z|+|r^\delta z-re_1|-r\big)-4r^{1/2+\varepsilon/2}\ge\mu r\big(\sqrt{\zeta^2+r^{2\beta'-2}}+\sqrt{(1-\zeta)^2+r^{2\beta'-2}}-1\big)-4r^{1/2+\varepsilon/2}.$$

Now,

$$\sqrt{\zeta^2+r^{2\beta'-2}}-\zeta=\frac{r^{2\beta'-2}}{\sqrt{\zeta^2+r^{2\beta'-2}}+\zeta}\ge r^{2\beta'-2}/8,$$

and similarly, $\sqrt{(1-\zeta)^2 + r^{2\beta'-2}} - (1-\zeta) \ge r^{2\beta'-2}/8$. Hence, $\ell(o, re_1) - \mu r \ge r^{1/2+\varepsilon/2}$, provided that $r > 0$ is sufficiently large. To complete the proof we note that by (G1) and (G2) for $r \ge 1$ the complements of the events $E_r^{(1)}$ occur whp and by (2) the complements of the events $E_r^{(3,a)}, E_r^{(3,b)}$ and $E_r^{(2)} \setminus (E_r^{(1)} \cup E_r^{(3,a)} \cup E_r^{(3,b)})$ occur whp. □

Using Proposition 29 we can now prove Theorem 2.

*Proof of Theorem* 2. Observe that if $x_1 \in Q_1(o)$, $x_2 \in \mathbb{R}^d$, $\gamma \in R(x_1, x_2)$ and $y \in \gamma$ and $z \in \mathbb{Z}^d$ are such that $x_2 \in Q_1(z)$ and $\mathsf{dist}(y, [x_1, x_2]) \ge |x_1 - x_2|^{3/4+\varepsilon}$, then

$$\mathsf{dist}(y, [x_1, x_2]) \ge |x_1 - x_2|^{3/4+\varepsilon} \ge \big(|z| - 2\sqrt{d}\big)^{3/4+\varepsilon} \ge |z|^{3/4+\varepsilon/2},$$

where the last inequality holds provided that $|z|$ is sufficiently large. Therefore, using Proposition 29 in conjunction with the isotropy assumption and the Borel-Cantelli lemma proves the claim. □

## 6.3. **Shortest-path trees**

We conclude this section by explaining how the framework of [16] can be used to see that Theorem 2 gives rise to non-trivial implications on the geometry of so-called shortest-path trees. First, we need to resolve a technical issue related to non-uniqueness of geodesics. Suppose $G$ forms a random segment process in $\mathbb{R}^d$ and $x_1, x_2 \in G^{(0)}$ are such that there exist several distinct paths of minimal length $\ell(x_1, x_2)$ connecting $x_1$ and $x_2$, where $G^{(0)}$ denotes the set of all endpoints of segments of $G$. In Section 4, we used the lexicographic ordering to select one of these paths. Although this rule is compatible with respect to translations of $G$, it is in general incompatible with respect to rotations. In order to preserve isotropy, we introduce a new selection rule involving additional randomness. To be more precise, we assign U([0, 1]) distributed, i.i.d. weights $\{U_v\}_{v\in G^{(0)}}$ to the endpoints of segments of $G$. These weights are assumed to be independent of $G$. Then, for each finite path $\gamma$ in $G$ starting and ending at some vertex of $G$, let $\sigma(\gamma)$ denote the sum of weights associated with the endpoints of segments occurring in $\gamma$. Finally, among all paths $\gamma$ connecting $x_1, x_2$ and such that $\nu_1(\gamma) = \ell(x_1, x_2)$ let $\rho(x_1, x_2)$ denote the path $\gamma$ with minimal value of $\sigma(\gamma)$.

Now, we introduce shortest-path trees, which loosely speaking can be thought of as the union of all shortest Euclidean paths emanating from a given point on the underlying segment process $G$. To be more precise, the *shortest-path tree* $\mathsf{spt}(G, \eta)$ *associated with $G$ and rooted at $\eta \in G$* is defined as a random segment process on the vertex set $\{\eta\} \cup G^{(0)}$. Two nodes $x, y \in \{\eta\} \cup G^{(0)}$ are connected by an edge in $\mathsf{spt}(G, \eta)$ if and only if the line segment $[x, y]$ is a subset of the geodesic $\rho(\eta, x)$, or of the geodesic $\rho(\eta, y)$. An illustration for the shortest-path tree on a Delaunay tessellation is shown in Figure 5. Here, we only draw shortest paths from the vertices to the root. The dots mark points for which the shortest path to the root is not unique.

From [16] we recall two further concepts. First, we consider the notion of asymptotic omnidirectionality.

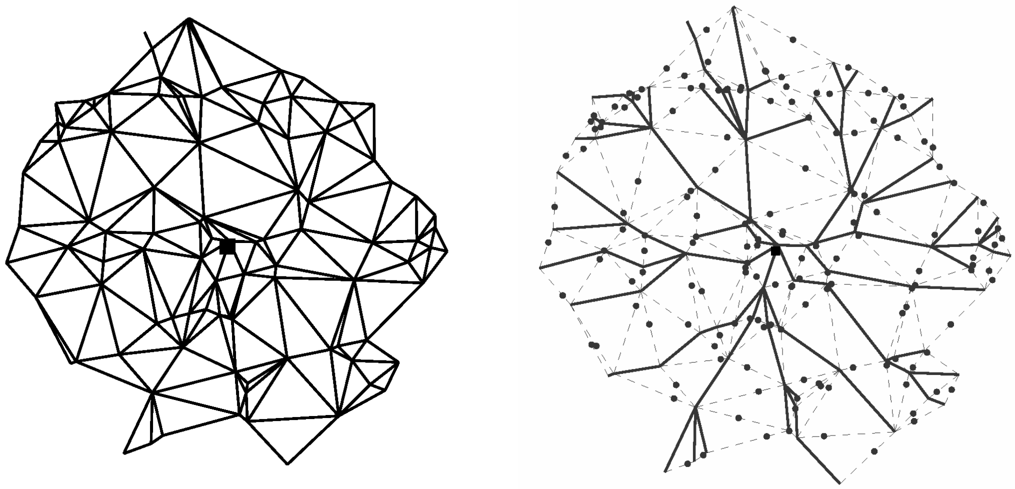
**Definition 30.** A locally finite set $\varphi \subset \mathbb{R}^d$ is said to be *asymptotically omnidirectional* if for all $k \ge 1$, the set $\{q/|q| : q \in \varphi, |q| > k\}$ is dense in the unit sphere $\partial B_1(o) = \{x \in \mathbb{R}^d : |x| = 1\}$.

Next, we discuss a.s. asymptotic omnidirectionality for the random segment processes under consideration.

**Lemma 31.** *Let $G$ be a stationary, isotropic and ergodic random segment process. Then, $G^{(0)}$ is a.s. asymptotically omnidirectional.*

*Proof.* For $v \in \partial B_1(o)$ and $\varepsilon \in (0, 1/2)$, let $A_{v,\varepsilon}$ denote the event that $\{q/|q| : q \in G^{(0)}\} \subset \partial B_1(o)$ does not contain any accumulation points in $\partial B_1(o) \cap B_\varepsilon(v)$. Isotropy implies $\mathbb{P}(A_{v,\varepsilon}) = \mathbb{P}(A_{e_1,\varepsilon})$ for any $v \in \partial B_1(o)$, and $\#G^{(0)} = \infty$ yields $\mathbb{P}(A_{e_1,\varepsilon}) < 1$. Ergodicity then shows $\mathbb{P}(A_{e_1,\varepsilon}) = 0$. Since $\varepsilon > 0$ was arbitrary, this proves the claim. □

For $x \in \mathbb{R}^d$ and $w \in [0, \pi/2)$ let $C(x, w) = \{y \in \mathbb{R}^d : |\angle([o, x], [o, y])| \le w\}$ denote the cone with apex $o$, axis $ox$ and angle $w$. Moreover, for $u, u' \in t^{(0)}$ let $t^{\mathsf{out}}(u, u')$ denote the set of all $u'' \in t^{(0)}$ such that the unique path on $t$ connecting $u''$ to $u$ contains $u'$. Then, we recall the definition of $\delta$-straightness from [16].



(a) Cut-out of Poisson-Delaunay tessellation and distinguished root

(b) Shortest-path tree

FIGURE 5. Construction of the shortest-path tree on a Poisson-Delaunay tessellation

**Definition 32.** For $\delta > 0$ a tree $t \subset \mathbb{R}^d$ is called *$\delta$-straight* at $u \in t^{(0)}$ if

$$t^{\mathsf{out}}(u, u') \subset u + C(u' - u, |u' - u|^\delta)$$

for all $u' \in t$ with $|u'|$ sufficiently large.

In order to verify $\delta$-straightness, the following deterministic result is useful.

**Lemma 33.** *Let $d \geq 2$, $\delta \in (0, 1/4)$. Then, there exists a constant $c > 0$ with the following property. Let $q_1, \dots, q_n \in \mathbb{R}^d$ be any sequence of distinct points in $\mathbb{R}^d$, such that*

1. $|q_i| \geq 3^{1/\delta}$ *for all* $i \in \{1, \dots, n\}$,
2. $|q_{j+1} - q_j| \leq |q_j|^{3/4}$ *for all* $j \in \{1, \dots, n\}$,
3. $\mathsf{dist}(q_j, [o, q_k]) \leq |q_k|^{1-\delta}$ *for all* $j, k \in \{1, \dots, n\}$ *with* $j < k$.

*Then,* $|\angle([o, q_j], [o, q_k])| \leq c|q_j|^{-\delta}$ *for all* $j, k \in \{1, \dots, n\}$ *with* $j < k$.

*Proof.* Lemma 33 constitutes a minor generalization of [16, Lemma 2.7] and the proof of the latter result in fact yields the presented more general claim. □

In particular, combining Lemma 33 with Corollary 2, we see that if $G$ is a stationary, isotropic, ergodic random segment process in $\mathbb{R}^d$ which satisfies (2) and conditions (G1) and (G2), then with probability 1 for every point $\eta \in G$ the shortest path tree $\mathsf{spt}(G, \eta)$ associated with $G$ and rooted at $\eta$ is $\delta$-straight at $\eta$.

For a subset $B \subset \mathbb{R}^d$ and a direction $v \in \partial B_1(o)$ we say that $B$ *has asymptotic direction* $v$ if

$$\lim_{\substack{|x| \to \infty \\ x \in B}} x/|x| = v.$$

Now, we deduce from [16, Proposition 2.8] and the above discussion the following result that will be used in Section 7.

**Corollary 34.** *Let $G$ be a stationary, isotropic and ergodic random segment process in $\mathbb{R}^d$, which satisfies* (2) *as well as conditions* (G1) *and* (G2). *Furthermore, write $T = \mathsf{spt}(G^*, o)$ for the shortest path tree with respect to the origin $o$. Then, with probability* 1, *the tree $T$ exhibits the following properties.*

1. *Every semi-infinite path in $T$ starting from $o$ has an asymptotic direction.*
2. *For every $v \in \partial B_1(o)$ there exists at least one semi-infinite path in $T$ starting from $o$ with asymptotic direction $v$.*
3. *The set $V$ of all $v$ such that there exists more than one semi-infinite path starting from $o$ with asymptotic direction $v$ is dense in $\partial B_1(o)$.*

## 7. Competition interfaces

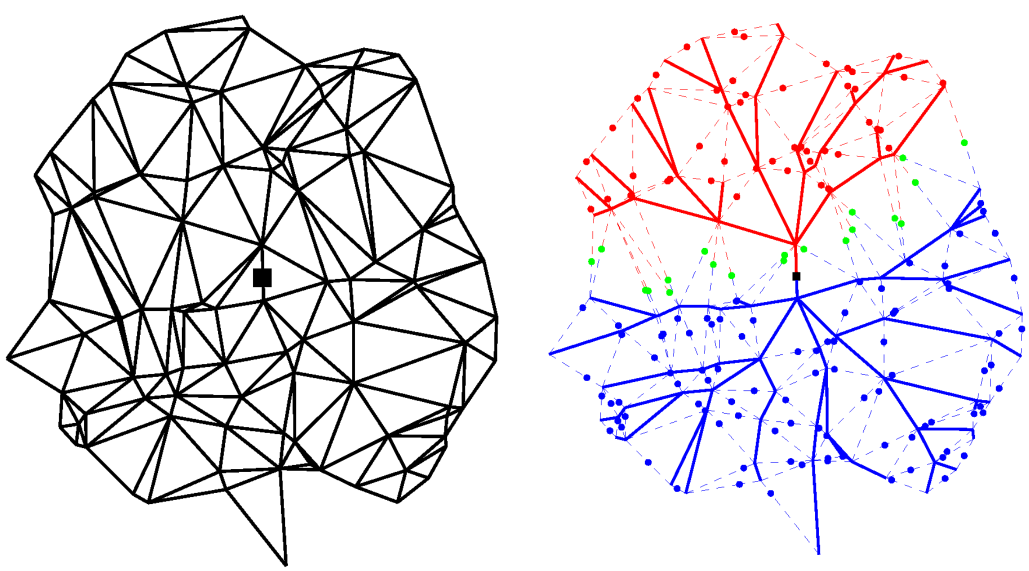
In the present section, we consider further implications of Theorem 1 for the case of planar segment processes. We explain to what extent modifications of the results on competition interfaces considered in [2] can be applied in the current setting. In the following, we assume that $G$ denotes a planar, stationary, isotropic and ergodic random segment process which satisfies (2) as well as conditions (G1) and (G2). Moreover, let $T = \mathsf{spt}(G^*, o)$ denote the shortest-path tree on the *Palm version* $G^*$ of $G$, i.e., on the random segment process whose distribution is determined by

$$\mathbb{E}h(G^*) = \frac{1}{\mathbb{E}\nu_1(G \cap [0,1]^2)} \mathbb{E} \int_{G \cap [0,1]^2} h(G - x)\nu_1(\mathrm{d}x),$$

where $h : \mathbb{G} \to [0, \infty)$ is any $\mathcal{G}$-measurable function. Note that since the origin lies in the interior of an edge of $G^*$ with probability 1, the tree $T$ can be decomposed into two subtrees $T_1$ and $T_2$ rooted at $o$. We first introduce the notion of competition interfaces based on [2, Definition 1].

**Definition 35.** The *competition interface* $\Gamma \subset G^*$ is defined as the subset consisting of all $x \in G^*$ such that for every $\varepsilon > 0$ there exist $y_1, y_2 \in B_\varepsilon(x) \cap G^*$ with $\rho(o, y_1) \cap \rho(o, y_2) = \{o\}$.

An illustration of the competition interface can be seen in Figure 6.



(a) Cut-out of Poisson-Delaunay tessellation and distinguished root

(b) Two subtrees of shortest-path tree (red and blue) and competition interface (green)

Figure 6. Construction of the competition interface on a Poisson-Delaunay tessellation

In the following, for any interval $I \subset \partial B_1(o)$ let $C_I = \{x \in \mathbb{R}^2 \mid \angle(e_1, [o, x]) \in I\}$ denote the planar sector of points whose angle with the $x$-axis is contained in $I$, where we identify $\partial B_1(o)$ with $[0, 2\pi] \mod 2\pi$. Then, we derive the following analog of [2, Proposition 9].

**Proposition 36.** *If both subtrees $T_1$ and $T_2$ are unbounded, then there exists a partition of the competition interface $\Gamma \subset G^*$ into subsets $\Gamma_1, \Gamma_2 \subset \Gamma$ such that both $\Gamma_1$ and $\Gamma_2$ admit an asymptotic direction. In particular, there exist random intervals $I_1, I_2 \subset [0, 2\pi]$ such that $\operatorname{int} I_1 \cap \operatorname{int} I_2 = \emptyset$, $I_1 \cup I_2 = \partial B_1(o)$ and for all $\delta > 0$ there exists an a.s. finite random $r_0 > 0$ such that $G^* \cap C_{I_i \ominus B_\delta(o)} \setminus B_{r_0}(o) \subset T_i$, and $T_i \setminus B_{r_0}(o) \subset C_{I_i \oplus B_\delta(o)}$ for all $i \in \{1, 2\}$.*

*Proof.* For the convenience of the reader, we recall the main ideas presented in [2, Proposition 9]. For $i \in \{1, 2\}$ let $\gamma_{i,h}$ and $\gamma_{i,\ell}$ denote the (trigonometrically) highest and lowest semi-infinite path contained in the subtree $T_i$. Note that the competition interface $\Gamma$ is contained in the union of two subsets $M_1, M_2 \subset \mathbb{R}^2$ enclosed by $\gamma_{1,h} \cup \gamma_{2,\ell}$ on the one hand and by $\gamma_{2,h} \cup \gamma_{1,\ell}$ on the other hand, see Figure 7. Furthermore, we conclude from Corollary 34 that $\gamma_{1,h}, \gamma_{2,h}, \gamma_{1,\ell}$ and $\gamma_{2,\ell}$ admit asymptotic directions $\theta_1, \theta_2, \theta_1'$ and $\theta_2'$, respectively. Also note that $\theta_1 = \theta_2'$ and $\theta_2 = \theta_1'$. Indeed, if $\theta_1 < \theta_2'$ then Corollary 34 would imply the existence of a semi-infinite path $\gamma''$ with asymptotic direction $\theta'' = (\theta_1 + \theta_2')/2$ and by the choice of $\gamma_{1,h}$ and $\gamma_{2,\ell}$, this semi-infinite path could be contained neither in $T_1$ nor in $T_2$. Therefore, $\Gamma \cap M_1$ admits the asymptotic direction $\theta_1$, whereas $\Gamma \cap M_2$ admits the asymptotic direction $\theta_2$.

To prove the remaining claims, we put $I_1 = [\theta_1', \theta_1]$ and $I_2 = [\theta_2', \theta_2]$, where the intervals are to be considered mod $2\pi$. We may restrict our attention to the case $I_1$ without loss of generality. To prove the first claim, let $\delta > 0$ be arbitrary. The curve $\gamma_{1,\ell} \cup \gamma_{1,h} \setminus (\gamma_{1,\ell} \cap \gamma_{1,h})$ subdivides $\mathbb{R}^2$ into two closed sets $A_1^{(1)}$ and $A_2^{(1)}$. Exactly one of these sets, say $A_1^{(1)}$, is disjoint from $T_2$, see Figure 7. Then, by definition of $\theta_1, \theta_1'$, all $x \in G^* \cap C_{[\theta_1' + \delta, \theta_1 - \delta]}$ with $|x|$ sufficiently large are contained in $A_1^{(1)}$. We omit the proof of the last claim, since it is shown using similar arguments. □

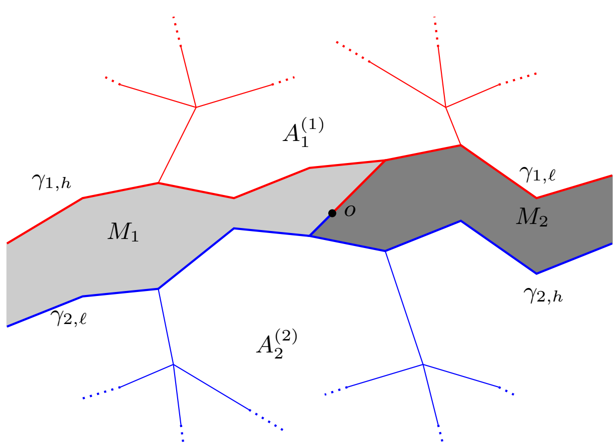



FIGURE 7. Subtrees $T_1$ (red) and $T_2$ (blue); geodesics $\gamma_{1,\ell}$, $\gamma_{1,h}$, $\gamma_{2,\ell}$ and $\gamma_{2,h}$; domains $M_1$ (light gray) and $M_2$ (dark gray)

**Remark.** The second part of Proposition 36 remains true in the case, where one of the subtrees, say $T_1$, is bounded. Indeed, then we put $I_1 = \emptyset$, $I_2 = [0, 2\pi)$ and make the convention that $C_\emptyset = \{o\}$.

**Example** Let $G^* \subset \mathbb{R}^2$ denote the Palm version of an isotropic planar Poisson line tessellation $G$ with intensity $\lambda > 0$. It is well-known that $G^*$ can be obtained from $G$ by adding an isotropic line $\ell_0$ through the origin. The origin separates $\ell_0$ into two rays and each of the subtrees $T_1$, $T_2$ contains precisely one of these rays. In particular, both $T_1$ and $T_2$ are unbounded with probability 1.

For general random segment processes, it is difficult to determine the probability of the event that both subtrees $T_1, T_2$ are unbounded. One can show that for Poisson-Voronoi and Poisson-Delaunay graphs one of the subtrees $T_1$, $T_2$ is bounded with positive probability. However, preliminary Monte Carlo simulations indicate that this appears to be a rather pathological event. We conclude this section by expressing a sufficient condition implying that with positive probability both subtrees $T_1, T_2$ are unbounded. First, we need a variant of the classical result on the uniqueness of semi-infinite paths in the two-dimensional case, which is based on the framework developed in [19] (see also [8, 15] for further applications). For the convenience of the reader, we present the details.

**Proposition 37.** *Let $v \in \partial B_1(o)$ be arbitrary. Then, with probability 1, for every $x \in G^{(0)}$ there exists exactly one semi-infinite path in $\mathsf{spt}(G, x)$ with asymptotic direction $v$ starting from $x$.*

*Proof.* If there exist two distinct semi-infinite paths $\gamma_1, \gamma_2$ in $T = \mathsf{spt}(G, x)$ starting from $x \in G^{(0)}$ and with asymptotic direction $v$, then $x_0$ denotes the last vertex common to both $\gamma_1$ and $\gamma_2$ and let $x_1$ be the successor vertex of $x_0$ in $\gamma_1$. Furthermore, we consider the subtree $T^{\mathsf{out}}(x_0, x_1)$ of $T$ introduced in Definition 32. As the semi-infinite sub-path of $\gamma_1$ starting from $x_1$ is contained in $T^{\mathsf{out}}(x_0, x_1)$, we conclude that $T^{\mathsf{out}}(x_0, x_1)$ is unbounded. The trigonometrically lowest and trigonometrically highest semi-infinite path in $T^{\mathsf{out}}(x_0, x_1)$ is denoted by $\gamma_\ell(x_0, x_1)$ and $\gamma_h(x_0, x_1) \subset T^{\mathsf{out}}(x_0, x_1)$ , respectively. As either $\gamma_\ell(x_0, x_1)$ or $\gamma_h(x_0, x_1)$ lies between $\gamma_1$ and $\gamma_2$ we also conclude that at least one of $\gamma_\ell(x_0, x_1)$ and $\gamma_h(x_0, x_1)$ admits the asymptotic direction $v$. Denoting by $D(v)$ the event that there exists a point in $G^{(0)}$ admitting two semi-infinite paths with asymptotic direction $v$ and by $S(v)$ the family of all semi-infinite paths with asymptotic direction $v$, an application of Fubini's theorem yields

$$\int_0^{2\pi} \mathbb{P}\big(D(v)\big)\mathrm{d}v = \mathbb{E}\nu_1\big(\{v \in \partial B_1(o) : D(v)\}\big) \le \mathbb{E} \sum_{x_0, x_1 \in X} 1_{T^{\mathsf{out}}(x_0,x_1) \text{ is unbounded}} \int_0^{2\pi} 1_{\{\gamma_\ell(x_0,x_1), \gamma_h(x_0,x_1)\} \cap S(v) \ne \emptyset} \mathrm{d}v,$$

which equals 0, as the integrand can be non-zero for at most two values in $[0, 2\pi)$. In particular, there exists $v \in \partial B_1(o)$ with $\mathbb{P}\big(D(v)\big) = 0$ and isotropy then completes the proof. □

It is useful to have a more specific description of the uniquely determined geodesic with a given direction.

**Lemma 38.** *Let $v \in \partial B_1(o)$ and $x \in G$ be such that there exists a unique semi-infinite path $\gamma$ in $\mathsf{spt}(G, x)$ with asymptotic direction $v$ starting from $x$. Furthermore, let $\{\zeta_n\}_{n \ge 1}$ be any semi-infinite path in $G$ with asymptotic direction $v$ starting from $x$. Then $\gamma$ is the unique semi-infinite path contained in $\bigcup_{n\ge 0} \rho(x, \zeta_n)$ starting from $x$. Additionally, for every $z \in \gamma$ it holds that $z \in \rho(x, \zeta_n)$ for all sufficiently large $n \ge 0$.*

*Proof.* From Corollary 2 and Lemma 33 we conclude that for all $\varepsilon > 0$ there exist $r, n_0 \ge 1$ such that for every $n \ge n_0$ the subset $\rho(x, \zeta_n) \setminus B_r(x)$ is contained in the cone at $x$ with axis direction $v$ and angle $\varepsilon$. In particular, $\bigcup_{n\ge 0} \rho(x, \zeta_n)$ admits the asymptotic direction $v$. Hence, our assumption implies that $\gamma$ is the unique semi-infinite path starting from $x$ that is contained in $\bigcup_{n\ge 0} \rho(x, \zeta_n)$. To verify the second claim, suppose that there exist infinitely many positive integers $n_i$ such that $z \notin \rho(x, \zeta_{n_i})$. Then, by the same argument as before, $\bigcup_{i\ge 0} \rho(x, \zeta_{n_i})$ admits the asymptotic direction $v$, so that $\gamma$ forms the unique semi-infinite path starting from $x$ that is contained in $\bigcup_{i\ge 0} \rho(x, \zeta_{n_i})$. In particular, $z \in \rho(x, \zeta_{n_i})$ for some $i \ge 1$. □

In the next result, we express a sufficient condition for the property that simultaneous unboundedness of the subtrees $T_1$ and $T_2$ occurs with positive probability.

**Proposition 39.** *Suppose that $\mathbb{P}\big(\bigcap_{x,y\in G^{(0)}} \{\#R(x, y) = 1\}\big) = 1$, i.e., with probability 1 any two vertices of $G$ are connected by a unique geodesic. Then, with positive probability, both subtrees $T_1$ and $T_2$ of $\mathsf{spt}(G^*, o)$ are unbounded.*

*Proof.* Let $I$ denote the subset of all $x \in G$ for which both subtrees of $\mathsf{spt}(G, x)$ are unbounded. By definition of the Palm version, it suffices to show that $I$ contains a segment of positive length with probability 1. If

$\gamma$ is a path in $G$ and $x, y \in G$ lie on $\gamma$, then it is convenient to write $\gamma(x, y)$ for the sub-path of $\gamma$ starting from $x$ and ending at $y$. If $\gamma$ is a semi-infinite path, we write $\gamma(x)$ for the semi-infinite sub-path starting from $x$. Let $x \in G^{(0)}$ be any point such that if $\gamma_+$ and $\gamma_-$ denote the unique semi-infinite paths starting from $x$ with asymptotic directions $e_1$ and $-e_1$, respectively, then $\gamma_+$ and $\gamma_-$ intersect only at $x$. We can find such a point $x$ by starting from an arbitrary element $x_0 \in G^{(0)}$ and considering the last point that is common to the semi-infinite paths starting from $x_0$ with asymptotic directions $-e_1$ and $e_1$, respectively. Furthermore, let $x'$ denote the successor vertex of $x$ in $\gamma_+$ and let $\gamma'_-$ denote the semi-infinite path with direction $-e_1$ starting from $x'$. See Figure 8 for an illustration of the configuration.

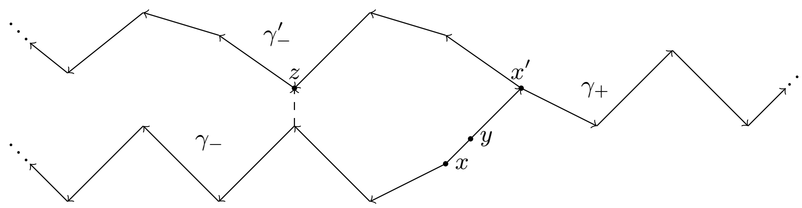



FIGURE 8. Semi-infinite geodesics $\gamma_-$, $\gamma'_-$ and $\gamma_+$

Then, for every $y \in (x, x')$ the shortest-path tree $\mathsf{spt}(G, y)$ contains the semi-infinite path $\gamma_+$, so that it suffices to show that there exists $y \in (x, x')$ such that the path $[y, x] \cup \gamma_-$ is contained in $\mathsf{spt}(G, y)$. Uniqueness of geodesics implies that $[x, x'] \cup \gamma'_-$ cannot form a semi-infinite path in $\mathsf{spt}(G, x)$. In particular, there exists a vertex $z \in \gamma'_-$ such that $\nu_1(\rho(x, z)) < |x - x'| + \nu_1(\gamma'_-(x', z))$. Therefore, we can choose $y \in (x, x')$ such that $|y - x| + \nu_1(\rho(x, z)) < |y - x'| + \nu_1(\gamma'_-(x', z))$. In particular, writing $\gamma_- = \{\zeta_n\}_{n \ge 0}$ we obtain for all $n \ge 0$ with $z \in \rho(x', \zeta_n)$ that

$$|y - x| + \nu_1(\rho(x, \zeta_n)) \le |y - x| + \nu_1(\rho(x, z)) + \nu_1(\rho(z, \zeta_n)) < |y - x'| + \nu_1(\gamma'_-(x', z)) + \nu_1(\rho(z, \zeta_n)),$$

which equals $|y - x'| + \nu_1(\rho(x', \zeta_n))$. Lemma 38 now implies that the path $[y, x] \cup \gamma_-$ is contained in $\mathsf{spt}(G, y)$. □

We conclude the present paper by providing an explicit example fitting into the framework described above. More precisely, we show that the Delaunay tessellation and the relative neighborhood graph on a suitable family of point processes satisfy the condition of Proposition 39. Indeed, we recall the following result from [9, Lemma 3.1]

**Lemma 40.** *Let $X$ be a stationary point process in $\mathbb{R}^d$ such that for every $n \ge 1$ the $n$th factorial moment measure of $X$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^{nd}$. Furthermore, let $k \ge 1$ and let $\{c_{i,j}\}_{1 \le i < j \le k}$ be such that $c_{i,j} \ne 0$ for some $i < j$. Then, the event that there exist pairwise distinct $x_1, \ldots, x_k \in X$ with $\sum_{1 \le i < j \le k} c_{i,j} |x_i - x_j| = 0$ occurs with probability $0$.*

For the Poisson-Voronoi model, the absolute continuity of the $n$th factorial moment measure corresponding to the point process of vertices is a non-trivial issue, which would requires a separate proof. The probability density of the second factorial moment measure is explicitly determined in [12], but it seems difficult to generalize the results to factorial moment measures of arbitrary order.

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