# Stationary Apollonian packings 

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#### Abstract

The notion of stationary Apollonian packings in the $d$-dimensional Euclidean space is introduced as a mathematical formalization of so-called random Apollonian packings and rotational random Apollonian packings, which constitute popular grain packing models in physics. Apart from dealing with issues of existence and uniqueness in the entire Euclidean space, asymptotic results are provided for the growth durations and it is shown that the packing is space-filling with probability 1 , in the sense that the Lebesgue measure of its complement is zero. Finally, the phenomenon is studied that grains arrange in clusters and properties related to percolation are investigated.


Keywords dense packing • fractal germ-grain model • continuum percolation • growth model
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## 1 Introduction

The task of creating dense packings of non-overlapping grains is a classical problem which continues to be a highly topical and important research question more than 2000 years after its initial appearance [1, 20, 23]. As noted in [8, so-called random Apollonian packings can be used for modeling dense packings of objects occurring in applications ranging from the study of tree crowns in dense forests [15, 25] to the analysis of structural properties in porous materials [12, 21, 26]. To create such packings one starts either from an initial set of objects with a specified size or with an initial set of germs from which grains start to grow radially at equal speed. Whenever two growing grains meet, both cease to grow. Then, iteratively, new germs are added, which grow until hitting an already existing grain. In order to provide more realistic models, the initial set of germs is usually placed at random. A realization of a two-dimensional random Apollonian packing is shown in Figure 1 . Denser packings can be achieved in so-called rotational random Apollonian packings, where each grain is rotated around its germ so as to maximize the time of growth until some other grain is hit. We refer the reader to [7] for a detailed analysis of this model.

From a mathematical perspective, it is an undesirable property of these packings that their distribution could depend on the size of the sampling window or on properties of the initial seed.

[^0]

Fig. 1: Realization of a planar random Apollonian packing

Therefore, in the present paper we study a framework where points arrive in the entire Euclidean space $\mathbb{R}^{d}$ according to a space-time point process of germs in $\mathbb{R}^{d} \times[0, \infty)$ which is stationary in its spatial component. Although random Apollonian packings are based on a seemingly simple growth mechanism, when considering extensions to the entire Euclidean plane, the size of a considered grain potentially depends on the configuration of the point process in arbitrarily far away regions. Indeed, in order to determine the growth duration of a considered grain, one requires some information on the growth durations of surrounding germs. A priori it is not clear that these iterated dependencies can be resolved and do not continue indefinitely. Hence, there is no simple expression that would allow one to compute the growth duration of a considered grain by taking into account the configuration of the process of grains in its surrounding. Due to this complicated dependency structure, the issue of existence of extensions of Apollonian packings to the entire Euclidean space is a non-trivial one.

The growth protocol described above is of lilypond type. These growth protocols have been investigated for almost 20 years, see e.g. [3, 4, 5, 9, 11, 13, 17], and by now there is a well-established toolkit that can be used to tackle the question of existence. Loosely speaking, this suggests to define stationary Apollonian packings as the packing that results from letting each grain grow until a certain growth stopping time is reached, which may vary from one grain to another. This family of growth-stopping times should satisfy two important properties. First, there are no overlappings between grains, and second the growth of any grain is stopped by getting into contact with some other grain. The first question considered in this paper deals with providing suitable sufficient conditions that not only imply the existence of a family of growth-stopping times satisfying the two constraints mentioned above, but also that this family is unique.

In a locally finite setting, i.e., if there are only a finite number of germs arriving in any bounded region of $\mathbb{R}^{d}$, the corresponding model is investigated in [9], where in addition to issues of existence and uniqueness, also questions of percolation are studied and a central limit theorem is established. In the present paper, after adapting classical methods to derive existence and uniqueness conditions for stationary Apollonian packings, we move in a slightly different direction. Our primary goal is to investigate effects that are distinctive for non-locally finite settings, i.e., for configurations where in any bounded region of the Euclidean space an infinite number of germs are born. For this purpose, we will assume that the process of germs is a spatially homogeneous Poisson point process, which is independently marked using certain star-shaped grain shapes. The precise conditions on these shapes will be discussed in Section 3 .

As mentioned above, there is no simple formula expressing the growth duration of a given germ in terms of properties of germs in its surrounding. Nevertheless, in the present paper, we show that it is possible to describe some asymptotic properties of these growth durations as the time of birth
tends to infinity. As time proceeds, the growth duration of newly-born germs should decrease, since available (pore) space becomes narrower, so that it takes less time until an already existing grain is hit. When considering a bounded sampling window, we show that growth durations decay at a polynomial speed, in the sense that we provide both upper and lower polynomial bounds for growth durations. We also provide numerical evidence for the conjecture that the upper bound is sharp.

Since in the setup considered in this paper germs arrive incessantly, the pore space left by stationary Apollonian packings is shrinking as time proceeds. Note that large holes in the pore space cannot persist for a long period of time, as grains induced by newly-arriving germs will fill up these holes quickly. Hence, it is natural to expect that stationary Apollonian packings are spacefilling, in the sense that almost surely, the pore-space volume tends to 0 as time tends to infinity. This will be shown in Section 5 .

Finally, we consider questions of percolation. Formally, percolation of stationary Apollonian packings can be captured by considering percolation on a directed graph on the process of germs, where an edge is drawn from one germ to another if the grain corresponding to the latter one stops the growth of the former one. Again, absence of percolation in lilypond models (see, e.g. [4, 11]) provides some useful intuition, but still care has to be taken. Indeed, the non-local finiteness induces effects that are not present in the classical setup. First, we show that there is absence of oriented percolation. This is in accordance with intuition, as directed edges tend to point in the direction of larger grains. Hence, the infinitely many small grains that are added in the non-locally finite model do not play a role. For non-oriented percolation, the rationale is different. Since bounded sampling windows typically contain an infinite number of germs, we cannot hope for the absence of percolation in the sense of obtaining cluster sizes consisting of finitely many grains. In fact, for a packing of balls, we show under homogeneous Poisson assumptions that, with probability 1, every cluster consists of infinitely many balls. On the other hand, we show that the reasoning for lilypond models can indeed be adapted to prove absence of infinite-volume clusters. In fact, we conjecture absence of percolation in the stronger sense that every cluster forms a bounded subset of $\mathbb{R}^{d}$.

Our paper is organized as follows. First, in Section 2, we introduce some basic notation and state our main results. Next, in Section 3, we derive a sufficient condition for existence and uniqueness of stationary Apollonian packings. In Section 4, we show under Poisson assumptions that the growth time of a grain whose germ had arrived at time $t>0$ decays according to a power law in $t$ and we provide rigorous upper and lower bounds for the exponent depending only on the dimension and the rate at which new germs enter the system. We also discuss the results of Monte Carlo simulations which have been performed to study the dependence of this exponent on the rate at which new germs arrive. Section 5 is devoted to the space-filling property of stationary Apollonian packings. In Section 6, we observe that in stationary Apollonian packings, grains of the final configuration arrange in clusters and we consider percolation-type properties including finiteness of the volume covered by each cluster and absence of percolation after any finite amount of time. We also provide simulation results for the number of connected components in a bounded sampling window depending on the rate at which new germs arrive. Finally, in Section 7, we present further conjectures and possible directions of future research.

## 2 Main results

Before analyzing various properties of stationary Apollonian packings, the first step is to prove a rigorous existence and uniqueness result. In contrast to classical random Apollonian packings where germs are added sequentially to a bounded sampling window and each grain grows until it touches one of the previously determined grains, we consider a model defined in the entire Euclidean space where several grains may grow simultaneously. In particular, the size of a given grain not only depends on the space-time location of earlier germs, but can also be influenced by germs which appear at a later point in time. As explained in Section 1, the well-definedness of such a model is a non-trivial issue, and we establish a sufficient condition for existence and uniqueness of stationary Apollonian packings based on the absence of a specific variant of descending chains. To make this precise, we first introduce some basic notation.

Let $d \geq 2$ be any fixed integer. We define $\mathbb{R}^{d,+}=\mathbb{R}^{d} \times[0, \infty)$. Let $Q_{r}(\xi)=\xi+[-r / 2, r / 2]^{d}$ denote the $d$-dimensional cube of side length $r>0$ centered at $\xi \in \mathbb{R}^{d}$. For any $B \subset \mathbb{R}^{d}$ we write int $B$ and $\partial B$ for the topological interior and the topological boundary of $B$, respectively. By $\mathcal{S}$ we denote the set of all compact, star-shaped subsets $B \subset \mathbb{R}^{d}$ such that 1) the star center is given by the origin $o \in \mathbb{R}^{d}$ and 2) int $B=\cup_{t<1} t B$, where $t B=\{t b: b \in B\}$. Note that the second condition has important implications. For instance, it allows us to conclude that for any $B \in \mathcal{S}$ we have $B \subset$ int $s B$ for any $s>1$. Additionally, if $\xi_{1}, \xi_{2} \in \mathbb{R}^{d}$ and $B_{1}, B_{2} \in \mathcal{S}$ satisfy $\left(\xi_{1}+\operatorname{int} B_{1}\right) \cap\left(\xi_{2}+\operatorname{int} B_{2}\right)=\emptyset$, then $\left(\xi_{1}+B_{1}\right) \cap\left(\xi_{2}+\operatorname{int} B_{2}\right)=\emptyset$. Furthermore, we put $\mathbb{R}^{d,+, \mathcal{S}}=\mathbb{R}^{d,+} \times \mathcal{S}$ and say that $\varphi \subset \mathbb{R}^{d,+, \mathcal{S}}$ is locally finite if $\varphi \cap(B \times \mathcal{S})$ is finite for every bounded Borel set $B \subset \mathbb{R}^{d,+}$. For $r_{0}>0$ we let $\mathbb{N}\left(r_{0}\right)$ denote the family of all locally finite subsets $\varphi$ of $\mathbb{R}^{d,+, \mathcal{S}}$ which are $r_{0}$-bounded in the sense that $\ell \subset Q_{r_{0}}(o)$ for all $(\xi, \tau, \ell) \in \varphi$. Finally, we put $\mathbb{N}^{*}=\cup_{r_{0}>0} \mathbb{N}\left(r_{0}\right)$.

If there is no collision with other grains, then we assume that any grain grows homothetically and linearly in time, where the center of the homothety is given by the spatial location of the associated germ. To be more precise, define the function $g: \varphi \times[0, \infty) \rightarrow \mathcal{B}\left(\mathbb{R}^{d}\right)$ by

$$
g((\xi, \tau, \ell), t)= \begin{cases}\xi+(t-\tau) \ell & \text { if } t \geq \tau \\ \emptyset & \text { otherwise }\end{cases}
$$

where $\mathcal{B}\left(\mathbb{R}^{d}\right)$ denotes the $\sigma$-algebra of Borel sets in $\mathbb{R}^{d}$. For $x \in \varphi$ write $\varphi_{x} \subset \varphi$ for the subset consisting of $x$ and those $y=\left(\eta, \sigma, \ell^{\prime}\right) \in \varphi$ with $\eta \in \operatorname{int} g(x, \sigma)$. In a sense, germs from $\varphi_{x}$ are not important for the growth of the grain at $x$. Indeed, the interaction between grains will be defined in such a way that any germ $y=\left(\eta, \sigma, \ell^{\prime}\right) \in \varphi_{x}$ does not influence the growth of the grain at $x$ : at time $\sigma$ either the grain corresponding to $x$ does not grow any longer, or the appearance of the germ $y$ does not play a role, since its spatial coordinate $\eta$ is covered by the interior of the grain at $x$.

Next, we define the notion of a family of growth-stopping times, which is closely related to the kind of dynamic lilypond models that has been introduced in 9]. These growth-stopping times are used to describe two key features of stationary Apollonian packings. On the one hand, they constitute hard-core particle packings in the sense that no two grains can overlap. That is, once a grain is in contact with some other grain, it stops growing. On the other hand, for any grain there exists a stopping neighbor provided that this grain does not grow for an unbounded amount of time and that its germ is not covered by an existing grain. Loosely speaking, the stopping neighbor can be thought of as the first grain that will get into contact with the given grain.

Definition 1 A function $f: \varphi \rightarrow[0, \infty]$ with $f(\xi, \tau, \ell) \geq \tau$ for all $(\xi, \tau, \ell) \in \varphi$ is said to define a family of $\varphi$-growth-stopping times if the following two conditions are satisfied.
(H) Hard-core property. (int $g(x, f(x))) \cap g(y, f(y))=\emptyset$ for all $x \in \varphi$ and $y \in \varphi \backslash \varphi_{x}$.
(N) Existence of stopping neighbors. If $x=(\xi, \tau, \ell) \in \varphi$ is such that $f(x)<\infty$, then there exists $y \in \varphi \backslash \varphi_{x}$ such that
(a) $\xi \in \operatorname{int} g(y, \min \{\tau, f(y)\})$, or
(b) $f(y) \leq f(x)$ and $g(x, f(x)) \cap g(y, f(y)) \neq \emptyset$.

In other words, the hard-core property says that grains cannot overlap. It also ensures that the spatial coordinate $\eta$ of a germ $y=\left(\eta, \sigma, \ell^{\prime}\right) \in \varphi$ can be covered by the interior of the grain at $x \in \varphi$ only if $y \in \varphi_{x}$. An element $y \in \varphi \backslash \varphi_{x}$ as in property ( N ) is called stopping neighbor of $x$. The existence of stopping neighbors means that if a grain does not grow for an unbounded amount of time and its germ is not covered by an existing grain, then there exists another grain with smaller or equal growth-stopping time, and which is in contact with the considered grain. Furthermore, for any given $\varphi$, we briefly say family of growth-stopping times instead of family of $\varphi$-growth-stopping times. As mentioned above, existence and uniqueness of growth-stopping times are non-trivial issues. We will derive a sufficient condition based on a specific notion of descending chains.

Definition 2 Let $\varphi \in \mathbb{N}^{*}$. A sequence $\left\{x_{n}\right\}_{n \geq 1}$ of elements in $\varphi$ is said to form a strong descending chain if there exists a sequence $\left\{t_{n}\right\}_{n \geq 1}$ of non-negative numbers such that

$$
\begin{equation*}
t_{n}>t_{n+1} \text { for all } n \geq 1 \tag{i}
\end{equation*}
$$

(ii) $\quad x_{n_{1}} \neq x_{n_{2}}$ for all $n_{1}, n_{2} \geq 1$ with $n_{1} \neq n_{2}$,
(iii) $g\left(x_{n}, t_{n}\right) \cap g\left(x_{n+1}, t_{n}\right) \neq \emptyset$ and $g\left(x_{n}, t_{n+1}\right) \cap g\left(x_{n+1}, t_{n+1}\right)=\emptyset$ for all $n \geq 1$.

For questions of percolation, a weaker form of descending chains will also play an important role. Since this concept is not needed to state our main results, we defer its introduction to Section 3.2, In Section 3, we prove the following result on the existence and uniqueness of limit configurations.
Theorem 1 Let $\varphi \in \mathbb{N}^{*}$ and assume that $\varphi$ does not contain strong descending chains. Then, there exists a unique family of $\varphi$-growth-stopping times $f_{\varphi}: \varphi \rightarrow[0, \infty]$.

Theorem 1 builds on earlier work regarding the well-definedness of various models of stochastic geometry in the whole space. As already mentioned in Section 1, the growth protocol underlying the stationary Apollonian packings is of lilypond type and for lilypond systems existence and uniqueness results have been thoroughly investigated in literature (see, e.g. [9, 11, 13]). However, the idea behind imposing the absence of descending chains, namely that the configuration in a bounded sampling window should not be influenced by points that are arbitrarily far away, also appears in the investigation of sandpile models [10]. Indeed, in [10] the absence of 'infinite backwards chains of topplings' is used to establish the well-definedness of limit configurations of topplings.

Next, we show that the sufficient condition in Theorem 1 is fulfilled with probability 1 for a large class of spatially stationary marked point processes. If $r_{0}>0$ and $(\Omega, \mathcal{F}, \mathbb{P})$ is any probability space, a function $\Phi: \Omega \rightarrow \mathbb{N}\left(r_{0}\right)$ is said to be a marked point process in $\mathbb{R}^{d,+, \mathcal{S}}$ if $\#(\Phi \cap B)$ is an integer-valued random variable for every Borel set $B \subset \mathbb{R}^{d,+, \mathcal{S}}$ whose projection to $\mathbb{R}^{d,+}$ is bounded. In the following, we say that $\Phi$ is spatially stationary, if the distribution of the marked point process $\{(\xi+\eta, t, L)\}_{(\xi, t, L) \in \Phi}$ does not depend on the choice of $\eta \in \mathbb{R}^{d}$. Additionally, we say that $\Phi$ is $m$ dependent if for any bounded Borel sets $A, B \subset \mathbb{R}^{d}$ that are of distance at least $m$ the marked point processes $\Phi \cap(A \times[0, \infty) \times \mathcal{S})$ and $\Phi \cap(B \times[0, \infty) \times \mathcal{S})$ are independent.
Theorem 2 Let $\Phi$ be a marked point process in $\mathbb{R}^{d,+, \mathcal{S}}$. Assume that $\Phi$ is spatially stationary and $m$-dependent for some $m \geq 1$. Then, almost surely, $\Phi$ does not contain a strong descending chain.

After having established a sufficient condition for the a.s. absence of strong descending chains in $\Phi$, we address the issue of percolation of stationary Apollonian packings. First, it is convenient to associate with each $\varphi \in \mathbb{N}^{*}$ a directed graph $G(\varphi)$ encoding the stopping-neighbor relation.

Definition 3 Let $\varphi \in \mathbb{N}^{*}$ and suppose that $\varphi$ does not admit strong descending chains. Define a directed graph $G(\varphi)$ on the vertex set $\varphi$ as follows. For $x, y \in \varphi$ an edge is drawn from $x$ to $y$ if and only if $y$ constitutes a stopping neighbor of $x$. Furthermore, $G^{\prime}(\varphi)$ denotes the undirected graph on the vertex set $\varphi$, where $x, y \in \varphi$ are connected by an edge if and only if there is an edge from $x$ to $y$ in $G(\varphi)$ or there is an edge from $y$ to $x$ in $G(\varphi)$.

Our goal is to analyze various properties of the connected components of $G(\Phi)$ and $G^{\prime}(\Phi)$ for a suitable class of spatially stationary marked point processes $\Phi$. A realization of the graph $G^{\prime}(\Phi)$ is shown in Figure 2 .

First, we consider the problem of oriented percolation in the directed graph $G(\Phi)$, recalling that oriented percolation is said to occur in a directed graph if it contains an infinite directed self-avoiding path. In the following, we say that the marked point process $\Phi$ is independently marked if it can be represented as $\left\{\left(x_{i}, L_{i}\right)\right\}_{i \geq 1}$, where $\left\{x_{i}\right\}_{i \geq 1}$ is a sequence of random vectors in $\mathbb{R}^{d,+}$ and $\left\{L_{i}\right\}_{i \geq 1}$ denotes a family of independent and identically distributed $r_{0}$-bounded random elements of $\mathcal{S}$ that are independent of $\left\{x_{i}\right\}_{i \geq 1}$. We also need the notion of second factorial moment measures of point processes, see e.g. [24].

Theorem 3 Let $m \geq 1$ and $\Phi$ be an independently marked point process, where the projection of $\Phi$ to $\mathbb{R}^{d,+}$ is a spatially stationary, $m$-dependent point process in $\mathbb{R}^{d,+}$ with absolutely continuous second factorial moment measure. Then, with probability 1 , there is no oriented percolation in $G(\Phi)$.

The issue of non-oriented percolation is more involved. Therefore, we now require additionally that with probability 1 each $x \in \Phi$ admits precisely one stopping neighbor $y \in \Phi$. In Section 6.3, we will present specific examples of stationary point processes satisfying this condition. In the following, for any $x \in \Phi$ let $C_{x} \subset G^{\prime}(\Phi)$ be the connected component of $G^{\prime}(\Phi)$ containing $x$.


Fig. 2: Realization of the graph $G^{\prime}(\Phi)$ and the underlying Apollonian packing

Theorem 4 Let $m \geq 1$ and $\Phi$ be an independently marked point process, where the projection of $\Phi$ to $\mathbb{R}^{d,+}$ is a spatially stationary, $m$-dependent point process in $\mathbb{R}^{d,+}$ with absolutely continuous second factorial moment measure. Furthermore, assume that with probability 1 each $x \in \Phi$ admits precisely one stopping neighbor $y \in \Phi$. Then, with probability 1 , the cluster volume $\nu_{d}\left(\bigcup_{y \in C_{x}} g\left(y, f_{\Phi}(y)\right)\right)$ is finite for all $x \in \Phi$, where $\nu_{d}$ denotes the Lebesgue measure in $\mathbb{R}^{d}$.

Theorem 4 can be seen as an analogue of the absence of percolation in classical lilypond models 4, 11. However, when investigating not the volume, but the number of grains in a cluster, then the behavior is radically different in the sense that with probability 1 , all clusters percolate. We prove this claim for a specific model, where the grains are constant and equal to $B_{1}(o)$, the unit ball in $\mathbb{R}^{d}$ which is centered at the origin.

Theorem 5 Consider the marked point process $\Phi=\Psi \times B_{1}(o)$, where $\Psi$ is a homogeneous Poisson point process in $\mathbb{R}^{d,+}$ with intensity $\lambda>0$. Then, almost surely, for every $x \in \Phi$ there exist infinitely many $y=\left(\eta, \sigma, B_{1}(o)\right) \in C_{x}$ with $f_{\Phi}(y)>\sigma$.

Next, we study the dependence of the growth duration of grains on the time of arrival of the corresponding germ. We will see that asymptotically this duration is contained within some polynomial bounds. In order to derive these bounds, we assume that $\Phi$ is an independently marked Poisson point process such that the intensity function $\lambda: \mathbb{R}^{d,+} \rightarrow[0, \infty)$ of the underlying unmarked Poisson point process is given by

$$
\begin{equation*}
\lambda(\xi, \tau)=\lambda(1+\tau)^{\alpha} \tag{1}
\end{equation*}
$$

for some $\alpha>-1$ and $\lambda>0$. Note that for $\alpha=0$, we obtain a stationary space-time Poisson point process in $\mathbb{R}^{d,+}$. But by choosing $\alpha$ to be either strictly negative or strictly positive, we can also consider models where the rate at which new germs appear either decreases or increases over time.

First, we derive an upper bound. For $\varepsilon \in(0,1), t>0$ let $E_{t, \varepsilon}$ denote the event that there exists $x=(\xi, \tau, \ell) \in \Phi \cap\left(Q_{1}(o) \times[t, \infty) \times \mathcal{S}\right)$ with $f_{\Phi}(x)-\tau \geq \tau^{-\alpha_{1}+\varepsilon}$, where $\alpha_{1}=(\alpha+1) / d$.
Theorem 6 Let $\Phi$ be an independently marked Poisson point process such that the intensity function $\lambda: \mathbb{R}^{d,+} \rightarrow[0, \infty)$ of the underlying unmarked Poisson point process is given by (1). Furthermore, assume that there exists $r_{1}>0$ such that $Q_{r_{1}}(o) \subset L$ for all $(\xi, \tau, L) \in \Phi$. Then, for every $\varepsilon \in(0,1)$ there exist $t_{1}, c_{1}>0$ such that $\mathbb{P}\left(E_{t, \varepsilon}\right) \leq \exp \left(-t^{c_{1}}\right)$ for all $t \geq t_{1}$.

In particular, using the Borel-Cantelli lemma shows that with probability 1 there exists a random time $T_{1} \in(0, \infty)$ such that

$$
\begin{equation*}
f_{\Phi}(x)-\tau \leq \tau^{-\alpha_{1}+\varepsilon} \text { for all } x=(\xi, \tau, \ell) \in \Phi \cap\left(Q_{1}(o) \times\left[T_{1}, \infty\right) \times \mathcal{S}\right) \tag{2}
\end{equation*}
$$

Next, we derive a lower bound on the growth duration, which corresponds to (2). Here and in the following, let $S O_{d}$ denote the group of rotations in $\mathbb{R}^{d}$. Of course, the definition of independently marked point processes can easily be modified to allow marks in $S O_{d}$ instead of marks in $\mathcal{S}$. Furthermore, we put $\alpha_{2}=(1+1 / d)(\alpha+1)$.

Theorem 7 Let $\Phi^{\prime}=\left\{\left(\xi_{i}, \tau_{i}, \Theta_{i}\right)\right\}_{i \geq 1}$ be an independently $S O_{d}$-marked Poisson point process such that the intensity function $\lambda: \mathbb{R}^{d,+} \rightarrow[0, \infty)$ of the underlying unmarked Poisson point process is given by (11). Put $\Phi=\left\{\left(\xi_{i}, \tau_{i}, \Theta_{i}(A)\right)\right\}$, where $A=\left\{z \in \mathbb{R}^{d}: \beta(z) \leq 1\right\} \in \mathcal{S}$ is the unit ball with respect to a certain norm $\beta(\cdot)$ on $\mathbb{R}^{d}$. Let $\varepsilon \in(0,1)$ be arbitrary. Then with probability 1 , there exists a random time $T_{2}<\infty$ such that $f_{\Phi}(x)-\tau \geq \tau^{-\alpha_{2}-\varepsilon}$ for all $x=(\xi, \tau, \ell) \in \Phi \cap\left(Q_{1}(o) \times\left[T_{2}, \infty\right) \times \mathcal{S}\right)$ such that $f_{\Phi}(x)>\tau$.

Finally, under the same assumptions as in Theorem 7, we show that the stationary Apollonian packing $\operatorname{AP}(\Phi)$ defined by

$$
\operatorname{AP}(\Phi)=\bigcup_{x \in \Phi} \operatorname{int} g\left(x, f_{\Phi}(x)\right)
$$

is a.s. space-filling. To be more precise, considering the pore space $\mathbb{R}^{d} \backslash \mathrm{AP}(\Phi)$ as a stationary random closed set [24], we show that with probability 1 its Lebesgue measure is equal to 0 .

Theorem 8 Let $\Phi$ be an independently marked point process that is constructed as in Theorem 7. Then, the packing $\operatorname{AP}(\Phi)$ is a.s. space-filling, i.e., $\mathbb{P}\left(\nu_{d}\left(\mathbb{R}^{d} \backslash \operatorname{AP}(\Phi)\right)=0\right)=1$.

## 3 Existence and uniqueness

Section 3.1 is devoted to the proof of Theorem 1. In Section 3.2, we relate absence of $\ell$-descending chains to absence of percolation in specific graphs and also show that the almost sure absence of percolation holds for a large class of independently marked point processes, which contains e.g. spatially homogeneous Poisson point processes with absolutely continuous intensity function. We also discuss a specific family of not necessarily independently marked models consisting of stationary approximations to rotational random Apollonian packings.

### 3.1 A sufficient condition based on absence of descending chains

As observed in [9, Apollonian packings are intimately related to the well-studied lilypond models initially introduced in [11]. Various techniques have been established to prove existence and uniqueness of lilypond models on different levels of generality. For our purposes, the approaches described in [6, 9, 13] turn out to be most suitable. However, the main difference from [9] lies in the assumptions on the space-time process of germs. While this process is assumed to be spatially locally finite in [9], we deal with configurations that are locally finite only in the space-time domain. Considering finite-time approximations and using further adaptations, the techniques developed in [6, 9] are powerful enough to derive the desired results in the present setting. Still, to make our presentation self-contained, we provide a detailed proof of Theorem 1. We first consider models truncated at a finite time $t_{0}>0$, where it will be convenient to use the abbreviations $\mathbb{R}^{d, t_{0}, \mathcal{S}}=\mathbb{R}^{d} \times\left[0, t_{0}\right] \times \mathcal{S}$ and $\mathbb{N}^{t_{0}, *}=\left\{\varphi \in \mathbb{N}^{*}: \varphi \subset \mathbb{R}^{d, t_{0}, \mathcal{S}}\right\}$. Furthermore, for the proof it is convenient to consider some variants of conditions $(\mathrm{H})$ and (N), which are less intuitive but more convenient for proofs. Later, in Lemma 10, we will see that when allowing also $t_{0}=\infty$, then these new conditions are equivalent to conditions $(\mathrm{H})$ and $(\mathrm{N})$.

Definition 4 Let $t_{0}>0$ and $\varphi \in \mathbb{N}^{t_{0}, *}$. Then, a function $f: \varphi \rightarrow\left[0, t_{0}\right]$ with $f(\xi, \tau, \ell) \geq \tau$ for all $(\xi, \tau, \ell) \in \varphi$ is said to define a family of $\left(t_{0}, \varphi\right)$-growth-stopping times if the following two conditions are satisfied.
$\left(\mathrm{H}^{\prime}\right) \quad$ Hard-core property. $(\operatorname{int} g(x, f(x))) \cap g(y, \min \{f(x), f(y)\})=\emptyset$ for all $x \in \varphi$ and $y \in \varphi \backslash \varphi_{x}$.

Existence of stopping neighbors. For all $x \in \varphi$ with $f(x)<t_{0}$ there exists $y \in \varphi \backslash \varphi_{x}$ with $g(x, f(x)) \cap g(y, \min \{f(x), f(y)\}) \neq \emptyset$.

For Lemmas 119 and Corollaries $1 / 2$ we assume that $t_{0}>0$ and $\varphi \in \mathbb{N}^{t_{0}, *}$. If $t_{0}$ and $\varphi$ are given, then we also say family of growth-stopping times instead of family of $\left(t_{0}, \varphi\right)$-growth-stopping times. Note that most arguments in the present section can be extended to the case, where the $r_{0}$-boundedness assumption of the grains is replaced by the weaker condition that the system of grains $\{\ell+\xi\}_{x=(\xi, \tau, \ell) \in \varphi}$ defines a locally finite family of compact sets. Nevertheless, in order to make the presentation more accessible, we restrict ourselves to the $r_{0}$-bounded case.

Another very useful property of growth-stopping times - which initially has been observed in 13 - is their possible interpretation as fixed points of a specific operator $T_{t_{0}}:\left[0, t_{0}\right]^{\varphi} \rightarrow\left[0, t_{0}\right]^{\varphi},\left[0, t_{0}\right]^{\varphi}$ denoting the family of functions from $\varphi$ to $\left[0, t_{0}\right]$. Given a proposal $h: \varphi \rightarrow\left[0, t_{0}\right]$ for a family of growth-stopping times and a germ $x \in \varphi$ the value of the new function $T_{t_{0}} h$ evaluated at $x$ is given by the largest time $t$ such that the grain $g(x, t)$ intersects neither another growing grain nor any grain that has stopped growing at the time described by the function $h$. More precisely, for $h: \varphi \rightarrow\left[0, t_{0}\right]$ define the function $T_{t_{0}} h: \varphi \rightarrow\left[0, t_{0}\right]$ by

$$
\left(T_{t_{0}} h\right)(x)=\sup \left\{t \in\left[0, t_{0}\right]: g(x, t) \cap \cup_{y \in \varphi \backslash \varphi_{x}} g(y, \min \{t, h(y)\})=\emptyset\right\}
$$

Additionally, let $\left(S_{t_{0}} h\right)(x)$ denote the set of all points of $\varphi$ where the above supremum is assumed, i.e., all $y=\left(\eta, \sigma, \ell^{\prime}\right) \in \varphi \backslash \varphi_{x}$ such that

$$
g\left(x, T_{t_{0}} h(x)\right) \cap g\left(y, \min \left\{T_{t_{0}} h(x), h(y)\right\}\right) \neq \emptyset
$$

Our interest in $T_{t_{0}}$ is based on the following observation.
Lemma 1 Let $f \in\left[0, t_{0}\right]^{\varphi}$ with $f(\xi, \tau, \ell) \geq \tau$ for all $(\xi, \tau, \ell) \in \varphi$. Then, $f$ defines a family of growth-stopping times if and only if $T_{t_{0}} f=f$. Additionally, under these equivalent conditions, if $x \in \varphi$ is such that $f(x)<t_{0}$, then every element of $S_{t_{0}} f(x)$ constitutes a stopping neighbor of $x$.

Proof. First, assume $T_{t_{0}} f=f$ and observe that the hard-core property is immediately implied by the definition of $T_{t_{0}}$. Furthermore, if $x \in \varphi$ is such that $f(x)<t_{0}$, then clearly any $y \in\left(S_{t_{0}} f\right)(x)$ forms a stopping neighbor of $x$.

To prove the other direction, assume that $f$ defines a family of growth-stopping times and let $x \in \varphi$ be arbitrary. For the inequality $T_{t_{0}} f(x) \leq f(x)$ assume for the sake of deriving a contradiction that there exists $t>0$ with $t \in\left(f(x), T_{t_{0}} f(x)\right)$ and let $y \in \varphi$ be a stopping neighbor of $x$. Then $g(x, t) \cap g(y, \min \{t, f(y)\}) \neq \emptyset$, which contradicts the definition of $T_{t_{0}} f(x)$. To show $T_{t_{0}} f(x) \geq f(x)$ assume that there exists $t>0$ with $t \in\left(T_{t_{0}} f(x), f(x)\right)$ and let $y \in S_{t_{0}} f(x)$. In particular, using condition 2) of the definition of $\mathcal{S}$ we conclude that $g\left(x, T_{t_{0}} f(x)\right) \subset \operatorname{int} g(x, t)$, so that (int $\left.g(x, t)\right) \cap$ $g(y, \min \{t, f(y)\}) \neq \emptyset$. However, this yields a contradiction to the hard-core property.

We note the following useful properties of $T_{t_{0}}$ and refer the reader to [13, Proposition 3.1] for a detailed discussion of the respective statements for lilypond models.

Lemma 2 Let $h_{1}, h_{2} \in\left[0, t_{0}\right]^{\varphi}$ be such that $h_{1}(x) \leq h_{2}(x)$ for all $x \in \varphi$. Then $T_{t_{0}} h_{1}(x) \geq T_{t_{0}} h_{2}(x)$ for all $x \in \varphi$.

Proof. Applying the definition of $T_{t_{0}}$ shows immediately that $T_{t_{0}} h_{1}(x) \geq T_{t_{0}} h_{2}(x)$ for all $x \in \varphi$.
Lemma 3 Let $\left(h_{n}\right)_{n \geq 1}$ be a sequence of functions $h_{n}: \varphi \rightarrow\left[0, t_{0}\right]$ with $h_{n}(\xi, \tau, \ell) \geq \tau$ for all $n \geq 1$ and $(\xi, \tau, \ell) \in \varphi$. If $\left(h_{n}\right)_{n \geq 1}$ converges pointwise to a function $h: \varphi \rightarrow\left[0, t_{0}\right]$, then $\left(T_{t_{0}} h_{n}\right)_{n \geq 1}$ converges pointwise to $T_{t_{0}} h$.

Proof. Let $x \in \varphi$ be arbitrary. To prove $T_{t_{0}} h(x) \geq \limsup _{n \rightarrow \infty} T_{t_{0}} h_{n}(x)$, assume there exists $t>0$ with $t \in\left(T_{t_{0}} h(x), \lim \sup _{n \rightarrow \infty} T_{t_{0}} h_{n}(x)\right)$. Choose $y=\left(\eta, \sigma, \ell^{\prime}\right) \in \varphi \backslash \varphi_{x}$ such that

$$
\begin{equation*}
(\operatorname{int} g(x, t)) \cap g(y, \min \{t, h(y)\}) \neq \emptyset \tag{3}
\end{equation*}
$$

Then, from $t<\lim \sup _{n \rightarrow \infty} T_{t_{0}} h_{n}(x)$ we conclude $g(x, t) \cap g\left(y, \min \left\{t, h_{n}(y)\right\}\right)=\emptyset$ for infinitely many $n \geq 1$, which implies $(\operatorname{int} g(x, t)) \cap g(y, \min \{t, h(y)\})=\emptyset$. However, the latter identity contradicts (3).

To prove $T_{t_{0}} h(x) \leq \liminf _{n \rightarrow \infty} T_{t_{0}} h_{n}(x)$, assume $t \in\left(\liminf _{n \rightarrow \infty} T_{t_{0}} h_{n}(x), T_{t_{0}} h(x)\right)$ for a suitable $t>0$. Since $t>\liminf _{n \rightarrow \infty} T_{t_{0}} h_{n}(x)$, we may choose $y=\left(\eta, \sigma, \ell^{\prime}\right) \in \varphi \backslash \varphi_{x}$ such that

$$
\begin{equation*}
g(x, t) \cap g\left(y, \min \left\{t, h_{n}(y)\right\}\right) \neq \emptyset \tag{4}
\end{equation*}
$$

for infinitely many $n \geq 1$. In particular, $g(x, t) \cap g(y, \min \{t, h(y)\}) \neq \emptyset$. However, from $T_{t_{0}} h(x)>t$ we conclude $g(x, t) \cap g(y, \min \{t, h(y)\})=\emptyset$, contradicting (4).

In the following, we construct a family of growth-stopping times $f: \varphi \rightarrow\left[0, t_{0}\right]$ by providing a family of functions $\left(f^{n}\right)_{n \geq 0}$ such that $\left(f^{2 n}\right)_{n \geq 0}$ converges to $f$ from above and $\left(f^{2 n+1}\right)_{n \geq 0}$ converges to $f$ from below. To be more precise, for $n=0$ we put $f^{0}=t_{0}$ and for $n \geq 0$ we define recursively $f^{n+1}=T_{t_{0}} f^{n}$. By definition of $g$ and $T_{t_{0}}$, we have $\tau \leq f^{n}(x) \leq t_{0}$ for all $n \geq 0$ and $x=(\xi, \tau, \ell) \in \varphi$. For the proof of the uniqueness of growth-stopping times, the following result is useful.

Lemma 4 If $f \in\left[0, t_{0}\right]^{\varphi}$ is a family of growth-stopping times, then $f^{2 n+1}(x) \leq f(x) \leq f^{2 n}(x)$ for all $n \geq 0$ and $x \in \varphi$.

Proof. The assertion follows immediately from Lemmas 1 and 2 .
The convergence of the sequences $\left(f^{2 n}\right)_{n \geq 0}$ and $\left(f^{2 n+1}\right)_{n \geq 0}$ is based on the following result.
Lemma 5 For every $n \geq 0$ and $x \in \varphi$, it holds that
(i) $f^{2 n}(x) \geq f^{2 n+2}(x)$,
(ii) $f^{2 n+1}(x) \leq f^{2 n+3}(x)$,
(iii) $f^{2 n+1}(x) \leq \min \left\{f^{2 n}(x), f^{2 n+2}(x)\right\}$.

Proof. Properties (i)-(iii) follow immediately from Lemma 2 by induction.
In particular, Lemma 5 yields functions $f^{-}, f^{+}: \varphi \rightarrow\left[0, t_{0}\right]$ such that for every $x \in \varphi$,
(i) $\lim _{n \rightarrow \infty} f^{2 n}(x)=f^{+}(x)$,
(ii) $\lim _{n \rightarrow \infty} f^{2 n+1}(x)=f^{-}(x)$,
(iii) $f^{-}(x) \leq f^{+}(x)$.

We also note an immediate corollary to Lemma 3 .
Corollary 1 The identities $f^{-}(x)=T_{t_{0}} f^{+}(x)$ and $f^{+}(x)=T_{t_{0}} f^{-}(x)$ hold for all $x \in \varphi$.
The next step in the construction of a family of growth-stopping times consists in deriving a suitable sufficient condition that implies $f^{-}(x)=f^{+}(x)$ for all $x \in \varphi$. To achieve this goal, we need the following two auxiliary results, which show that growth-stopping times decrease when passing to stopping neighbors.

Lemma 6 Let $h, h^{\prime} \in\left[0, t_{0}\right]^{\varphi}$ be such that $T_{t_{0}} h^{\prime}=h$ and $h^{\prime}=T_{t_{0}} h$. If $x=(\xi, \tau, \ell) \in \varphi$ and $y \in S_{t_{0}} h^{\prime}(x)$ are such that $\xi \notin \operatorname{int} g\left(y, \min \left\{\tau, h^{\prime}(y)\right\}\right)$, then $h^{\prime}(y) \leq h(x)$.

Proof. We assume $h^{\prime}(y)>h(x)$ for the sake of deriving a contradiction. First, we deduce from $T_{t_{0}} h^{\prime}(x)=h(x)$ that $h(x) \geq \tau$. Next, if $x \in \varphi_{y}$, then $\xi \in \operatorname{int} g(y, \tau)$ (note that the roles of $x$ and $y$ are switched in comparison to the definition in Section 22 . Thus, $h^{\prime}(y)>h(x) \geq \tau$ would contradict our assumption $\xi \notin \operatorname{int} g\left(y, \min \left\{\tau, h^{\prime}(y)\right\}\right)$. Therefore, we may assume $x \notin \varphi_{y}$. By definition of $y$, we have $g(x, h(x)) \cap g(y, h(x)) \neq \emptyset$ and since $x \notin \varphi_{y}$ the assumption $h^{\prime}(y)=T_{t_{0}} h(y)$ implies (int $\left.g\left(y, h^{\prime}(y)\right)\right) \cap g(x, h(x))=\emptyset$. However, taken together, these two relations yield a contradiction to $h(x)<h^{\prime}(y)$.

Corollary 2 Let $f \in\left[0, t_{0}\right]^{\varphi}$ be a family of growth-stopping times. If $x=(\xi, \tau, \ell), y \in \varphi$ are such that $y$ is a stopping neighbor of $x$ and $\xi \notin \operatorname{int} g(y, \min \{\tau, f(y)\})$, then $f(y) \leq f(x)$.

Proof. Assume that $f(y)>f(x)$. Then, using $f(x) \geq \tau$ and $\xi \notin \operatorname{int} g(y, \min \{\tau, f(y)\})$, we conclude that $x \notin \varphi_{y}$. Next, since $y$ is a stopping neighbor of $x$, we have $g(x, f(x)) \cap g(y, f(x)) \neq \emptyset$. Hence, $\operatorname{int}(g(y, f(y))) \cap g(x, f(x)) \neq \emptyset$ which contradicts $\left(\mathrm{H}^{\prime}\right)$.

The following result describes in greater detail the consequences of $f^{-}(x)<f^{+}(x)$.
Lemma 7 Let $x \in \varphi$ be such that $f^{-}(x)<f^{+}(x)$. Then for all $y \in S_{t_{0}} f^{+}(x)$,
(i) $f^{-}(y)<\min \left\{f^{-}(x), f^{+}(y)\right\}$,
(ii) $\quad g\left(x, f^{-}(x)\right) \cap g\left(y, f^{-}(x)\right) \neq \emptyset$,
(iii) $g\left(x, f^{-}(y)\right) \cap g\left(y, f^{-}(y)\right)=\emptyset$.

Proof. Throughout the proof, we write $x=(\xi, \tau, \ell)$. First, assume that $\xi \in \operatorname{int} g\left(y, \min \left\{\tau, f^{+}(y)\right\}\right)$, so that $f^{-}(x)=\tau$. We can use Corollary 1 to deduce that $T_{t_{0}} f^{-}(x)=f^{+}(x)$. Hence, applying the definition of $T_{t_{0}}$ gives

$$
\begin{equation*}
\left(\operatorname{int} g\left(x, f^{+}(x)\right)\right) \cap g\left(y, \min \left\{f^{+}(x), f^{-}(y)\right\}\right)=\emptyset, \tag{5}
\end{equation*}
$$

so that $f^{-}(y)<\tau$. For the second part of (i) we observe that $f^{+}(y)=f^{-}(y)$ would imply $\xi \in$ int $g\left(y, f^{-}(y)\right)$, which contradicts (5). Property (ii) is clear by the choice of $y$. Finally, (iii) follows immediately from $f^{-}(x)=\tau$ and (i).

It remains to consider the case where $\xi \notin \operatorname{int} g\left(y, \min \left\{\tau, f^{+}(y)\right\}\right)$. By Lemma 6, we have $f^{+}(y) \leq$ $f^{-}(x)$, so that $f^{-}(y)<f^{-}(x)$ follows once $f^{-}(y)<f^{+}(y)$ is verified. For the latter, observe that $f^{-}(y)=f^{+}(y)$ would yield a contradiction to the relations $g\left(x, f^{-}(x)\right) \cap g\left(y, f^{+}(y)\right) \neq$ $\emptyset,\left(\operatorname{int} g\left(x, f^{+}(x)\right)\right) \cap g\left(y, f^{-}(y)\right)=\emptyset$ and our assumption $f^{-}(x)<f^{+}(x)$. From $g\left(x, f^{-}(x)\right) \cap$ $g\left(y, f^{+}(y)\right) \neq \emptyset$ we deduce $g\left(x, f^{-}(x)\right) \cap g\left(y, f^{-}(x)\right) \neq \emptyset$, which is (ii). Finally, we also have (int $\left.g\left(x, f^{-}(x)\right)\right) \cap g\left(y, f^{+}(y)\right)=\emptyset$, which implies (iii), since $f^{-}(y)<f^{-}(x)$ and $f^{-}(y)<f^{+}(y)$.

Lemma 7 can be used to prove that if there exists $x \in \varphi$ with $f^{-}(x)<f^{+}(x)$, then $\varphi$ contains a strong descending chain.

Lemma 8 If there exists $x \in \varphi$ with $f^{-}(x)<f^{+}(x)$, then $\varphi$ contains a strong descending chain.
Proof. We define the sequence $\left\{x_{n}\right\}_{n \geq 1}$ with $x_{n} \in \mathbb{R}^{d, t_{0}, \mathcal{S}}$ for $n \geq 1$ recursively by putting $x_{1}=x$ and by choosing $x_{n+1} \in \varphi$ to be an arbitrary element from $S_{t_{0}} f^{+}\left(x_{n}\right)$. Here $S_{t_{0}} f^{+}\left(x_{n}\right) \neq \emptyset$, since Lemma 7 implies that $f^{-}\left(x_{n}\right)<f^{+}\left(x_{n}\right) \leq t_{0}$. Furthermore, we put $t_{n}=f^{-}\left(x_{n}\right)$. Then Lemma 7 shows that $\left\{x_{n}\right\}_{n \geq 1}$ forms a strong descending chain using the sequence of times $\left\{t_{n}\right\}_{n \geq 1}$.

As corollary, we obtain existence and uniqueness of $\left(t_{0}, \varphi\right)$-growth-stopping times.
Corollary 3 If $\varphi$ does not contain strong descending chains, then there exists a unique family of $\left(t_{0}, \varphi\right)$-growth-stopping times $f_{t_{0}, \varphi}: \varphi \rightarrow\left[0, t_{0}\right]$.

Proof. Lemma 8 shows that $f^{-}(x)=f^{+}(x)$ for all $x \in \varphi$, so that Corollary 1 yields $f^{+}=T_{t_{0}} f^{+}$. Hence, by Lemma 1, $f_{t_{0}, \varphi}=f^{+}$defines a family of $\left(t_{0}, \varphi\right)$-growth-stopping times. Regarding uniqueness, if $f$ is any family of $\left(t_{0}, \varphi\right)$-growth stopping times, then Lemma 4 yields $f=f^{+}$.

To pass from $\mathbb{R}^{d, t_{0}, \mathcal{S}}$ to $\mathbb{R}^{d,+, \mathcal{S}}$ we use the following compatibility result.
Lemma 9 Let $0<t_{0}^{\prime}<t_{0}$, and $f \in\left[0, t_{0}\right]^{\varphi}$ be a family of $\left(t_{0}, \varphi\right)$-growth-stopping times. Then $f^{\prime}=\min \left\{f, t_{0}^{\prime}\right\}$ defines a family of $\left(t_{0}^{\prime}, \varphi \cap \mathbb{R}^{d, t_{0}^{\prime}, \mathcal{S}}\right)$-growth-stopping times.

Proof. Since the hard-core property is immediate, it suffices to show existence of stopping neighbors with respect to $\varphi \cap \mathbb{R}^{d, t_{0}^{\prime}}$. So let $x=(\xi, \tau, \ell) \in \varphi \cap \mathbb{R}^{d, t_{0}^{\prime}, \mathcal{S}}$ be such that $f(x)<t_{0}^{\prime}$ and let $y=$ $\left(\eta, \sigma, \ell^{\prime}\right) \in \varphi \backslash \varphi_{x}$ denote a stopping neighbor of $x$ with respect to $\varphi$. We distinguish two cases and first assume $\xi \in \operatorname{int} g(y, \min \{\tau, f(y)\})$. Then, $y \in \varphi \cap \mathbb{R}^{d, t_{0}^{\prime}, \mathcal{S}}$ and $\min \{\tau, f(y)\}=\min \left\{\tau, f^{\prime}(y)\right\}$ which shows that $y$ is a stopping neighbor of $x$ with respect to $\varphi \cap \mathbb{R}^{d, t_{0}^{\prime}, \mathcal{S}}$. On the other hand, if $\xi \notin \operatorname{int} g(y, \min \{\tau, f(y)\})$ then we may apply Corollary 2 to deduce $\sigma \leq f(y) \leq f(x)<t_{0}^{\prime}$. Hence, $y$ is contained in $\varphi \cap \mathbb{R}^{d, t_{0}^{\prime}, \mathcal{S}}$ and therefore forms a stopping neighbor of $x$ in $\varphi \cap \mathbb{R}^{d, t_{0}^{\prime}, \mathcal{S}}$.

It can be checked that the result of Lemma 9 is also true in the case $t_{0}=\infty$, i.e., when $\varphi \subset \mathbb{R}^{d,+, \mathcal{S}}$. Here, we say that a function $f: \varphi \rightarrow[0, \infty]$ with $f(\xi, \tau, \ell) \geq \tau$ for all $(\xi, \tau, \ell) \in \varphi$ defines a family of $(\infty, \varphi)$-growth-stopping times if the following two conditions are satisfied.
$\left(\mathrm{H}^{\prime \prime}\right)$ Hard-core property. $(\operatorname{int} g(x, f(x))) \cap g(y, \min \{f(x), f(y)\})=\emptyset$ for all $x \in \varphi$ and $y \in \varphi \backslash \varphi_{x}$.
$\left(\mathrm{N}^{\prime \prime}\right)$ Existence of stopping neighbors. For all $x \in \varphi$ with $f(x)<\infty$ there exists $y \in \varphi \backslash \varphi_{x}$ with $g(x, f(x)) \cap g(y, \min \{f(x), f(y)\}) \neq \emptyset$.
Before proving Theorem 1, we show that conditions (H) and (N) are equivalent to conditions $\left(\mathrm{H}^{\prime \prime}\right)$ and $\left(\mathrm{N}^{\prime \prime}\right)$.

Lemma 10 Let $\varphi \in \mathbb{N}^{*}$ and $f \in[0, \infty]^{\varphi}$ be such that $f(\xi, \tau, \ell) \geq \tau$ for all $(\xi, \tau, \ell) \in \varphi$. Then, conditions $(\mathrm{H})$ and $(\mathrm{N})$ are equivalent to conditions $\left(\mathrm{H}^{\prime \prime}\right)$ and $\left(\mathrm{N}^{\prime \prime}\right)$.

Proof. Assume that conditions $(\mathrm{H})$ and $(\mathrm{N})$ hold. Condition $\left(\mathrm{H}^{\prime \prime}\right)$ is a direct consequence of condition (H). To verify condition ( $\mathrm{N}^{\prime \prime}$ ), let $x \in \varphi$ be such that $f(x)<\infty$ and let $y \in \varphi$ be as in condition (N). First, if $\xi \in \operatorname{int} g(y, \min \{\tau, f(y)\})$, then $g(x, f(x)) \cap g(y, \min \{f(x), f(y)\})$ contains $\xi$. On the other hand, if $f(y) \leq f(x)$ and $g(x, f(x)) \cap g(y, f(y)) \neq \emptyset$, then $g(x, f(x)) \cap g(y, \min \{f(x), f(y)\}) \neq \emptyset$.

For the other direction, assume that conditions $\left(\mathrm{H}^{\prime \prime}\right)$ and $\left(\mathrm{N}^{\prime \prime}\right)$ hold. To verify condition $(\mathrm{H})$, let $x=(\xi, \tau, \ell) \in \varphi$ and $y=\left(\eta, \sigma, \ell^{\prime}\right) \in \varphi \backslash \varphi_{x}$ be arbitrary. Condition (H) is satisfied if $f(x)=\tau$, so that we may assume $f(x)>\tau$. If $f(y) \leq f(x)$, then condition $\left(\mathrm{H}^{\prime \prime}\right)$ shows that (int $\left.g(x, f(x))\right) \cap$ $g(y, f(y))=\emptyset$, so that we can restrict to the case $f(y)>f(x)$. In particular, (int $g(x, f(x))) \cap$ $g(y, f(x))=\emptyset$, which shows that $x \notin \varphi_{y}$. Hence, we can again apply condition ( $\mathrm{H}^{\prime \prime}$ ) (with reversed roles of $x$ and $y$ ) to conclude the verification of condition (H). Finally, we verify condition (N). Let $x=(\xi, \tau, \ell) \in \varphi$ be such that $f(x)<\infty$ and let $y \in \varphi \backslash \varphi_{x}$ be as in condition ( $\mathrm{N}^{\prime \prime}$ ). If $y$ is such that $\xi \notin \operatorname{int} g(y, \min \{\tau, f(y)\})$, then we can argue as in the proof of Corollary 2 that $f(y) \leq f(x)$. Indeed, in order to derive a contradiction, we assume that $f(y)>f(x)$. Hence, condition ( $\mathrm{N}^{\prime \prime}$ ) gives $g(x, f(x)) \cap g(y, f(x)) \neq \emptyset$. In particular, $\operatorname{int}(g(y, f(y))) \cap g(x, f(x)) \neq \emptyset$ which contradicts $\left(\mathrm{H}^{\prime \prime}\right)$.

Proof of Theorem 1. Using Corollary 3, we see that for all $n \geq 1$ there exists a unique ( $n, \varphi \cap \mathbb{R}^{d, n, \mathcal{S}}$ )-growth-stopping time $f_{n, \varphi \cap \mathbb{R}^{d, n}, \mathcal{S}}$. Define a function $f: \varphi \rightarrow[0, \infty]$ by

$$
\begin{equation*}
f(x)=\lim _{\substack{n \rightarrow \infty \\ n \geq n_{0}}} f_{n, \varphi \cap \mathbb{R}^{d, n, \mathcal{S}}}(x) \tag{6}
\end{equation*}
$$

where $n_{0} \geq 1$ is chosen such that $x \in \varphi \cap \mathbb{R}^{d, n_{0}, \mathcal{S}}$. Using Lemma 9 we see that the limit (6) exists (or converges to $\infty$ ) and (using Lemma 10) that $f: \varphi \rightarrow[0, \infty]$ constitutes a well-defined $\varphi$-growth-stopping time. We also conclude from Lemma 9 and the subsequent remark that any $\varphi$-growth-stopping time $f^{\prime}$ satisfies $\min \left\{f^{\prime}(x), n\right\}=f_{n, \varphi \cap \mathbb{R}^{d, n, s}}(x)$ for all $x \in \varphi$, which proves uniqueness.

### 3.2 Descending chains and dependent percolation

In Section 3.1 we solved the problem of existence and uniqueness of stationary Apollonian packings under the condition of absence of strong descending chains. Hence, it is worthwhile to verify this condition for a large class of spatially stationary point processes.

In the case of convex grains, general sufficient conditions have been derived for a closely related variant of descending chains in [9], which requires suitable bounds on the factorial moment measures of the underlying point process. We propose a further method which is especially useful in situations with finite range of dependence. To be more precise, the a.s. absence of strong descending chains holds for spatially stationary $m$-dependent marked point processes, see Proposition 2 below. This observation will be useful when constructing approximations to optimally rotated Apollonian packings in Section 3.3 .

We follow a similar approach to [14] and relate the existence of descending chains to percolation in specific graphs on the vertex set $\varphi$. The absence of percolation in these graphs may be proven within the dependent percolation framework of [19].

Definition 5 Let $b \geq 0, \varepsilon>0$ and $\varphi \in \mathbb{N}^{*}$. Then, define a graph $G^{s}(\varphi, b, \varepsilon)$ on $\varphi$ as follows. Two vertices $x, y \in \varphi$ are connected by an edge in $G^{\mathrm{s}}(\varphi, b, \varepsilon)$ if and only if $g(x, b+\varepsilon) \cap g(y, b+\varepsilon) \neq \emptyset$ and $g(x, b) \cap g(y, b)=\emptyset$.

We say that a (directed) graph percolates if there exists a (directed) self-avoiding path consisting of infinitely many vertices. The following result clarifies the relation between percolation of $G^{\mathrm{s}}(\varphi, b, \varepsilon)$ and strong descending chains.
Lemma 11 Let $\varphi \in \mathbb{N}^{*}$. If $\varphi$ admits a strong descending chain, then there exists $b \geq 0$ such that $G^{s}(\varphi, b, \varepsilon)$ percolates for all $\varepsilon>0$.

Proof. Let $\left\{x_{n}\right\}_{n \geq 1}$ be a strong descending chain for some sequence $\left\{t_{n}\right\}_{n \geq 1}$ and let $\varepsilon>0$ be arbitrary. Since $\left\{t_{n}\right\}_{n \geq 1}$ forms a strictly decreasing sequence converging to some limit $b$ and since $g\left(x_{n}, t_{n}\right) \cap g\left(x_{n+1}, t_{n}\right) \neq \emptyset$ for all $n \geq 1$, there exists $n_{0} \geq 1$ such that $g\left(x_{n}, b+\varepsilon\right) \cap g\left(x_{n+1}, b+\varepsilon\right) \neq \emptyset$ for all $n \geq n_{0}$. Moreover, from $t_{n}>t_{n+1}$ and $g\left(x_{n}, t_{n+1}\right) \cap g\left(x_{n+1}, t_{n+1}\right)=\emptyset$ for all $n \geq 1$ we conclude $g\left(x_{n}, b\right) \cap g\left(x_{n+1}, b\right)=\emptyset$ for all $n \geq 1$. Hence, $x_{n}$ and $x_{n+1}$ are connected by an edge in $G^{\mathrm{s}}(\varphi, b, \varepsilon)$ for all $n \geq n_{0}$.

It will be convenient to use the abbreviation $Q_{r}^{t \mathcal{S}}(\xi)$ for $Q_{r}(\xi) \times[0, t] \times \mathcal{S}$, where $r, t>0$ and $\xi \in \mathbb{R}^{d}$. The following proposition constitutes a useful auxiliary result in proving the a.s. absence of strong descending chains.

Proposition 1 Let $b \geq 0, r_{0}, t>0$ and $\Phi$ be an $m$-dependent spatially stationary marked point process in $\mathbb{R}^{d,+, \mathcal{S}}$. Then, there exists $\varepsilon>0$ such that with probability 1 the graph $G^{s}\left(\Phi \cap \mathbb{R}^{d, t, \mathcal{S}}, b, \varepsilon\right)$ does not percolate.
Proof. We define a site-percolation process $Y=\left\{Y_{z}\right\}_{z \in \mathbb{Z}^{d}}$ as follows, where we put $\Phi^{t}=\Phi \cap \mathbb{R}^{d, t, \mathcal{S}}$. Say that $z$ is open, i.e., $Y_{z}=1$ if and only if there exist $x \in \Phi^{t} \cap Q_{1}^{t, \mathcal{S}}(z)$ and $y \in \Phi^{t}$ such that $x, y$ are connected by an edge in $G^{s}\left(\Phi^{t}, b, \varepsilon\right)$. For $\varepsilon \in(0,1)$ this process clearly exhibits finite range of dependence and we claim that the probability that a site is open can be made as small as desired if $\varepsilon>0$ is chosen sufficiently small. Once this claim is proven, using [19, Theorem 0.0 ], we conclude that the site process $Y$ a.s. does not percolate provided that $\varepsilon>0$ is sufficiently small. Therefore, also the graph $G^{s}\left(\Phi^{t}, b, \varepsilon\right)$ does not percolate. In order to show that the probability that $o$ is open tends to 0 as $\varepsilon \rightarrow 0$, we first note that by $r_{0}$-boundedness of the grains, there exist only finitely many pairs $x, y \in \Phi^{t}$ with $x \in Q_{1}^{t, \mathcal{S}}(o), g(x, b+1) \cap g(y, b+1) \neq \emptyset$ and $g(x, b) \cap g(y, b)=\emptyset$. For each such pair choose some (random) $\varepsilon_{x, y}>0$ such that $g\left(x, b+\varepsilon_{x, y}\right) \cap g\left(y, b+\varepsilon_{x, y}\right)=\emptyset$. Then, for every $\varepsilon>0$ smaller than the minimum of these finitely many values there do not exist $x \in \Phi \cap Q_{1}^{t, \mathcal{S}}(z)$ and $y \in \Phi$ such that $x, y$ are connected by an edge in $G^{s}\left(\Phi^{t}, b, \varepsilon\right)$.

Theorem 2] on the almost sure absence of strong descending chains for $m$-dependent marked point processes is now obtained from Proposition 1 .

Proof of Theorem 2. It suffices to show that for every $t>0$ with probability 1 , the set $\Phi^{t}=\Phi \cap \mathbb{R}^{d, t, \mathcal{S}}$ does not contain a strong descending chain. The proof is similar to [14, Thereom 2.2], but we provide the details for the convenience of the reader. Consider a function $h:[0, \infty) \rightarrow[0, \infty)$ with $h(b)=b+\varepsilon(b)$, where $\varepsilon(b)>0$ is chosen such that $G^{s}\left(\Phi^{t}, b, \varepsilon(b)\right)$ a.s. does not percolate. Clearly, this function satisfies the condition of [14, Lemma 2.1], so that there exists a countable set $\mathbb{C} \subset[0, \infty)$ with $\bigcup_{b \in \mathbb{C}}[b, b+\varepsilon(b))=[0, \infty)$. If $\Phi^{t}$ admits a strong descending chain $\left(x_{n}, t_{n}\right)_{n \geq 1}$, then put $b=\lim _{n \rightarrow \infty} t_{n}$. Choose $b_{0} \in \mathbb{C}$ and $\varepsilon_{0}>0$ such that $\left[b, b+\varepsilon_{0}\right) \subset\left[b_{0}, b_{0}+\varepsilon\left(b_{0}\right)\right)$. Since $G^{\mathrm{s}}\left(\Phi^{t}, b_{0}, \varepsilon\left(b_{0}\right)\right) \supset G^{\mathrm{s}}\left(\Phi^{t}, b, \varepsilon_{0}\right)$ we conclude from Lemma 11 that $G^{\mathrm{s}}\left(\Phi^{t}, b_{0}, \varepsilon\left(b_{0}\right)\right)$ percolates. In particular, we obtain

$$
\begin{aligned}
\mathbb{P}\left(\Phi^{t} \text { admits a strong descending chain }\right) & \leq \mathbb{P}\left(\cup_{b_{0} \in \mathbb{C} \cap[0, \infty)} G^{\mathbf{s}}\left(\Phi^{t}, b_{0}, \varepsilon\left(b_{0}\right)\right) \text { percolates }\right) \\
& \leq \sum_{b_{0} \in \mathbb{C} \cap[0, \infty)} \mathbb{P}\left(G^{s}\left(\Phi^{t}, b_{0}, \varepsilon\left(b_{0}\right)\right) \text { percolates }\right) .
\end{aligned}
$$

Since $\mathbb{P}\left(G^{s}\left(\Phi^{t}, b_{0}, \varepsilon\left(b_{0}\right)\right)\right.$ percolates $)=0$, this completes the proof.

For existence and uniqueness of stationary Apollonian packings, absence of strong descending chains is a sufficient condition. However, as we will see in Section 6, for percolation-type questions absence of another kind of descending chains is highly relevant.

Definition 6 Let $\varphi \in \mathbb{N}^{*}$. A sequence $\left\{x_{n}\right\}_{n \geq 1}$ of elements in $\varphi$ is said to form a weak descending chain if there exists a sequence $\left\{t_{n}\right\}_{n \geq 1}$ such that
(i) $\quad t_{n} \geq t_{n+1}$ for all $n \geq 1$,
(ii) $\quad x_{n_{1}} \neq x_{n_{2}}$ for all $n_{1}, n_{2} \geq 1$ with $n_{1} \neq n_{2}$,
(iii) $g\left(x_{n}, t_{n}\right) \cap g\left(x_{n+1}, t_{n}\right) \neq \emptyset$ and $g\left(x_{n}, t_{n+1}\right) \cap \operatorname{int} g\left(x_{n+1}, t_{n+1}\right)=\emptyset$ for all $n \geq 1$.

Note that in contrast to a strong descending chain, in a weak descending chain the sequence of times $\left\{t_{n}\right\}_{n \geq 1}$ could be eventually constant. Furthermore, if $\left\{x_{n}\right\}_{n \geq 1}$ constitutes a strong descending chain, then $\left\{x_{n}\right\}_{n \geq 1}$ forms also a weak descending chain.

As before, to verify absence of weak descending chains, it is useful to investigate percolation of a specific graph.

Definition 7 Let $b \geq 0, \varepsilon>0$ and $\varphi \in \mathbb{N}^{*}$. Then, define a directed graph $G^{\mathrm{w}}(\varphi, b, \varepsilon)$ on the vertex set $\varphi$ as follows. For $x, y \in \varphi$ an edge is drawn from $x$ to $y$ if
(i) $\quad g(x, b+\varepsilon) \cap g(y, b+\varepsilon) \neq \emptyset$, and
(ii) $\quad g(x, b) \cap \operatorname{int} g(y, b)=\emptyset$.

Note that in comparison to the second condition in Definition 5, the second condition in Definition 7 is weaker since it only requires $g(x, b) \cap \operatorname{int} g(y, b)=\emptyset$ instead of $g(x, b) \cap g(y, b)=\emptyset$. As in Lemma 11, we may now prove the following result.

Lemma 12 Let $\varphi \in \mathbb{N}^{*}$. If $\varphi$ contains a weak descending chain, then there exists $b \geq 0$ such that $G^{\mathrm{w}}(\varphi, b, \varepsilon)$ percolates for all $\varepsilon>0$.

Proof. Let $\left\{x_{n}\right\}_{n \geq 1}$ be a weak descending chain for some sequence $\left\{t_{n}\right\}_{n \geq 1}$ and let $\varepsilon>0$ be arbitrary. Since $\left\{t_{n}\right\}_{n \geq 1}$ forms a (not necessarily strictly) decreasing sequence converging to some limit $b$ and since $g\left(x_{n}, t_{n}\right) \cap g\left(x_{n+1}, t_{n}\right) \neq \emptyset$ for all $n \geq 1$, there exists $n_{0} \geq 1$ such that $g\left(x_{n}, b+\varepsilon\right) \cap$ $g\left(x_{n+1}, b+\varepsilon\right) \neq \emptyset$ for all $n \geq n_{0}$. Moreover, from $t_{n} \geq t_{n+1}$ and $g\left(x_{n}, t_{n+1}\right) \cap \operatorname{int} g\left(x_{n+1}, t_{n+1}\right)=\emptyset$ for all $n \geq 1$ we conclude $g\left(x_{n}, b\right) \cap \operatorname{int} g\left(x_{n+1}, b\right)=\emptyset$ for all $n \geq 1$. Hence, $x_{n}$ is connected to $x_{n+1}$ by a directed edge in $G^{\mathrm{w}}(\varphi, b, \varepsilon)$ for all $n \geq n_{0}$.

Similarly to Proposition 1 and Theorem 2, one now establishes the a.s. absence of weak descending chains. However, a close inspection of the proof of Proposition 1 shows that we need to make the additional assumption that for every $b \geq 0$ with probability 1 , there do not exist $x, y \in \Phi$ such that $g(x, b) \cap g(y, b) \neq \emptyset$ and $g(x, b) \cap \operatorname{int} g(y, b)=\emptyset$. The proof of the following result is omitted, since it would be a simple repetition of the arguments presented in Proposition 1 and Theorem 2 .

Corollary 4 Let $\Phi$ be an m-dependent spatially stationary marked point process in $\mathbb{R}^{d,+, \mathcal{S}}$. Furthermore, assume that for every $b>0$ with probability 1 there do not exist $x \in \Phi$ and $y \in \Phi \backslash \Phi_{x}$ such that $g(x, b) \cap g(y, b) \neq \emptyset$ and $g(x, b) \cap \operatorname{int} g(y, b)=\emptyset$. Then, almost surely, $\Phi$ does not contain a weak descending chain.

We conclude this section by verifying the condition in Corollary 4 for independently marked, $m$-dependent point processes.

Lemma 13 Let $b \geq 0$ and $\Phi$ be an independently marked spatially stationary point process in $\mathbb{R}^{d,+, \mathcal{S}}$. Furthermore, assume that the second factorial moment measure of the underlying unmarked point process is absolutely continuous. Then, with probability 1 , there do not exist $x, y \in \Phi$ such that $g(x, b) \cap g(y, b) \neq \emptyset$ and $g(x, b) \cap \operatorname{int} g(y, b)=\emptyset$.

Proof. Let $N_{b}$ denote the number of elements $x, y \in \Phi$ such that $g(x, b) \cap g(y, b) \neq \emptyset$ and $g(x, b) \cap$ int $g(y, b)=\emptyset$. Then, it suffices to show that $\mathbb{P}\left(N_{b}=0\right)=1$. If $x=(\xi, \tau, L), y=\left(\eta, \sigma, L^{\prime}\right) \in \Phi$, are such that $g(x, b) \cap g(y, b) \neq \emptyset$ and $g(x, b) \cap \operatorname{int} g(y, b)=\emptyset$, then $\sigma=b-d^{L^{\prime}}(g(x, b), \eta)$, where
$d^{L^{\prime}}(g(x, b), \eta)=\min \left\{r \geq 0:\left(\eta+r L^{\prime}\right) \cap g(x, b) \neq \emptyset\right\}$ denotes the smallest $r \geq 0$ such that the $\eta+r L^{\prime}$ hits the grain $g(x, b)$. Letting $\alpha(\cdot, \cdot)$ denote the density of the second factorial moment measure and $\mathbb{A}$ the distribution of the typical mark, Campbell's formula implies that

$$
\mathbb{E} N_{b} \leq \int_{\mathcal{S}} \int_{\mathcal{S}} \int_{\mathbb{R}^{d}} \int_{0}^{b} \int_{\mathbb{R}^{d}} \int_{0}^{b} \alpha((\xi, \tau),(\eta, \sigma)) 1_{\sigma=b-d^{L^{\prime}}(\xi+(b-\tau) L, \eta)} \mathrm{d} \tau \mathrm{~d} \xi \mathrm{~d} \sigma \mathrm{~d} \eta \mathbb{A}(\mathrm{~d} L) \mathbb{A}\left(\mathrm{d} L^{\prime}\right)
$$

and the latter expression vanishes, since the Lebesgue measure of the set $\{(\eta, \sigma, \xi, \tau): \sigma=b-$ $\left.d^{L^{\prime}}(\xi+(b-\tau) L, \eta)\right\}$ is 0.

### 3.3 Examples

In Theorem 2 and Corollary 4 we provided explicit sufficient conditions for absence of strong and weak descending chains in independently marked point processes. In particular, these conditions hold for spatially stationary Poisson point processes.

Proposition 2 Let $\alpha \in \mathbb{R}, A \in \mathcal{S}$ and $\Phi^{\prime}=\left\{\left(\xi_{i}, \tau_{i}, \Theta_{i}\right)\right\}_{i \geq 1}$ be an independently $S_{d}$-marked Poisson point process such that the intensity function $\lambda: \mathbb{R}^{d,+} \rightarrow[0, \infty)$ of the underlying unmarked Poisson point process is spatially constant. Then, almost surely, $=\left\{\left(\xi_{i}, \tau_{i}, \Theta_{i}(A)\right\}\right.$ contains neither strong nor weak descending chains.

Proof. The absence of descending chains follows from Theorem 2, Corollary 4 and Lemma 13 ,
Example 1 In the following sections, we consider intensity functions of the form $\lambda: \mathbb{R}^{d,+} \rightarrow[0, \infty)$ given by $\lambda(\xi, \tau)=\lambda(1+\tau)^{\alpha}$ for some $\alpha>-1$ and $\lambda>0$. Increasing the value of $\alpha$ increases the speed at which new germs appear. Interesting special cases include $\alpha=0$, where the space-time intensity measure of germs is proportional to ( $d+1$ )-dimensional Lebesgue measure or $\alpha=-1+\varepsilon$ with small $\varepsilon>0$, where the number of germs in a bounded sampling window is infinite but increases very slowly in time. Furthermore, the parameter $\alpha$ may also yield additional flexibility that could be useful when fitting the model to real data. In the following, we restrict our attention to the case $\alpha>-1$, since for $\alpha<-1$ the number of grains arriving in any bounded sampling window is almost surely finite. Additionally, a more detailed analysis of the critical case $\alpha=-1$ would be worthwhile.

Apart from the case of independent marks, Theorem 2 also covers $m$-dependent marks and as an application of this general framework we consider stationary approximations to rotational random Apollonian packings. Recall that in the latter model, which is investigated in 7], germs are added sequentially to a bounded sampling window and for each germ the corresponding grain is rotated so as to maximize the time until an already existing grain is hit. When trying to create a stationary variant of this packing, it is already difficult to define a suitable optimization criterion. Indeed, since grains may still grow while further germs arrive, complex dependencies between the optimal positioning of grains arise.

Let $A \in \mathcal{S}$ be fixed and $\psi \subset \mathbb{R}^{d,+}$ be locally finite. We propose a family of stationary packings, where at each point $x=(\xi, \tau) \in \psi$ an approximation to the optimal rotation is determined by inspecting a suitable neighborhood of $x$. Possible rotations of $A$ are restricted to a finite (but arbitrarily large) set of rotations $U \subset S O_{d}$. To be more precise, for $b>0$ a fixed positive number, we consider the space-time neighborhood $\psi \cap\left(Q_{b}(\xi) \times[0, \tau+b]\right)$ of $x$ and write $\varphi \cap\left(Q_{b}(\xi) \times[\tau+b]\right)=$ $\left\{x_{1}, \ldots, x_{n}\right\}$ for some $n \geq 1$. First, recall that for $n \geq 1, k \in\{1, \ldots, n\}$ and $\mathbf{t}=\left(t_{1}, \ldots, t_{n}\right) \in[0, \infty)^{n}$ the $k$ th order statistic $\operatorname{ord}_{k, n}(\mathbf{t})$ is the $k$ th smallest element of $\mathbf{t}$, i.e.,

$$
\operatorname{ord}_{1, n}(\mathbf{t}) \leq \operatorname{ord}_{2, n}(\mathbf{t}) \leq \cdots \leq \operatorname{ord}_{n, n}(\mathbf{t}) .
$$

For $k \in\{1, \ldots, n\}$ we construct recursively a subset $U_{k} \subset U^{n}$ as follows. For $\boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{n}\right) \in U^{n}$ let $f_{\boldsymbol{\theta}}$ be the family of growth-stopping times associated with $\left\{y_{i, \theta}\right\}_{1 \leq i \leq n}=\left\{\left(x_{i}, \theta_{i}(A)\right)\right\}_{1 \leq i \leq n}$. For $k=1$ we define $U_{1} \subset U^{n}$ to be the set of all $\boldsymbol{\theta} \in U^{n}$ maximizing $\operatorname{ord}_{1, n}\left(f_{\boldsymbol{\theta}}\left(y_{1, \boldsymbol{\theta}}\right), \ldots, f_{\boldsymbol{\theta}}\left(y_{n, \boldsymbol{\theta}}\right)\right)$. For $k>1$ we define $U_{k} \subset U_{k-1}$ to be the set of all $\boldsymbol{\theta} \in U_{k-1}$ that maximize $\operatorname{ord}_{k, n}\left(f_{\boldsymbol{\theta}}\left(y_{1, \boldsymbol{\theta}}\right), \ldots, f_{\boldsymbol{\theta}}\left(y_{n, \boldsymbol{\theta}}\right)\right)$.

Finally, let $\theta_{x}^{\text {opt }, U, b} \in U_{n}$ denote an element of $U_{n}$ chosen according to some deterministic rule. Note that the construction of the family of marks $\left(\theta_{x}^{\mathrm{opt}, U, b}\right)_{x \in \varphi}$ is invariant with respect to spatial translation of the underlying locally finite set $\psi$ and that for $x \in \psi$ the value of $\theta_{x}^{\text {opt }, U, b}(A)$ is determined by $\psi \cap\left(Q_{b}(\xi) \times[0, \infty)\right)$. Thus, Theorem 2 yields the following result, where we put $\psi^{\mathrm{opt}, U, b}=\left\{\left(x, \theta_{x}^{\mathrm{opt}, U, b}(A)\right)\right\}_{x \in \psi}$.
Corollary 5 Let $b>0, A \in \mathcal{S}, U \subset S O_{d}$ be finite and $\Psi \subset \mathbb{R}^{d,+}$ be a spatially stationary and $m$-dependent point process in $\mathbb{R}^{d,+}$. Then, a.s. $\Psi^{\circ p t, U, b}$ does not contain strong $\ell^{\circ p t, U, b}$-descending chains.

It would be interesting to investigate if the marks $\theta^{\text {opt }, U_{n}, b_{n}}(A)$ converge to some random marking as $b_{n} \rightarrow \infty$ and for suitably increasing $U_{n} \subset S O_{d}$. This limit would then qualify as stationary extension of the rotational random Apollonian packing.

## 4 Asymptotics for growth durations

In the present section, we study the dependence of the growth duration of grains on the time of arrival of the corresponding germ. We will see that under Poisson assumptions this quantity exhibits a power-law decay in time and we provide rigorous bounds on the corresponding exponent. We also performed Monte Carlo simulations to obtain more precise information on the dependence of this exponent on the speed at which new germs arrive. The simulation results are given in Section 4.2.

### 4.1 Rigorous bounds on the exponent

We begin by providing an elementary proof of Theorem 6, which is based on the observation that a grain can only grow for a long time if there is a large space-time environment of the corresponding germ that does not contain any further points.

Proof of Theorem 6. If $r>0$ and $x=(\xi, \tau, \ell) \in \Phi$ are such that $\tau>1$ and $f_{\Phi}(x)>r+\tau$, then $\Phi \cap Q_{r_{1} r}^{\tau, \mathcal{S}}(\xi)=\emptyset$. For $t>0$ let $N_{t}$ denote the number of elements $x=(\xi, \tau, \ell) \in \Phi \cap\left(Q_{1}(o) \times[t, \infty) \times \mathcal{S}\right)$ such that $\Phi \cap Q_{r_{1} \tau^{-\alpha_{1}+\varepsilon}}^{\tau, \mathcal{S}}(\xi)=\emptyset$. Using the Slivnyak-Mecke formula, we compute for all sufficiently large $t>1$,

$$
\begin{aligned}
\mathbb{P}\left(N_{t}>0\right) & \leq \mathbb{E} N_{t} \\
& =\lambda \int_{Q_{1}(o)} \int_{t}^{\infty}(1+\tau)^{\alpha} \mathbb{P}\left(\Phi \cap Q_{r_{1} \tau^{-\alpha_{1}+\varepsilon}}^{\tau, \mathcal{S}}(\xi)=\emptyset\right) \mathrm{d} \tau \mathrm{~d} \xi \\
& =\lambda \int_{Q_{1}(o)} \int_{t}^{\infty}(1+\tau)^{\alpha} \exp \left(-\lambda r_{1}^{d} \tau^{-(\alpha+1)+d \varepsilon} \int_{0}^{\tau}(1+\sigma)^{\alpha} \mathrm{d} \sigma\right) \mathrm{d} \tau \mathrm{~d} \xi \\
& =\lambda \int_{t}^{\infty}(1+\tau)^{\alpha} \exp \left(-\lambda(\alpha+1)^{-1} r_{1}^{d} \tau^{-(\alpha+1)+d \varepsilon}\left((1+\tau)^{\alpha+1}-1\right)\right) \mathrm{d} \tau \\
& \leq \int_{t}^{\infty} \exp \left(-\lambda(\alpha+1)^{-1} 2^{-1} r_{1}^{d} \tau^{d \varepsilon}\right) \mathrm{d} \tau
\end{aligned}
$$

Since the latter expression is at most $\exp \left(-t^{d \varepsilon / 2}\right)$, this proves the claim.
To prove a rigorous lower bound on the growth duration, we need a couple of auxiliary results. We assume additionally that $\Phi=\left\{\left(\xi_{i}, \tau_{i}, \Theta_{i}(A)\right\}_{i \geq 1}\right.$, where $\left\{\left(\xi_{i}, \tau_{i}, \Theta_{i}\right)\right\}_{i \geq 1}$ is an independently $S O_{d^{-}}$ marked Poisson point process. First, we derive a more refined upper bound on the growth duration of grains which arrive rather early. We fix $r_{1}, r_{2}>0$ such that $B_{r_{1}}(o) \subset \operatorname{int} A$ and $A \subset B_{r_{2}}(o)$, where $B_{r}(o)=\left\{\xi \in \mathbb{R}^{d}:|\xi| \leq r\right\}$ denotes the ball with radius $r>0$ in $\mathbb{R}^{d}$ centered at the origin, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^{d}$. In the following, it will also be convenient to use the abbreviation $B_{r}^{t}(o)=B_{r}(o) \times[0, t]$, where $r, t>0$. Finally, for $t>0$ let $E_{1, t}$ denote the event that there exists $x=(\xi, \tau, \ell) \in \Phi \cap \mathbb{R}^{d, t, \mathcal{S}}$ with $\left(f_{\Phi}(x)-\tau\right) r_{2} \geq \max \{|\xi| / 2, t\}$.

Lemma 14 There exists $t_{2}>0$ such that $\mathbb{P}\left(E_{1, t}\right) \leq \exp \left(-t^{d}\right)$ for all $t \geq t_{2}$.
Proof. Let $C_{x, r}=x+\left\{(\eta, \sigma) \in \mathbb{R}^{d, r}: \eta \in B_{r r_{1}}(o) \backslash B_{\sigma r_{1}}(o)\right\}$ denote the complement inside $x+B_{r r_{1}}^{r}(o)$ of the cone of height $r>0$, base $B_{r r_{1}}(o)$ and apex $x \in \mathbb{R}^{d,+}$. Then, letting $\kappa_{d}=\nu_{d}\left(B_{1}(o)\right)$ denote the volume of the unit ball in $\mathbb{R}^{d}$, we obtain for every measurable function $u:[0, \infty) \rightarrow[0, \infty)$

$$
\begin{aligned}
\mathbb{E} \# & \left\{x=(\xi, \tau) \in \Phi \cap \mathbb{R}^{d, t, \mathcal{S}}: \Phi \cap\left(C_{x, u(|\xi|)} \times \mathcal{S}\right)=\emptyset\right\} \\
& =\lambda \int_{\mathbb{R}^{d}} \int_{0}^{t}(1+\tau)^{\alpha} \exp \left(-\lambda \kappa_{d} r_{1}^{d} \int_{0}^{u(|\xi|)}\left(u(|\xi|)^{d}-\sigma^{d}\right)(1+\tau+\sigma)^{\alpha} \mathrm{d} \sigma\right) \mathrm{d} \tau \mathrm{~d} \xi \\
& \leq \lambda \int_{\mathbb{R}^{d}} \int_{0}^{t}(1+\tau)^{\alpha} \exp \left(-\lambda \kappa_{d} 2^{-1}\left(u(|\xi|) r_{1}\right)^{d} \int_{0}^{u(|\xi|) / 2}(1+\tau+\sigma)^{\alpha} \mathrm{d} \sigma\right) \mathrm{d} \tau \mathrm{~d} \xi \\
& \leq \lambda \int_{\mathbb{R}^{d}} \int_{0}^{t}(1+\tau)^{\alpha} \exp \left(-\lambda \kappa_{d} 2^{-2}\left(u(|\xi|) r_{1}\right)^{d} u(|\xi|)(1+\tau+u(|\xi|) / 2)^{\min \{0, \alpha\}}\right) \mathrm{d} \tau \mathrm{~d} \xi \\
& \leq \lambda 2^{|\alpha|} t^{1+|\alpha|} \int_{\mathbb{R}^{d}} \exp \left(-\lambda \kappa_{d} 2^{-2} r_{1}^{d} u(|\xi|)^{d+1}(1+t+u(|\xi|) / 2)^{\min \{0, \alpha\}}\right) \mathrm{d} \xi
\end{aligned}
$$

Observe that if $x=(\xi, \tau, \ell) \in \Phi \cap \mathbb{R}^{d, t, \mathcal{S}}$ is such that $\left(f_{\Phi}(x)-\tau\right) r_{2} \geq \max \{|\xi| / 2, t\}$, then $\Phi \cap$ $\left(C_{x, r_{2}^{-1} \max \{|\xi| / 2, t\}} \times \mathcal{S}\right)=\emptyset$. Hence, putting $c_{1}=\lambda \kappa_{d} 2^{-2} r_{1}^{d} r_{2}^{-d-1}$,

$$
\begin{aligned}
\mathbb{P}\left(E_{1, t}\right)\left(\lambda 2^{|\alpha|} t^{1+|\alpha|}\right)^{-1} \leq & \int_{B_{2 t}(o)} \exp \left(-c_{1} t^{d+1}\left(1+t+t /\left(2 r_{2}\right)\right)^{\min \{0, \alpha\}}\right) \mathrm{d} \xi \\
& +\int_{\mathbb{R}^{d} \backslash B_{2 t}(o)} \exp \left(-c_{1} 2^{-d-1}|\xi|^{d+1}\left(1+t+|\xi| /\left(4 r_{2}\right)\right)^{\min \{0, \alpha\}}\right) \mathrm{d} \xi
\end{aligned}
$$

We derive bounds for these two summands separately, which are valid for all sufficiently large $t>0$. For the first one we have

$$
\begin{align*}
& \int_{B_{2 t}(o)} \exp \left(-c_{1} t^{d+1}\left(1+t+t /\left(2 r_{2}\right)\right)^{\min \{0, \alpha\}}\right) \mathrm{d} \xi \\
& \quad=\kappa_{d} 2^{d} t^{d} \exp \left(-c_{1} t^{d+1}\left(1+\left(1+1 /\left(2 r_{2}\right)\right) t\right)^{\min \{0, \alpha\}}\right) \\
& \quad \leq \kappa_{d} 2^{d} t^{d} \exp \left(-c_{1}\left(2+1 /\left(2 r_{2}\right)\right)^{\min \{0, \alpha\}} t^{d+1+\min \{0, \alpha\}}\right) \tag{7}
\end{align*}
$$

whereas, for the second one we compute

$$
\begin{align*}
& \int_{\mathbb{R}^{d} \backslash B_{2 t}(o)} \exp \left(-c_{1} 2^{-d-1}|\xi|^{d+1}\left(1+t+|\xi| /\left(4 r_{2}\right)\right)^{\min \{0, \alpha\}}\right) \mathrm{d} \xi \\
& \quad \leq \int_{\mathbb{R}^{d} \backslash B_{2 t}(o)} \exp \left(-c_{1} 2^{-d-1}\left(2+1 /\left(4 r_{2}\right)\right)^{\min \{0, \alpha\}}|\xi|^{d+1+\min \{0, \alpha\}}\right) \mathrm{d} \xi \\
& \quad=d \kappa_{d} \int_{2 t}^{\infty} r^{d-1} \exp \left(-c_{1} 2^{-d-1}\left(2+1 /\left(4 r_{2}\right)\right)^{\min \{0, \alpha\}} r^{d+1+\min \{0, \alpha\}}\right) \mathrm{d} r \tag{8}
\end{align*}
$$

Combining (7) and (8) completes the proof of Lemma 14
Additionally, we need a similar upper bound for grains that arrive rather late. For $t>0$, $\varepsilon \in(0,1)$ denote by $E_{2, \varepsilon, t}$ the event that there exists $x=(\xi, \tau, \ell) \in \Phi \cap\left(\mathbb{R}^{d} \times[t, \infty) \times \mathcal{S}\right)$ with $\left(f_{\Phi}(x)-\tau\right) r_{2} \geq u_{\varepsilon, \tau}(|\xi|)$, where $u_{\varepsilon, \tau}:[0, \infty) \rightarrow[0, \infty)$ denotes the function defined by

$$
u_{\varepsilon, \tau}(r)= \begin{cases}\tau^{-\alpha_{1}+\varepsilon} & \text { if } r \leq 4 \\ r / 2 & \text { otherwise }\end{cases}
$$

Lemma 15 Let $\varepsilon \in(0,1)$ be arbitrary. Then, there exists $t_{3}>0$ such that for every $t \geq t_{3}$,

$$
\mathbb{P}\left(E_{2, \varepsilon, t}\right) \leq \exp \left(-t^{\min \{d \varepsilon,(\alpha+1) / 2\} / 2}\right)
$$

Proof. First, for all sufficiently large $t>0$,

$$
\begin{aligned}
\mathbb{E} & \#\left\{x \in \Phi \cap\left(\mathbb{R}^{d} \times[t, \infty) \times \mathcal{S}\right): \Phi \cap\left(B_{\left(r_{1} / r_{2}\right) u_{\varepsilon, \tau}(|\xi|)}^{\tau}(\xi) \times \mathcal{S}\right)=\emptyset\right\} \\
& =\lambda d \kappa_{d} \int_{t}^{\infty}(1+\tau)^{\alpha} \int_{0}^{\infty} r^{d-1} \exp \left(-\lambda \kappa_{d}\left(r_{1} / r_{2}\right)^{d} u_{\varepsilon, \tau}(r)^{d} \int_{0}^{\tau}(1+\sigma)^{\alpha} \mathrm{d} \sigma\right) \mathrm{d} r \mathrm{~d} \tau \\
& \leq \lambda d \kappa_{d} \int_{t}^{\infty}(1+\tau)^{\alpha} \int_{0}^{\infty} r^{d-1} \exp \left(-\lambda \kappa_{d}\left(r_{1} / r_{2}\right)^{d} 2^{-1}(1+\alpha)^{-1} u_{\varepsilon, \tau}(r)^{d} \tau^{\alpha+1}\right) \mathrm{d} r \mathrm{~d} \tau
\end{aligned}
$$

Observe that if $x=(\xi, \tau, \ell) \in \Phi$ is such that $\left(f_{\Phi}(x)-\tau\right) r_{2} \geq u_{\varepsilon, \tau}(|\xi|)$, then $\Phi \cap\left(B_{\left(r_{1} / r_{2}\right) u_{\varepsilon, \tau}(|\xi|)}^{\tau}(\xi) \times\right.$ $\mathcal{S})=\emptyset$. Hence, putting $c_{2}=\lambda \kappa_{d}\left(r_{1} / r_{2}\right)^{d} 2^{-1}(1+\alpha)^{-1}, \mathbb{P}\left(E_{2, \varepsilon, t}\right)$ is bounded from above by

$$
\begin{aligned}
\mathbb{E} \# & \left\{x \in \Phi \cap\left(\mathbb{R}^{d} \times[t, \infty) \times \mathcal{S}\right): \Phi \cap\left(B_{\left(r_{1} / r_{2}\right) u_{\varepsilon, \tau}(|\xi|)}^{\tau}(\xi) \times \mathcal{S}\right)=\emptyset\right\} \\
\leq & \lambda d \kappa_{d} \int_{t}^{\infty}(1+\tau)^{\alpha} \int_{0}^{4} r^{d-1} \exp \left(-c_{2} \tau^{-\alpha-1+d \varepsilon} \tau^{\alpha+1}\right) \mathrm{d} r \mathrm{~d} \tau \\
& +\lambda d \kappa_{d} \int_{t}^{\infty}(1+\tau)^{\alpha} \int_{4}^{\infty} r^{d-1} \exp \left(-c_{2} 2^{-d} r^{d} \tau^{\alpha+1}\right) \mathrm{d} r \mathrm{~d} \tau
\end{aligned}
$$

As before, we derive bounds for these two summands separately. For the first one we have

$$
\begin{equation*}
\int_{t}^{\infty}(1+\tau)^{\alpha} \int_{0}^{4} r^{d-1} \exp \left(-c_{2} \tau^{-\alpha-1+d \varepsilon} \tau^{\alpha+1}\right) \mathrm{d} r \mathrm{~d} \tau \leq 4^{d+|\alpha|} \int_{t}^{\infty} \tau^{\alpha} \exp \left(-c_{2} \tau^{d \varepsilon}\right) \mathrm{d} \tau \tag{9}
\end{equation*}
$$

whereas for the second

$$
\begin{align*}
\int_{t}^{\infty}(1+\tau)^{\alpha} \int_{4}^{\infty} r^{d-1} \exp \left(-c_{2} 2^{-d} r^{d} \tau^{\alpha+1}\right) \mathrm{d} r \mathrm{~d} \tau & \leq 2^{|\alpha|} \int_{t}^{\infty} \exp \left(-c_{2} 2^{d} \tau^{\alpha+1}\right) \mathrm{d} \tau \\
& \leq \int_{t}^{\infty} \exp \left(-\tau^{(\alpha+1) / 2}\right) \mathrm{d} \tau \tag{10}
\end{align*}
$$

provided that $t>0$ is sufficiently large. Combining (9) and (10) completes the proof.
As a final preliminary result, we derive sufficient conditions implying that for sufficiently large system times, germs are always born after their stopping neighbors. This auxiliary result will also be used in Section 6. To be more precise, for $a, t>0$, let $E_{3, a, t}$ denote the event that there exist $x=(\xi, \tau, \ell) \in \Phi \cap\left(Q_{1}(o) \times[t-1, t] \times \mathcal{S}\right)$ and $y=\left(\eta, \sigma, \ell^{\prime}\right) \in \Phi$ such that $f_{\Phi}(x)-\tau \leq \tau^{-a}$ and $y$ is a stopping neighbor of $x$ with $\sigma \geq \tau$.
Lemma 16 Let $a>0$ and $\varepsilon \in(0,1)$ be arbitrary. Then, there exists $t_{4}>0$ such that $\mathbb{P}\left(E_{3, a, t}\right) \leq$ $t^{2 \alpha-(d+1) a+\varepsilon}$ for all $t \geq t_{4}$.

Proof. For $x=(\xi, \tau, \ell) \in \Phi \cap\left(Q_{1}(o) \times[t-1, t] \times \mathcal{S}\right)$ with $f_{\Phi}(x)-\tau \leq \tau^{-a}$ let $E_{3, a, t, x}$ denote the event that there exists a stopping neighbor $y=\left(\eta, \sigma, \ell^{\prime}\right)$ of $x$ with $\sigma \geq \tau$. We first claim that every $x=(\xi, \tau, \ell) \in \Phi \cap\left(Q_{1}(o) \times[t-1, t] \times \mathcal{S}\right)$ with $f_{\Phi}(x)-\tau \leq \tau^{-a}$ and for which there exists a stopping neighbor $y=\left(\eta, \sigma, \ell^{\prime}\right)$ of $x$ with $\sigma \geq \tau$ satisfies $\Phi \cap\left(B_{2 r_{2} \tau^{-a}}(\xi) \times\left(\tau, \tau+\tau^{-a}\right] \times \mathcal{S}\right) \neq \emptyset$. Indeed, $f_{\Phi}(y)-\sigma \leq f_{\Phi}(x)-\tau \leq \tau^{-a}$ implies $B_{r_{2} \tau^{-a}}(\xi) \cap B_{r_{2} \tau^{-a}}(\eta) \neq \emptyset$. Hence, for all $\varepsilon>0$ there exists $t_{4}>0$ such that for all $t>t_{4}$

$$
\begin{aligned}
\mathbb{P}\left(E_{3, a, t}\right) & \leq \mathbb{E} \#\left\{x \in \Phi \cap\left(Q_{1}(o) \times[t-1, t] \times \mathcal{S}\right): E_{3, a, t, x} \text { holds }\right\} \\
& \leq \lambda \int_{Q_{1}(o)} \int_{t-1}^{t}(1+\tau)^{\alpha} \mathbb{P}\left(\Phi\left(B_{2 r_{2} \tau^{-a}}(\xi) \times\left[\tau, \tau+\tau^{-a}\right] \times \mathcal{S}\right)>0\right) \mathrm{d} \tau \mathrm{~d} \xi
\end{aligned}
$$

By stationarity and the Markov inequality the integral in the last line is bounded from above by

$$
\begin{aligned}
\int_{t-1}^{t}(1+\tau)^{\alpha} \mathbb{E} \Phi\left(B_{2 r_{2} \tau^{-a}}(o) \times\left[\tau, \tau+\tau^{-a}\right] \times \mathcal{S}\right) \mathrm{d} \tau & \leq \int_{t-1}^{t}(1+\tau)^{\alpha} \kappa_{d} r_{2}^{d} 2^{d+|\alpha|} \tau^{\alpha-a(d+1)} \mathrm{d} \tau \\
& \leq t^{2 \alpha-(d+1) a+\varepsilon}
\end{aligned}
$$

Using Lemmas 14, 15 and 16 we can now prove Theorem 7. The idea is to make use of the observation that if a grain has a small but non-zero growth duration, then it is constrained to lie very closely to the boundary of its stopping neighbor. Recall that we assume, additionally, that $A=\left\{z \in \mathbb{R}^{d}: \beta(z) \leq 1\right\}$ is the unit ball with respect to a certain norm $\beta(\cdot)$ on $\mathbb{R}^{d}$. Furthermore, we also suppose that the process $\Phi$ is independently marked.

Proof of Theorem 7. For readability we write $f$ instead of $f_{\Phi}$ and $f_{t}$ instead of $\min \left\{t, f_{\Phi}\right\}$, where $t>0$. Moreover, without loss of generality we may assume $\varepsilon \in(0, \min \{1,(\alpha+1) / 2\})$. We first show that the number of $x=(\xi, \tau, \ell) \in \Phi \cap\left(Q_{1}(o) \times[0, \infty) \times \mathcal{S}\right)$ satisfying $0<f(x)-\tau<\tau^{-\alpha_{2}-\varepsilon}$ and having a stopping neighbor $y=\left(\eta, \sigma, \ell^{\prime}\right) \in \Phi$ with $\sigma \geq \tau$ is finite with probability 1 . Indeed, since $2 \alpha<(d+1) \alpha_{2}-1$ we may combine the estimate obtained in Lemma 16 with the Borel-Cantelli lemma to obtain the almost sure finiteness of the number of $x=(\xi, \tau, \ell) \in \Phi \cap\left(Q_{1}(o) \times[0, \infty) \times \mathcal{S}\right)$ satisfying $0<f(x)-\tau<\tau^{-\alpha_{2}-\varepsilon}$ and having a stopping neighbor $y=\left(\eta, \sigma, \ell^{\prime}\right) \in \Phi$ with $\sigma \geq \tau$.

It remains to consider the case, where $x=(\xi, \tau, \Theta(A)) \in \Phi$ admits a stopping neighbor $y=$ $\left(\eta, \sigma, \Theta^{\prime}(A)\right)$ with $\sigma<\tau$. We claim that then $\xi$ must lie close to the boundary of the grain associated with $y$. To be more precise, we assert that

$$
\xi \in \eta+\left(f_{\tau}(y)-\sigma+\rho \tau^{-\alpha_{2}-\varepsilon}\right) \Theta^{\prime}(A) \backslash\left(f_{\tau}(y)-\sigma\right) \Theta^{\prime}(A),
$$

where $\rho=1+r_{2} / r_{1}$. Indeed, the assumption $f(x)>\tau$ yields $\xi \notin \eta+\left(f_{\tau}(y)-\sigma\right) \Theta^{\prime}(A)$. On the other hand, from $f(x)-\tau<\tau^{-\alpha_{2}-\varepsilon}$ we conclude that $\left(\xi+\tau^{-\alpha_{2}-\varepsilon} \Theta(A)\right) \cap\left(\eta+\left(f_{\tau}(y)-\sigma+\tau^{-\alpha_{2}-\varepsilon}\right) \Theta^{\prime}(A)\right) \neq$ $\emptyset$. By the choice of $r_{1}, r_{2}$, we therefore obtain

$$
\left(\left(\Theta^{\prime}\right)^{-1}(\xi)+r_{2} r_{1}^{-1} \tau^{-\alpha_{2}-\varepsilon} A\right) \cap\left(\left(\Theta^{\prime}\right)^{-1}(\eta)+\left(f_{\tau}(y)-\sigma+\tau^{-\alpha_{2}-\varepsilon}\right) A\right) \neq \emptyset .
$$

Finally, since $A$ is the unit ball with respect to a norm, we conclude

$$
\xi-\eta \in\left(f_{\tau}(y)-\sigma+\rho \tau^{-\alpha_{2}-\varepsilon}\right) \Theta^{\prime}(A)
$$

where $\rho=1+r_{2} / r_{1}$. For $x=(\xi, \tau, \Theta(A)) \in \Phi$ we say that the event $E_{4, x}$ occurs if there exists $y=\left(\eta, \sigma, \Theta^{\prime}(A)\right) \in \Phi \cap \mathbb{R}^{d, \tau, \mathcal{S}}$ such that $\xi-\eta \in\left(f_{\tau}(y)-\sigma+\rho \tau^{-\alpha_{2}-\varepsilon}\right) \Theta^{\prime}(A) \backslash\left(f_{\tau}(y)-\sigma\right) \Theta^{\prime}(A)$. Observe that when using the notation $A_{y, \tau}^{(1)}=\eta+\left(A_{y, \tau}^{(2)} \oplus \rho \tau^{-\alpha_{2}-\varepsilon} \Theta^{\prime}(A)\right) \backslash A_{y, \tau}^{(2)}$, with $A_{y, \tau}^{(2)}=\left(f_{\tau}(y)-\sigma\right) \Theta^{\prime}(A)$, we see that $E_{4, x}$ can be written as $\left\{\xi \in \bigcup_{y \in \Phi \cap \mathbb{R}^{d, \tau, \mathcal{S}}} A_{y, \tau}^{(1)}\right\}$. Hence, for any $\varepsilon_{1} \in(0,1)$ and $t>2$,

$$
\begin{align*}
\mathbb{P}\left(\cup_{x \in \Phi \cap\left(Q_{1}(o) \times[t-1, t] \times \mathcal{S}\right)} E_{4, x}\right) \leq & \mathbb{E} \#\left\{x \in \Phi \cap\left(Q_{1}(o) \times[t-1, t] \times \mathcal{S}\right): E_{4, x}\right\} \\
= & \lambda \int_{Q_{1}(o)} \int_{t-1}^{t}(1+\tau)^{\alpha} \mathbb{P}\left(\xi \in \cup_{y \in \Phi \cap \mathbb{R}^{d, \tau, \mathcal{S}}} A_{y, \tau}^{(1)}\right) \mathrm{d} \tau \mathrm{~d} \xi \\
\leq & \lambda 2^{|\alpha|} t^{\alpha} \int_{t-1}^{t} \mathbb{P}\left(o \in \cup_{y \in \Phi \cap \mathbb{R}^{d, \tau^{\varepsilon_{1}}, \mathcal{S}}} A_{y, \tau}^{(1)}\right) \mathrm{d} \tau  \tag{11}\\
& +\lambda 2^{|\alpha|} t^{\alpha} \int_{t-1}^{t} \mathbb{P}\left(o \in \cup_{y \in \Phi \cap\left(\mathbb{R}^{d} \times\left[\tau^{\left.\left.\varepsilon_{1}, \tau\right] \times \mathcal{S}\right)}\right.\right.} A_{y, \tau}^{(1)}\right) \mathrm{d} \tau . \tag{12}
\end{align*}
$$

We first consider expression (11). If $y \in \Phi$ is $\operatorname{such}$ that $\left(f_{\tau}(y)-\sigma\right) r_{2} \leq \tau^{\varepsilon_{1}}$, then there exists a constant $c>0$ (not depending on $y$ or $\tau)$ such that

$$
\nu_{d}\left(A_{y, \tau}^{(1)}\right)=\nu_{d}(A)\left(\left(f_{\tau}(y)-\sigma+\rho \tau^{-\alpha_{2}-\varepsilon}\right)^{d}-\left(f_{\tau}(y)-\sigma\right)^{d}\right) \leq c \nu_{d}(A) \tau^{-\alpha_{2}-\varepsilon} \tau^{(d-1) \varepsilon_{1}}
$$

provided that $\tau>0$ is sufficiently large. Also observe that if $E_{1, \tau^{\varepsilon_{1}}}^{c}$ occurs, then $B_{1}(o) \cap A_{y, \tau}^{(1)}=\emptyset$ for all $y \in \Phi \cap \mathbb{R}^{d, \tau^{\varepsilon_{1}}, \mathcal{S}}$ with $y \notin B_{2 \tau^{\varepsilon_{1}}}^{\tau_{1}^{\varepsilon_{1}}}(o) \times \mathcal{S}$ and, moreover, $\left(f_{\tau}(y)-\sigma\right) r_{2} \leq \tau^{\varepsilon_{1}}$ for all $y \in \Phi \cap B_{2 \tau^{\varepsilon_{1}}}^{\tau_{1}}(o)$. Hence,

$$
\begin{aligned}
\mathbb{P}\left(o \in \cup_{y \in \Phi \cap \mathbb{R}^{d, \tau^{\varepsilon}}, \mathcal{S}} A_{y, \tau}^{(1)}\right) & =\kappa_{d}^{-1} \mathbb{E} \nu_{d}\left(B_{1}(o) \cap \cup_{y \in \Phi \cap \mathbb{R}^{d, \tau^{\varepsilon_{1}}, \mathcal{S}}} A_{y, \tau}^{(1)}\right) \\
& \leq \kappa_{d}^{-1} \mathbb{E} \nu_{d}\left(\cup_{y \in \Phi \cap\left(B_{\left.2 \tau^{\varepsilon_{1}}(o) \times \mathcal{S}\right)}^{\tau_{1}}\left(A_{y, \tau}^{(1)}\right) 1_{E_{1, \tau}^{c}}{ }^{\varepsilon_{1}}\right.}+\mathbb{P}\left(E_{\left.1, \tau^{\varepsilon_{1}}\right)}\right)\right. \\
& \leq \kappa_{d}^{-1} \mathbb{E} \sum_{y \in \Phi \cap\left(B_{2 \tau^{\varepsilon_{1}}}^{\tau_{1} \varepsilon_{1}}(o) \times \mathcal{S}\right)} \nu_{d}\left(A_{y, \tau}^{(1)}\right) 1_{E_{1, \tau^{c}}^{c}}+\mathbb{P}\left(E_{1, \tau^{\varepsilon_{1}}}\right) .
\end{aligned}
$$

For the first summand we compute

$$
\begin{aligned}
\mathbb{E} \sum_{y \in \Phi \cap\left(B_{2 \tau^{\varepsilon_{1}}}^{\tau_{1}}(o) \times \mathcal{S}\right)} \nu_{d}\left(A_{y, \tau}^{(1)}\right) 1_{E_{1, \tau}^{c}}^{c} & \leq c \nu_{d}(A) \tau^{-\alpha_{2}-\varepsilon} \tau^{(d-1) \varepsilon_{1}} \mathbb{E} \Phi\left(B_{2 \tau^{\varepsilon_{1}}}^{\tau_{1}^{\varepsilon_{1}}}(o) \times \mathcal{S}\right) \\
& \leq \lambda c \nu_{d}(A) \tau^{-\alpha_{2}-\varepsilon+(d-1) \varepsilon_{1}} \int_{B_{2 \tau_{1}(o)}} \int_{0}^{\tau^{\varepsilon_{1}}}(1+\sigma)^{\alpha} \mathrm{d} \sigma \mathrm{~d} \eta \\
& \leq \lambda c \nu_{d}(A) \kappa_{d}(\alpha+1)^{-1} 2^{d+1+|\alpha|} \tau^{-\alpha_{2}+\varepsilon_{1}(2 d+\alpha)-\varepsilon}
\end{aligned}
$$

For any $\varepsilon_{1} \in(0, \varepsilon /(4 d+2 \alpha))$ we use the inequality $\alpha_{2} \geq \alpha+1$ to deduce that

$$
\begin{equation*}
\lambda 2^{|\alpha|} t^{\alpha} \int_{t-1}^{t} \mathbb{P}\left(o \in \cup_{y \in \Phi \cap\left(\mathbb{R}^{d, \tau^{\varepsilon}, \mathcal{S}}\right)} A_{y, \tau}^{(1)}\right) \mathrm{d} \tau \leq t^{-1-\varepsilon / 4} \tag{13}
\end{equation*}
$$

for all sufficiently large $t>0$. Next, we derive a suitable upper bound for the expression $\mathbb{P}(o \in$ $\left.\bigcup_{y \in \Phi \cap\left(\mathbb{R}^{d} \times\left[\tau^{\varepsilon_{1}}, \tau\right] \times \mathcal{S}\right)} A_{y, \tau}^{(1)}\right)$ appearing in 12). If $y=\left(\eta, \sigma, \Theta^{\prime}(A)\right) \in \Phi$ is such that $\left(f_{\tau}(y)-\sigma\right) r_{2} \leq$ $\sigma^{-\alpha_{1}+\varepsilon_{1}}$ and $\sigma<\tau$, then there exists a constant $c^{\prime}>0$ (not depending on $y$ or $\tau$ ) such that

$$
\nu_{d}\left(A_{y, \tau}^{(1)}\right)=\nu_{d}(A)\left(\left(f_{\tau}(y)-\sigma+\rho \tau^{-\alpha_{2}-\varepsilon}\right)^{d}-\left(f_{\tau}(y)-\sigma\right)^{d}\right) \leq c^{\prime} \nu_{d}(A) \tau^{-\alpha_{2}-\varepsilon} \sigma^{-(d-1) \alpha_{1}+(d-1) \varepsilon_{1}}
$$

for all sufficiently large $\tau>0$. Also observe that if $E_{2, \varepsilon_{1}, \tau^{\varepsilon_{1}}}^{c}$ occurs, then $B_{1}(o) \cap A_{y, \tau}^{(1)}=\emptyset$ for all $y \in \Phi \cap\left(\left(\mathbb{R}^{d} \backslash B_{4}(o)\right) \times\left[\tau^{\varepsilon_{1}}, \tau\right] \times \mathcal{S}\right)$ and, moreover, $\left(f_{\tau}(y)-\sigma\right) r_{2} \leq \sigma^{-\alpha_{1}+\varepsilon_{1}}$ for all $y \in$ $\Phi \cap\left(B_{4}(o) \times\left[\tau^{\varepsilon_{1}}, \tau\right] \times \mathcal{S}\right)$. As before,

$$
\begin{aligned}
\mathbb{P}\left(o \in \bigcup_{y \in \Phi \cap\left(\mathbb{R}^{d} \times\left[\tau^{\varepsilon_{1}}, \tau\right] \times \mathcal{S}\right)} A_{y, \tau}^{(1)}\right) & \leq \kappa_{d}^{-1} \mathbb{E} \nu_{d}\left(B_{1}(o) \cap \bigcup_{y \in \Phi \cap\left(\mathbb{R}^{d} \times\left[\tau^{\varepsilon_{1}}, \tau\right] \times \mathcal{S}\right)} A_{y, \tau}^{(1)}\right) \\
& \leq \kappa_{d}^{-1} \mathbb{E} \nu_{d}\left(\bigcup_{y \in \Phi \cap\left(B_{4}(o) \times\left[\tau^{\varepsilon_{1}}, \tau\right] \times \mathcal{S}\right)} A_{y, \tau}^{(1)}\right) 1_{E_{2, \varepsilon_{1}, \tau_{1}}^{c}}+\mathbb{P}\left(E_{\left.2, \varepsilon_{1}, \tau^{\varepsilon_{1}}\right)}\right) \\
& \leq \kappa_{d}^{-1} \mathbb{E} \sum_{y \in \Phi \cap\left(B_{4}(o) \times\left[\tau^{\left.\left.\varepsilon_{1}, \tau\right] \times \mathcal{S}\right)}\right.\right.} \nu_{d}\left(A_{y, \tau}^{(1)}\right) 1_{E_{2, \varepsilon_{1}, \tau_{1}}^{c}}+\mathbb{P}\left(E_{\left.2, \varepsilon_{1}, \tau^{\varepsilon_{1}}\right)}\right)
\end{aligned}
$$

and we also obtain that

$$
\begin{aligned}
\mathbb{E} \sum_{y \in \Phi \cap\left(B_{4}(o) \times\left[\tau^{\varepsilon_{1}}, \tau\right] \times \mathcal{S}\right)} \nu_{d}\left(A_{y, \tau}^{(1)}\right) 1_{E_{2, \varepsilon_{1}, \tau_{1} \varepsilon_{1}}^{c}} & \leq c^{\prime} \nu_{d}(A) \tau^{-\alpha_{2}-\varepsilon} \mathbb{E} \sum_{\left(\eta, \sigma, \ell^{\prime}\right) \in \Phi \cap\left(B_{4}(o) \times\left[\tau^{\varepsilon_{1}}, \tau\right]\right)} \sigma^{-(d-1) \alpha_{1}+(d-1) \varepsilon_{1}} \\
& \leq \lambda c^{\prime} \nu_{d}(A) \tau^{-\alpha_{2}-\varepsilon} \int_{\tau^{\varepsilon_{1}}}^{\tau} \int_{B_{4}(o)}(1+\sigma)^{\alpha} \sigma^{-(d-1) \alpha_{1}+(d-1) \varepsilon_{1}} \mathrm{~d} \eta \mathrm{~d} \sigma \\
& \leq \lambda c^{\prime} \nu_{d}(A) \tau^{-\alpha_{2}-\varepsilon} \kappa_{d} 4^{d} \int_{1}^{\tau}(1+\sigma)^{\alpha} \sigma^{-(d-1) \alpha_{1}+(d-1) \varepsilon_{1}} \mathrm{~d} \sigma
\end{aligned}
$$

which is at most $\tau^{-\alpha_{2}+\alpha_{1}-\varepsilon / 2}$ provided that $\tau>0$ is sufficiently large and that $\varepsilon_{1} \in(0, \varepsilon /(2 d-2))$. We recall $-\alpha_{2}+\alpha_{1}=-\alpha-1$ to deduce that

$$
\begin{equation*}
\lambda 2^{\alpha} t^{\alpha} \int_{t-1}^{t} \mathbb{P}\left(o \in \bigcup_{y \in \Phi \cap\left(\mathbb{R}^{d} \times\left[\tau^{\varepsilon_{1}}, \tau\right] \times \mathcal{S}\right)} A_{y, \tau}^{(1)}\right) d \tau \leq t^{-1-\varepsilon / 4} \tag{14}
\end{equation*}
$$

for all sufficiently large $t>0$. Finally, taking relations (13) and (14) into account, and using the Borel-Cantelli lemma completes the proof.

Remark. In the proof of Proposition 7 we subdivided $\Phi \cap\left(Q_{1}(o) \times[0, \infty) \times \mathcal{S}\right)$ into equidistant parts $\left\{\Phi \cap\left(Q_{1}(o) \times[n, n+1) \times \mathcal{S}\right)\right\}_{n \geq 0}$, computed suitable bounds for these parts and finally applied the Borel-Cantelli to obtain information on the global behavior. One could try to improve the results by considering different subdivisions, but it is not hard to check that e.g. the subdivision $\left\{\Phi \cap\left(Q_{1}(o) \times\left[\sum_{k=1}^{n} k^{\beta}, \sum_{k=1}^{n+1} k^{\beta}\right) \times \mathcal{S}\right)\right\}_{n \geq 0}$ with $\beta \in(-1, \infty)$ is optimal for $\beta=0$.

### 4.2 Simulation results

Using Theorems 6 and 7 we see that with probability 1 eventually the growth duration of a visible germ $(\xi, \tau)$ will be contained in the interval $\left(\tau^{-(\alpha+1)\left(d^{-1}+1\right)-\varepsilon}, \tau^{-(\alpha+1) / d+\varepsilon}\right)$. Before trying to determine the true value of the exponent in the power-law decay rigorously, it is reasonable to obtain estimates with the help of Monte Carlo simulations. For $t>0$ let $N(t)$ denote the (random) number of all visible germs which have arrived until time $t$ in the unit cube $Q_{1}(o)$. In other words, $N(t)=\#\left\{x \in \Phi \cap\left(Q_{1}^{t, \mathcal{S}}(o)\right): f_{\Phi}(x)>\tau\right\}$. In Figure 3. we show plots of $\log N(t)$ versus $\log t$ for various values of $\alpha$ and where the initial grain shape is deterministic and given by the unit disk $B_{1}(o)$ in $\mathbb{R}^{2}$.


Fig. 3: $\log N(t)$ versus $\log t$ for $\alpha=-1 / 2$ (black), $\alpha=0$ (green), $\alpha=1$ (red), $\alpha=2$ (blue) and $\alpha=3$ (orange).

From a conceptual point of view, it would be slightly more natural to provide plots of $\log t$ vs. $\log N(t)$. However, from a computational point of view, it makes sense to first fix a large number $N$ (in our simulations $N=10^{6}$ ), and then to simulate the models with varying parameter $\alpha$, until $N$ grains are visible. In Figure 3, we provide a plot of $\log N(t)$ versus $\log t$. In particular, for small values of $\alpha$ one can observe clearly that $N(t)$ is approximately of the form $C(\alpha) t^{a(\alpha)}$ for suitable $C(\alpha), a(\alpha) \geq 0$. For higher values of $\alpha$ this relationship is still plausible although it is also apparent, that more germs have to be created until the power law becomes visible.

In [8], numerical evidence is provided that the radius (which in our model is proportional to the growth time) of the $n$th visible grain is of the order $n^{-1 /\left(\alpha^{\prime}-1\right)}$, where $\alpha^{\prime} \approx 2.56$. Furthermore, the relation $\alpha^{\prime} \approx 2.56$ was observed to be universal in the sense that changing the speed at which grains grow does not have an effect on $\alpha^{\prime}$. The change of growth speed corresponds in our model to a change in the rate at which new germs appear. Making use of these results, we see that we can approximate the desired exponent of the power law corresponding to the grain radius at time $t$ by $-a(\alpha) /\left(\alpha^{\prime}-1\right)$. A table of estimated values of $a(\alpha)$ and $b(\alpha)=a(\alpha) /\left(\alpha^{\prime}-1\right)$ is shown in Table 4.2. The values for $a(\alpha)$ were fitted using linear regression, based on the last 500, 000 data points.

| $\alpha$ | -0.5 | 0 | 1 | 2 | 3 |
| ---: | :---: | :---: | :---: | :---: | :---: |
| $a(\alpha)$ | 0.391 | 0.781 | 1.57 | 2.33 | 3.05 |
| $b(\alpha) /(\alpha+1)$ | 0.501 | 0.501 | 0.503 | 0.498 | 0.489 |

Table 1: Estimated exponents $a(\alpha)$ and $b(\alpha)=a(\alpha) /\left(\alpha^{\prime}-1\right)$

Taking into account errors induced by finite sample size and by performing the simulation on a bounded torus instead of the entire Euclidean space, the last line in Table 4.2 suggests that the upper bound derived in Proposition 6 is in fact the true decay rate for the growth durations. To be more precise, in the following conjecture we assume that the marks of $\Phi$ are constant and given by some $A \in \mathcal{S}$.
Conjecture. As $t \rightarrow \infty$, the distribution of the random variable $t^{(\alpha+1) / d}\left(f_{\Phi \cup\{(o, t, A)\}}(o, t, A)-t\right)$ conditioned on the event $\left\{f_{\Phi \cup\{(o, t, A)\}}(o, t, A)>t\right\}$ converges to the distribution of a non-degenerate random variable.

## 5 Space-filling property

In this section, we prove Theorem 8, i.e., we show that stationary Apollonian packings are spacefilling in the sense that $\mathbb{P}\left(\nu_{d}\left(\mathbb{R}^{d} \backslash \mathrm{AP}(\Phi)\right)=0\right)=1$. In the proof, we show that there exists $q \in(0,1)$ such that with probability 1 , for every $t>0$ the final pore-space volume $\nu_{d}\left(Q_{1}(o) \backslash \mathrm{AP}(\Phi)\right)$ is smaller than $q$ times the volume of the pore space in the unit cube at time $t$.

In order to prepare the proof of Theorem 8 , we need to introduce certain auxiliary constructions. Let $t>0, \varphi \in \mathbb{N}^{*}$ and assume that the projection of $\varphi$ to $\mathcal{S}$ consists of rotations of a set $A \in \mathcal{S}$ that is the unit ball with respect to a certain norm on $\mathbb{R}^{d}$. Furthermore, assume that $\varphi$ does not contain strong descending chains. In the following, we write $\operatorname{AP}(\varphi, t)=\bigcup_{x \in \varphi} \operatorname{int} g\left(x, \min \left\{t, f_{\varphi}(x)\right\}\right)$ for the Apollonian packing observed at time $t>0$. In order to control the volume of the pore space $Q_{1}(o) \backslash \mathrm{AP}(\varphi, t)$, we choose a subdivision of $Q_{1}(o)$ into congruent subcubes of length $a=1 /(2 N-1)$, where $N \geq 1$ is some positive integer. In the proof we will investigate properties of this subdivision for large values of $N$. Without loss of generality, we may assume that $\nu_{d}\left(Q_{1}(o) \backslash \mathrm{AP}(\varphi, t)\right)>0$. Furthermore, the subfamily consisting of vacant subcubes not intersecting $\operatorname{AP}(\varphi, t)$ will play an important role. Therefore, we put

$$
S_{\mathrm{vac}}=\left\{z \in \mathbb{Z}^{d} \cap Q_{2 N-1}(o): Q_{a}(a z) \cap \operatorname{AP}(\varphi, t)=\emptyset\right\}
$$

Also define $\rho_{\max }=\sup _{\xi \in B_{\sqrt{d}}(o)} d_{A}(o, \xi), \rho_{\min }=\inf _{\xi \in B_{1}(o)} d_{A}(o, \xi), \rho=\left\lceil\rho_{\max } / \rho_{\text {min }}\right\rceil$ and $\rho^{\prime}=$ $d+3 \rho$, where $d_{A}$ denotes the metric induced by $A$. Furthermore, it is convenient to consider the subset $S_{\mathrm{vac}}^{\mathrm{int}}$ of $S_{\mathrm{vac}}$ consisting of those sites whose associated cube is neither close to the boundary of the cube $Q_{1}(o)$ nor to the boundary of a grain in $\operatorname{AP}(\varphi, t)$, i.e.,

$$
S_{\mathrm{vac}}^{\mathrm{int}}=\left\{z \in S_{\mathrm{vac}} \cap Q_{2 N-1-16 \rho^{\prime}}(o): Q_{16 \rho^{\prime} a}(a z) \cap \mathrm{AP}(\varphi, t)=\emptyset\right\} .
$$

When covering the pore space $Q_{1}(o) \backslash \mathrm{AP}(\varphi, t)$, we distinguish between big and small grains. For $x \in \varphi$,

$$
S_{x}=\left\{z \in S_{\text {vac }}^{\text {int }}: g\left(x, f_{\varphi}(x)\right) \cap Q_{a}(a z) \neq \emptyset\right\}
$$

denotes the family of all sites $z \in S_{\mathrm{vac}}^{\mathrm{int}}$ whose associated cube $Q_{a}(a z)$ admits non-empty intersection with the grain $g\left(x, f_{\varphi}(x)\right)$. Moreover,

$$
S_{\mathrm{big}}=\bigcup_{\substack{x=(\xi, \tau, \ell) \in \varphi \\ f_{\varphi}(x)-\tau \geq 4 \rho_{\max } a}} S_{x}
$$

denotes the family of all sites $z \in S_{\mathrm{vac}}^{\mathrm{int}}$ whose associated cube $Q_{a}(a z)$ intersects some grain with growth duration at least $4 \rho_{\text {max }} a$. We also consider the environment $S_{\text {big }}^{+}$consisting of all $z^{\prime} \in S_{\text {vac }}$ such that $z^{\prime} \in Q_{2 \rho^{\prime}}(z)$ for some $z \in S_{\mathrm{big}}$.

First, we derive a lower bound for the contributions $\nu_{d}\left(Q_{a}(a z) \cap \mathrm{AP}(\varphi) \backslash \mathrm{AP}(\varphi, t)\right)$ for $z \in S_{\text {big }}^{+}$.
Lemma 17 Let $t>0, N \geq 1$ and $\varphi \in \mathbb{N}^{*}$ and assume that the projection of $\varphi$ to $\mathcal{S}$ consists of rotations of a set $A \in \mathcal{S}$ that is the unit ball with respect to a certain norm on $\mathbb{R}^{d}$. Furthermore, put $a=1 /(2 N-1)$ and assume that $\varphi$ does not contain strong descending chains. Then,

$$
\sum_{z \in S_{\text {big }}^{+}} \nu_{d}\left(Q_{a}(a z) \cap \mathrm{AP}(\varphi) \backslash \mathrm{AP}(\varphi, t)\right) \geq a^{d} \# S_{\mathrm{big}}^{+} /\left(2 \rho^{\prime}+1\right)^{2 d}
$$

Proof. We claim that for every $z \in S_{\text {big }}$ there exists a site $h(z) \in S_{\text {vac }}$ such that $h(z) \in Q_{2 \rho^{\prime}}(z)$ and $Q_{a}(a h(z)) \subset \mathrm{AP}(\varphi) \backslash \mathrm{AP}(\varphi, t)$. This claim will yield the desired inequality, since

$$
\sum_{z \in S_{\mathrm{big}}^{+}} \nu_{d}\left(Q_{a}(a z) \cap \mathrm{AP}(\varphi) \backslash \mathrm{AP}(\varphi, t)\right) \geq a^{d} \# h\left(S_{\mathrm{big}}\right) \geq a^{d} \# S_{\mathrm{big}}^{+} /\left(2 \rho^{\prime}+1\right)^{2 d}
$$

where the second inequality uses $\left(2 \rho^{\prime}+1\right)^{d} \# h\left(S_{\mathrm{big}}\right) \geq \# S_{\mathrm{big}}$ and $\left(2 \rho^{\prime}+1\right)^{d} \# S_{\mathrm{big}} \geq \# S_{\mathrm{big}}^{+}$.
To prove the claim, we first choose any $x=(\xi, \tau, \ell) \in \varphi$ with $f_{\varphi}(x)-\tau \geq 4 \rho_{\max } a$ such that $z \in S_{x}$. Then, we distinguish two cases. If $d_{\ell}(a z, \xi) \geq 3 \rho_{\max } a$, then we put

$$
\eta=\xi+(a z-\xi)\left(1-3 \rho_{\max } a / d_{\ell}(a z, \xi)\right)
$$

and choose $h(z) \in \mathbb{Z}^{d}$ so that $\eta \in Q_{a}(a h(z))$. First, observe that

$$
|a h(z)-a z| \leq|a h(z)-\eta|+|\eta-a z| \leq \sqrt{d} a+3 \rho_{\max } a|\xi-a z| / d_{\ell}(a z, \xi) \leq a(\sqrt{d}+3 \rho)
$$

Next, we note that $Q_{a}(a h(z)) \subset \operatorname{AP}(\varphi)$, since for every $\zeta \in Q_{a}(a h(z))$,

$$
d_{\ell}(\xi, \zeta) \leq d_{\ell}(\xi, \eta)+d_{\ell}(\eta, a h(z))+d_{\ell}(a h(z), \zeta) \leq d_{\ell}(a z, \xi)-3 \rho_{\max } a+\rho_{\max } a+\rho_{\max } a
$$

which is at most $f_{\varphi}(x)-\tau$, since $Q_{a}(a z) \cap g\left(x, f_{\varphi}(x)\right) \neq \emptyset$. Finally, to show that $Q_{a}(a h(z)) \cap$ $g\left(x, \min \left\{t, f_{\varphi}(x)\right\}\right)=\emptyset$, we note that for every $\zeta \in Q_{a}(a h(z))$,

$$
\begin{aligned}
d_{\ell}(\xi, \zeta) & \geq d_{\ell}(\xi, \eta)-d_{\ell}(\eta, a h(z))-d_{\ell}(a h(z), \zeta) \geq d_{\ell}(a z, \xi)-3 \rho_{\max } a-\rho_{\max } a-\rho_{\max } a \\
& \geq \min \left\{f_{\varphi}(x), t\right\}-\tau+8 \rho^{\prime} \rho_{\min } a-5 \rho_{\max } a
\end{aligned}
$$

where the last inequality follows from $z \in S_{\mathrm{vac}}^{\mathrm{int}}$. Next, we prove the existence of a site $h(z)$ with the desired properties in the case $d_{\ell}(a z, \xi) \leq 3 \rho_{\max } a$. Note that since $z \in S_{\mathrm{vac}}^{\mathrm{int}}$, this can happen only if $\tau>t$. Choosing $h(z)=z$, for every $\zeta \in Q_{a}(a z)$ we obtain that

$$
d_{\ell}(\xi, \zeta) \leq d_{\ell}(\xi, a z)+d_{\ell}(a z, \zeta) \leq 3 \rho_{\max } a+\rho_{\max } a \leq f_{\varphi}(x)-\tau
$$

where the last inequality is due to the choice of $x$.
Next, we consider lower bounds for the contributions from cubes associated with sites in $S_{\mathrm{vac}} \backslash S_{\mathrm{big}}^{+}$. We show that in a non-vanishing proportion of these cubes newly arriving grains cover a substantial amount of volume. To be more precise, we introduce the subset $S_{\text {cent }}$ of $S_{\text {vac }}^{\text {int }}$ describing the set of sites $z \in \mathbb{Z}^{d}$ for which

$$
u_{z} \in Q_{a / 2}(a z) \text { and } \varphi \cap\left(Q_{(8 \rho+1) a}(a z) \times\left(t_{z}, t_{z}+\rho_{\min } a\right]\right)=\emptyset
$$

where $\left(u_{z}, t_{z}, \ell_{z}\right)$ are the coordinates of the first germ appearing in $\varphi \cap\left(Q_{(8 \rho+1) a}(a z) \times(t, \infty) \times \mathcal{S}\right)$. The importance of $S_{\text {cent }}$ is due to the observation that for every $z \in S_{\text {cent }} \backslash S_{\text {big }}^{+}$, the set $Q_{a}(a z) \cap$ $\mathrm{AP}(\varphi) \backslash \mathrm{AP}(\varphi, t)$ contains $B_{a /(4 \rho)}\left(u_{z}\right)$.
Lemma 18 Let $a, t>0, N \geq 1$ and $\varphi \subset \mathbb{R}^{d,+, S}$ be as in Lemma 17. Then, for every $z \in S_{\text {cent }} \backslash S_{\text {big }}^{+}$, the set $Q_{a}(a z) \cap \mathrm{AP}(\varphi) \backslash \mathrm{AP}(\varphi, t)$ contains a ball of radius $a /(4 \rho)$.

Proof. First, note that $\rho_{\min } a \ell_{z} \supset B_{a /(4 \rho)}(o)$, so that it suffices to show that $Q_{a}(a z) \cap g\left(x, f_{\varphi}(x)\right)=\emptyset$ for all $x \in \varphi \cap \mathbb{R}^{d, t_{z}+\rho_{\min } a} \backslash\left\{\left(u_{z}, t_{z}\right)\right\}$. Suppose we could find $x=(\xi, \tau, \ell) \in \varphi \cap \mathbb{R}^{d, t_{z}+\rho_{\min } a, \mathcal{S}} \backslash$ $\left\{\left(u_{z}, t_{z}, \ell_{z}\right)\right\}$ such that $Q_{a}(a z) \cap g\left(x, f_{\varphi}(x)\right) \neq \emptyset$. First note that $\xi \notin Q_{(8 \rho+1) a}(a z)$, since $z \in S_{\text {vac }}^{\text {int }}$ implies $\varphi \cap\left(Q_{(8 \rho+1) a}(a z) \times[0, t]\right)=\emptyset$ and $z \in S_{\text {cent }}$ yields $x \notin Q_{(8 \rho+1) a}(a z) \times\left[t, t_{z}+\rho_{\min } a\right]$. Thus,

$$
f_{\varphi}(x)-\tau \geq d_{\ell}\left(\xi, \partial Q_{a}(a z)\right) \geq 4 \rho \rho_{\min } a \geq 4 \rho_{\max } a
$$

which contradicts the assumption that $z \notin S_{\text {big }}$.
Hence, the next goal consists in deriving a lower bound for $\# S_{\text {cent }}$ in terms of $\# S_{\text {vac }}^{\text {int }}$. Whereas Lemmas 17 and 18 are purely deterministic, we will now need some randomness which will allow us to achieve the desired lower bound with the help of the law of large numbers.

Lemma 19 Let $t>0$, be as in Lemma 17 , and let $\Phi$ be an independently marked point process that is constructed as in Theorem 7. Then, there exists $c_{*}>0$ such that

$$
\lim _{a \rightarrow 0} \mathbb{P}\left(\# S_{\text {cent }} \geq c_{*} \# S_{\text {vac }}^{\mathrm{int}}\right)=1
$$

Proof. The proof is performed in two steps. First, choose a subset $S_{\text {ind }} \subset S_{\text {vac }}^{\text {int }}$ such that $\# S_{\text {ind }} \geq$ $(8 \rho+1)^{-d} \# S_{\text {vac }}^{\text {int }}$ and such that $\operatorname{int} Q_{(8 \rho+1) a}\left(a z_{1}\right) \cap \operatorname{int} Q_{(8 \rho+1) a}\left(a z_{2}\right)=\emptyset$ for all pairwise distinct $z_{1}, z_{2} \in S_{\text {ind }}$. Moreover, conditioned on $\Phi \cap \mathbb{R}^{d, t, \mathcal{S}}$ the events that $\left\{z \in S_{\text {cent }}\right\}_{z \in S_{\text {ind }}}$ are independent, identically distributed and each occurs with some probability $P_{a}>0$. Once we show that there exists a deterministic $p>0$ that does not depend on $t$ and satisfies $\mathbb{P}\left(\liminf _{a \rightarrow 0} P_{a} \geq p\right)=1$, the law of large numbers implies that

$$
\lim _{a \rightarrow 0} \mathbb{P}\left(\# S_{\text {cent }} \geq c_{*} \# S_{\text {vac }}^{\mathrm{int}}\right) \geq \lim _{a \rightarrow 0} \mathbb{P}\left(\# S_{\text {cent }} \geq p \# S_{\text {ind }} / 2\right)=1
$$

where $c_{*}=p(8 \rho+1)^{-d} / 2$, noting that the definitions of the sets $S_{\text {cent }}, S_{\mathrm{vac}}^{\mathrm{int}}$ and $S_{\text {ind }}$ depend on $a$.
In order to prove the existence of $p$, we establish lower bounds for the probabilities of the events $\left\{U_{o} \in Q_{a / 2}(o)\right\}$ and

$$
\left\{\Phi \cap\left(Q_{(8 \rho+1) a}(o) \times\left(T_{o}, T_{o}+\rho_{\min } a\right] \times \mathcal{S}\right)=\emptyset\right\}
$$

First, we note that the spatial homogeneity of $\Phi$ implies that $\mathbb{P}\left(U_{o} \in Q_{a / 2}(o)\right)=1 /(16 \rho+2)^{d}$. In order to compute a lower bound for the second probability, it is convenient to distinguish between the cases $\alpha \leq 0$ and $\alpha>0$. First, assume that $\alpha \leq 0$. Then, the rate at which new germs appear decreases in system time and we obtain

$$
\begin{aligned}
\mathbb{P}\left(\Phi \cap\left(Q_{(8 \rho+1) a}(o) \times\left(T_{o}, T_{o}+\rho_{\min } a\right] \times \mathcal{S}\right)=\emptyset\right) & \geq \mathbb{P}\left(\Phi \cap\left(Q_{(8 \rho+1) a}(o) \times\left(0, \rho_{\min } a\right] \times \mathcal{S}\right)=\emptyset\right) \\
& =\exp \left(-\lambda(8 \rho+1)^{d} a^{d} \int_{0}^{\rho_{\min } a}(1+\tau)^{\alpha} \mathrm{d} \tau\right)
\end{aligned}
$$

which is at least $\exp \left(-\lambda(8 \rho+1)^{d} a^{d} \rho_{\min } a\right)$ and tends to 1 as $a \rightarrow 0$. It remains to consider the case, where $\alpha>0$. Then, for every $b>0$,

$$
\begin{aligned}
\mathbb{P}\left(T_{o}>t+b\right) & =\exp \left(-\lambda(8 \rho+1)^{d} a^{d} \int_{t}^{t+b}(1+\tau)^{\alpha} \mathrm{d} \tau\right) \\
& =\exp \left(-\lambda(8 \rho+1)^{d} a^{d}\left((1+t+b)^{\alpha+1}-(1+t)^{\alpha+1}\right) /(\alpha+1)\right)
\end{aligned}
$$

In particular, choosing $b_{a, t}=\left(a^{-d}+(1+t)^{\alpha+1}\right)^{1 /(\alpha+1)}-(1+t)$, we see that $p_{0}=\inf _{t>0} \liminf _{a \rightarrow 0} \mathbb{P}\left(T_{o}-\right.$ $\left.t \leq b_{a, t}\right)>0$. Moreover, conditioned on the event $\left\{T_{o}-t \leq b_{a, t}\right\}$ if $a>0$ is sufficiently small, then the probability of the event $\left\{\Phi \cap\left(Q_{(8 \rho+1) a}(o) \times\left[T_{o}, T_{o}+\rho_{\text {min }} a\right] \times \mathcal{S}\right)=\emptyset\right\}$ is at least

$$
\begin{aligned}
& \exp \left(-\lambda(8 \rho+1)^{d} a^{d} \int_{T_{o}}^{T_{o}+\rho_{\min } a}(1+\tau)^{\alpha} \mathrm{d} \tau\right) \\
& \quad \geq \exp \left(-\lambda(8 \rho+1)^{d} a^{d} \int_{t+b_{a, t}}^{t+b_{a, t}+\rho_{\min } a}(1+\tau)^{\alpha} \mathrm{d} \tau\right) \\
& \quad=\exp \left(-\lambda(8 \rho+1)^{d} a^{d}\left(1+t+b_{a, t}\right)^{\alpha+1}\left(\left(1+\rho_{\min } a /\left(1+t+b_{a, t}\right)\right)^{\alpha+1}-1\right) /(\alpha+1)\right) \\
& \quad \geq \exp \left(-2 \lambda(8 \rho+1)^{d} a^{d}\left(1+t+b_{a, t}\right)^{\alpha+1}\right) \\
& \quad=\exp \left(-2 \lambda(8 \rho+1)^{d}\left(1+a^{d}(1+t)^{\alpha+1}\right)\right)
\end{aligned}
$$

and the latter expression is larger than some positive constant $p_{1}$ provided that $a \leq(1+t)^{-(\alpha+1) / d}$. In particular, choosing $p=p_{0} p_{1} /(16 \rho+2)^{d}$ proves the claim.

With the help of Lemmas 17 19, we can now prove Theorem 8 .

Proof of Theorem 8. Our goal is to show that there exists $q \in(0,1)$ such that with probability 1, for every $t>0$,

$$
\nu_{d}\left(Q_{1}(o) \backslash \mathrm{AP}(\Phi)\right) \leq q \nu_{d}\left(Q_{1}(o) \backslash \mathrm{AP}(\Phi, t)\right)
$$

Letting $t \rightarrow \infty$ will then complete the proof of Theorem 8 .
Using Lemma 18 , we see that for every $z \in S_{\text {cent }} \backslash S_{\text {big }}^{+}$,

$$
\nu_{d}\left(Q_{a}(a z) \cap \mathrm{AP}(\Phi) \backslash \mathrm{AP}(\Phi, t)\right) \geq \kappa_{d}(4 \rho)^{-d} a^{d}
$$

so that

$$
\sum_{z \in S_{\text {cent }} \backslash S_{\text {big }}^{+}} \nu_{d}\left(Q_{a}(a z) \cap \mathrm{AP}(\Phi) \backslash \mathrm{AP}(\Phi, t)\right) \geq \kappa_{d}(4 \rho)^{-d} a^{d} \#\left(S_{\text {cent }} \backslash S_{\text {big }}^{+}\right)
$$

Combining this with Lemma 17 yields

$$
\begin{equation*}
\sum_{z \in S_{\text {vac }}} \nu_{d}\left(Q_{a}(a z) \cap \mathrm{AP}(\Phi) \backslash \mathrm{AP}(\Phi, t)\right) \geq c_{1} a^{d} \# S_{\text {cent }} \tag{15}
\end{equation*}
$$

where $c_{1}=\min \left\{\kappa_{d}(4 \rho)^{-d},\left(2 \rho^{\prime}+1\right)^{-2 d}\right\}$.
From (15) and Lemma 19, we conclude that the event

$$
\nu_{d}\left(Q_{1}(o) \cap \mathrm{AP}(\Phi) \backslash \mathrm{AP}(\Phi, t)\right) \geq c_{1} a^{d} \# S_{\text {cent }} \geq c_{1} c_{*} a^{d} \# S_{\text {vac }}^{\mathrm{int}}
$$

occurs with a probability tending to 1 as $a$ tends to 0 . It remains to show that for all sufficiently small $a>0$,

$$
a^{d} \# S_{\mathrm{vac}}^{\mathrm{int}} \geq \nu_{d}\left(Q_{1}(o) \backslash \mathrm{AP}(\Phi, t)\right) / 2
$$

Indeed, as $\operatorname{AP}(\Phi, t)$ consists of a finite union of convex bodies, it is elementary that $\lim _{a \rightarrow 0} a^{d}\left(\# S_{\text {vac }}^{\text {int }}-\right.$ $\left.\# S_{\text {vac }}\right)=0$ and $\lim _{a \rightarrow 0} \nu_{d}\left(Q_{1}(o) \backslash \mathrm{AP}(\Phi, t)\right)-a^{d} \# S_{\text {vac }}=0$. Putting $q=1-c_{1} c_{2} / 2$, we see that for every $t>0$,

$$
\mathbb{P}\left(\nu_{d}\left(Q_{1}(o) \backslash \mathrm{AP}(\Phi)\right) \leq q \nu_{d}\left(Q_{1}(o) \backslash \mathrm{AP}(\Phi, t)\right)\right)=1,
$$

as desired.

## 6 Results on percolation

In this section, we investigate properties of connected components in stationary Apollonian packings and prove Theorems 3, 4 and 5 .

### 6.1 Oriented percolation

In the present subsection, we consider the problem of oriented percolation for the directed graph $G(\Phi)$. This result can be seen as an immediate extension of 9 to the case of germ configurations that are not spatially locally finite. It turns out that also in this more general setting, the absence of weak $\ell$-descending chains is the key to the proof of Theorem 3 .

Proof of Theorem 3. It is an immediate consequence of Corollary 2 that any infinite directed selfavoiding path $\gamma=\left(x_{i}\right)_{i \geq 1}$ in $G(\Phi)$ gives rise to the weak descending chain $\left(x_{i}, f_{\Phi}\left(x_{i}\right)\right)_{i \geq 1}$. Combining this observation with Corollary 4 proves Theorem 3 .

### 6.2 Non-oriented percolation

In Section 6.1 we have seen that for a large class of marked point processes $\Phi$ there is almost surely no oriented percolation in the graph $G(\Phi)$. However, when moving from the oriented to the unoriented case, the problem of percolation becomes more complicated and we will prove four results (Propositions 3 and 4, as well as Theorems 4 and 5).

First, some notation and assumptions need to be introduced. Let $\Phi$ be a spatially stationary marked point process in $\mathbb{R}^{d,+, \mathcal{S}}$ and assume that $\Phi$ does not admit strong descending chains and that with probability 1 each $x \in \Phi$ admits a unique stopping neighbor. For instance, the latter property is satisfied in models based on a Poisson point process of germs with independently marked, strictly convex and non-rotated grains. This will be shown in Section 6.3. Then, we let $h_{\Phi}: \Phi \rightarrow$ $\Phi$ denote the function which assigns to each $x \in \Phi$ its uniquely determined stopping neighbor. Furthermore, we say that $\left\{x, h_{\Phi}(x)\right\}$ defines a doublet if $h_{\Phi}\left(h_{\Phi}(x)\right)=x$ and note that if $G(\Phi)$ does not percolate, then for any $x \in \Phi$ there exists a uniquely determined doublet $\left\{x^{\prime}, h_{\Phi}\left(x^{\prime}\right)\right\}$ such that $h_{\Phi}^{n}(x) \subset\left\{x^{\prime}, h_{\Phi}\left(x^{\prime}\right)\right\}$ for all sufficiently large $n \geq 1$. Here $h_{\Phi}^{n}$ denotes the $n$-fold iteration of $h_{\Phi}$. We denote by $\Phi_{\text {doub }}$ the spatially stationary marked point process in $\mathbb{R}^{d,+, \mathcal{S}}$ consisting of all elements of the form $\min _{\text {lex }}\left\{x^{\prime}, h_{\Phi}\left(x^{\prime}\right)\right\}$, where $\left\{x^{\prime}, h_{\Phi}\left(x^{\prime}\right)\right\}$ forms a doublet in $\Phi$ and where $\min _{\text {lex }}$ denotes the lexicographical minimum. We can use $\Phi_{\text {doub }}$ to interpret the family of connected components of $G(\Phi)$ as a marked point process with centers in $\Phi_{\text {doub }}$ by associating with a connected component $C$ of $G(\Phi)$ the uniquely determined $x^{\prime} \in \Phi_{\text {doub }}$ such that $h_{\Phi}^{n}(x) \subset\left\{x^{\prime}, h_{\Phi}\left(x^{\prime}\right)\right\}$ for all $x \in C$ and all sufficiently large $n \geq 1$.

In the first result discussed in this section, we will use Lemma 16 to note that under suitable assumptions the process of clusters in $G(\Phi)$ is locally finite in the sense that the projection of its center process $\Phi_{\text {doub }}$ to $\mathbb{R}^{d}$ forms a stationary point process with finite intensity.

Proposition 3 Let $\Phi^{\prime}=\left\{\left(\xi_{i}, \tau_{i}, \Theta_{i}\right)\right\}_{i \geq 1}$ be an independently $S O_{d}$-marked Poisson point process such that the intensity function $\lambda: \mathbb{R}^{d,+} \rightarrow[0, \infty)$ of the underlying unmarked Poisson point process is given by (1) with $\alpha \in(-1,1 /(d-1))$. Put $\Phi=\left\{\left(\xi_{i}, \tau_{i}, \Theta_{i}(A)\right\}_{i \geq 1}\right.$, where $A \in \mathcal{S}$. It is also assumed that with probability 1 each $x \in \Phi$ admits a unique stopping neighbor. Then, $\mathbb{E} K<\infty$, where $K$ denotes the number of $(\xi, \tau, \ell) \in \Phi_{\text {doub }}$ such that $\xi \in Q_{1}(o)$.

Proof. Choose $r_{2}>1$ such that $A \subset B_{r_{2}}(o)$. We distinguish several cases. First, if $x=(\xi, \tau, \ell), y=$ $\left(\eta, \sigma, \ell^{\prime}\right) \in \Phi$ form a doublet with $\xi \in Q_{r_{2}}(o)$ and $\eta \notin Q_{6 r_{2}}(o)$, then $\left(f_{\Phi}(x)-\tau\right) r_{2} \geq r_{2}$, or $|\eta| \geq 2 \sigma$ and $\left(f_{\Phi}(y)-\sigma\right) r_{2} \geq|\eta| / 2$, or $|\eta| \leq 2 \sigma$ and $\left(f_{\Phi}(y)-\sigma\right) r_{2} \geq r_{2}$. Hence, Lemma 15 shows that the expected number of such doublets is finite. Therefore, it suffices to bound the expectation of the number $K^{\prime}$ of doublets formed by $x, y \in \Phi \cap\left(Q_{6 r_{2}}(o) \times[0, \infty) \times \mathcal{S}\right)$.

Since $\alpha \in(-1,1 /(d-1))$ there exists $\varepsilon>0$ such that $2 \alpha-((\alpha+1) / d-\varepsilon)(d+1)<-1$. We note that $K^{\prime}$ is at most $K_{1}+K_{2}$, where $K_{1}$ is the number of $x=(\xi, \tau, \ell) \in \Phi$ such that $\xi \in Q_{6 r_{2}}(o)$, $f_{\Phi}(x)-\tau \leq \tau^{-(\alpha+1) / d+\varepsilon}$ and $x$ admits a stopping neighbor $y=\left(\eta, \sigma, \ell^{\prime}\right)$ with $\sigma \geq \tau$, and where $K_{2}$ is the number of $x=(\xi, \tau, \ell) \in \Phi$ such that $\xi \in Q_{6 r_{2}}(o)$ and $f_{\Phi}(x)-\tau \geq \tau^{-(\bar{\alpha}+1) / d+\varepsilon}$. Then, Lemma 16 implies $\mathbb{E} K_{1}<\infty$, while Theorem 6 yields $\mathbb{E} K_{2}<\infty$.

Remarks. Note that the distribution of the number of doublets observed in a bounded sampling window depends on $\alpha$. Intuitively, we would expect that the number of doublets increases in $\alpha$. Indeed, if the arrival rate of germs increases, then it should be more probable that a growing grain will form a doublet with a newly arrived germ. In Figure 4 we provide numerical evidence for this observation, where for each parameter $\alpha \in\{-0.5,0,1,2,3\}$ we provide a graph for one simulation run that shows how the number of connected components evolves with the number of grains that have been added to the system. For $\alpha \in\{-0.5,0\}$ the number of clusters stabilizes already after the first few hundred grains. For $\alpha \in\{1,2\}$ it is hard to tell from the data whether the number of connected components converges or diverges, but in either case the convergence/divergence is rather slow. Finally, for $\alpha=3$ one can see more clearly a divergent behavior.

Next, we investigate the problem of percolation in Apollonian packings where growth is stopped after a finite amount of time. To state the result precisely, consider the following finite-time variant of Definition 3 ,


Fig. 4: Evolution of the number of connected components with increasing number of grains for $\alpha \in\{-0.5,0\}$ (left) and for $\alpha \in\{1,2,3\}$ (right).

Definition 8 Let $\varphi \in \mathbb{N}^{*}$ and assume that $\varphi$ does not admit strong descending chains. For each $t_{0}>0$, we define a directed graph $G\left(t_{0}, \varphi\right)$ on $\varphi$, where for $x, y \in \varphi$ an edge is drawn from $x$ to $y$ if and only if the graph $G(\varphi)$ contains the edge $(x, y)$ and $\max \left\{f_{\varphi}(x), f_{\varphi}(y)\right\} \leq t_{0}$. The undirected graph $G^{\prime}\left(t_{0}, \varphi\right)$ is defined similarly.

We now show that with probability 1 there is no percolation after a finite amount of time in the associated stationary Apollonian packing.
Proposition 4 Let $m \geq 1$ and $\Phi$ be an independently and spatially stationary marked point process in $\mathbb{R}^{d,+, \mathcal{S}}$, where the underlying unmarked point process is an $m$-dependent point process in $\mathbb{R}^{d,+}$ with absolutely continuous second factorial moment measure. Furthermore, assume that with probability 1 each $x \in \Phi$ admits precisely one stopping neighbor $y \in \Phi$. Then, for each $t_{0}>0$, with probability 1 the graph $G^{\prime}\left(t_{0}, \Phi\right)$ does not percolate.

Proof. Similar to [9, the statement can be proven by applying the mass-transport principle. First, observe that uniqueness of stopping neighbors implies that if $x, x^{\prime}, x^{\prime \prime} \in \Phi$ are such that $G(\Phi)$ contains an edge both from $x^{\prime}$ to $x$ and from $x^{\prime}$ to $x^{\prime \prime}$, then $x=x^{\prime \prime}$. In particular, it suffices to exclude the existence of a self-avoiding path $\gamma=\left(x_{i}\right)_{i \geq 0}$ where either there is an edge from $x_{i}$ to $x_{i+1}$ in $G(\Phi)$ for all $i \geq 0$ or there is an edge from $x_{i+1}$ to $x_{i}$ in $G(\Phi)$ for all $i \geq 0$. Since Theorem 3 , shows that the directed graph $G(\Phi)$ does not percolate almost surely, it remains to exclude the second option, where without loss of generality, we may assume that $x_{1} \in \Phi_{\text {doub }}$. In other words, by stationarity, we need to show that with probability 1 there does not exist an infinite path in $G\left(t_{0}, \Phi\right)$ which ends inside $Q_{1}(o)$.

Recall that $h_{\Phi}$ maps each $x \in \Phi$ to its unique stopping neighbor. By Theorem 3, with probability 1, the directed graph $G(\Phi)$ does not percolate. Hence, for each $x \in \Phi$ the set

$$
V(x)=\left\{y \in \Phi \mid h_{\Phi}^{j}(x)=y \text { for infinitely many } j \geq 0\right\}
$$

consists of precisely two elements. Define a function $\psi: \Phi \times \Phi \rightarrow\{0,1 / 2\}$ such that $\psi(x, y)$ equals $1 / 2$, if $y \in V(x)$ and equals 0 otherwise. Furthermore, define a function $\psi^{\prime}: \mathbb{Z}^{d} \times \mathbb{Z}^{d} \rightarrow[0, \infty)$ by

$$
\psi^{\prime}\left(z_{1}, z_{2}\right)=\sum_{x \in \Phi \cap Q_{1}^{t_{0}, \mathcal{S}}} \sum_{\left(z_{1}\right)} \sum_{y \in \Phi \cap Q_{1}^{t_{0}, \mathcal{S}}\left(z_{2}\right)} \psi(x, y)
$$

and note that by stationarity,

$$
\mathbb{E} \sum_{z \in \mathbb{Z}^{d}} \psi^{\prime}(z, o)=\mathbb{E} \sum_{z \in \mathbb{Z}^{d}} \psi^{\prime}(o, z)=\mathbb{E} \#\left(\Phi \cap\left(Q_{1}^{t_{0}, \mathcal{S}}(o)\right)\right)<\infty
$$

In particular, $\sum_{z \in \mathbb{Z}^{d}} \psi^{\prime}(z, o)$ is almost surely finite, so that with probability 1 there does not exist an infinite path in $G\left(t_{0}, \Phi\right)$ which ends inside $Q_{1}(o)$.

It is expected that the percolative behavior of $G^{\prime}(\Phi)$ is quite different from that of the graphs $G^{\prime}(t, \Phi)$. Indeed, in a homogeneous Poisson scenario with ball-shaped grains, Theorem 5 shows that with probability 1 , all connected components of $G^{\prime}(\Phi)$ consist of infinitely many points $x=(\xi, \tau, \ell) \in$ $\Phi$ satisfying $f_{\Phi}(x)>\tau$. Nevertheless, Theorem 4 shows that the connected components of $G^{\prime}(\Phi)$ are small in the sense that each covers only a finite volume.

Proof of Theorem 4. The proof is similar to the proof of Proposition 4. In particular, the function $V$ has the same meaning. Define a function $\psi^{\prime}: \mathbb{Z}^{d} \times \mathbb{Z}^{d} \rightarrow[0, \infty)$ by

$$
\psi^{\prime}\left(z_{1}, z_{2}\right)=\sum_{x \in \Phi} \sum_{y \in V(x)} 1_{y \in\left(Q_{1}\left(z_{2}\right) \times[0, \infty) \times \mathcal{S}\right)} \frac{\nu_{d}\left(g\left(x, f_{\Phi}(x)\right) \cap Q_{1}\left(z_{1}\right)\right)}{2} .
$$

Then, by stationarity,

$$
\mathbb{E} \sum_{z \in \mathbb{Z}^{d}} \psi^{\prime}(z, o)=\mathbb{E} \sum_{z \in \mathbb{Z}^{d}} \psi^{\prime}(o, z) \leq \nu_{d}\left(Q_{1}(o)\right) .
$$

In particular, $\mathbb{P}\left(\sum_{z \in \mathbb{Z}^{d}} \psi^{\prime}(z, o)<\infty\right)=1$. Note that $\nu_{d}\left(\bigcup_{x \in C_{y}} g\left(x, f_{\Phi}(x)\right)\right)=\infty$ for some $y \in \Phi$ with $V(y) \cap Q_{1}(o) \neq \emptyset$ would imply $\sum_{z \in \mathbb{Z}^{d}} \psi^{\prime}(z, o)=\infty$, which completes the proof.

Remark. We conjecture that Proposition 4 can be sharpened in the sense that with probability 1 , for every $x \in \Phi$ the cluster $\bigcup_{y \in C_{x}} g\left(y, f_{\Phi}(y)\right)$ forms a bounded subset of $\mathbb{R}^{d}$.

Finally, we prove Theorem 5. That is, we show that when measuring the size of clusters in terms of the number of constituting germs, then percolation occurs almost surely. For this purpose, we restrict ourselves to the special case, where $\Phi$ is an independently marked Poisson point process in $\mathbb{R}^{d,+, \mathcal{S}}$ such that 1 ) the intensity function of the underlying Poisson point process is constant both in space and time, and 2) the marks are constant and given by the unit ball in $\mathbb{R}^{d}$. Our idea of proof is to consider locations where growing balls get into contact. We show that close to any such location, there are infinitely many smaller balls attaching to one of these two balls. By the hard-core property, the two balls in contact provide a sufficient amount of protection against interaction from distant germs. To make this precise, we first state an elementary geometric auxiliary result. It is used to provide (i) a lower bound for the distance of balls to the contact location of two other balls and (ii) an upper bound for the radius of a ball subject to the non-overlapping condition with the balls in contact.

Lemma 20 Let $\xi, \xi^{\prime} \in \mathbb{R}^{d}$ with $\xi \neq \xi^{\prime}$ and put $r=\left|\xi-\xi^{\prime}\right|$. Furthermore, let $H$ denote the hyperplane through $\xi^{\prime}$ that is perpendicular to $\xi-\xi^{\prime}$ and let $\eta \in \mathbb{R}^{d} \backslash B_{r}(\xi)$ be in the same half space of $H$ as $\xi$.
(i) If $\left|\eta-\xi^{\prime}\right|>r / 2$, then $\left|\eta-\xi^{\prime}\right|-(|\eta-\xi|-r) \geq r / 4$.
(ii) If $\left|\eta-\xi^{\prime}\right| \leq r / 2$, then $|\eta-\xi|-r \leq|\eta| / 4$.

Proof. Without loss of generality, let $\xi^{\prime}=o$ and assume that $|\eta-\xi| \geq r$. First, we deal with claim (i). If the distance of $\eta$ from $H$ is at least $r / 2$, then $|\eta| \geq|\eta-\xi|$. Otherwise, let $\eta^{\prime}$ denote the projection of $\eta$ on $H$. Then, since $\max \left\{3\left|\eta^{\prime}\right|,\left|\eta^{\prime}-\xi\right|\right\} \geq r$,

$$
|\eta|-|\eta-\xi| \geq\left|\eta^{\prime}\right|-\left|\eta^{\prime}-\xi\right|=-\frac{r^{2}}{\left|\eta^{\prime}\right|+\left|\eta^{\prime}-\xi\right|} \geq-\frac{3 r}{4} .
$$

Next, consider claim (ii). Again, write $\eta^{\prime}$ for the projection of $\eta$ to $H$. Then,

$$
|\eta-\xi|-r \leq \frac{\left|\eta^{\prime}\right|^{2}}{\left|\eta^{\prime}-\xi\right|+r} \leq \frac{|\eta|^{2}}{2 r} \leq \frac{|\eta|}{4},
$$

which proves the second claim.

Proof of Theorem 5. We show a stronger statement in the sense that if we consider any two balls that are in contact with each other, then there exists an infinite number of balls attaching to the union of those balls. Let $t_{0}>0$ be arbitrary and let $x_{1}=\left(\xi_{1}, \tau_{1}, \ell_{1}\right), x_{2}=\left(\xi_{2}, \tau_{2}, \ell_{2}\right) \in \Phi \cap \mathbb{R}^{d, t_{0}, \mathcal{S}}$ be two germs whose associated balls are in contact, i.e., $\max \left\{f_{\Phi}\left(x_{1}\right), f_{\Phi}\left(x_{2}\right)\right\} \leq t_{0}$ and $g\left(x_{1}, f_{\Phi}\left(x_{1}\right)\right) \cap$ $g\left(x_{2}, f_{\Phi}\left(x_{2}\right)\right)=\left\{\xi_{12}\right\}$ for some $\xi_{12} \in \mathbb{R}^{d}$. Put $r_{1}=f_{\Phi}\left(x_{1}\right)-\tau_{1}, r_{2}=f_{\Phi}\left(x_{2}\right)-\tau_{2}, B=B_{r_{1}}\left(\xi_{1}\right) \cup B_{r_{2}}\left(\xi_{2}\right)$ and assume that $r_{1}, r_{2}>0$. Choose any $s_{1} \in\left(0, \min \left\{1, r_{1} / 4, r_{2} / 4\right\}\right)$ such that $\Phi \cap\left(B_{s_{1}}\left(\xi_{12}\right) \times\left[0, t_{0}\right] \times\right.$ $\mathcal{S})=\emptyset$. Let $y_{1}=\left(\eta_{1}, \sigma_{1}, \ell_{1}^{\prime}\right)$ be chosen as the germ in $\Phi \cap\left(\left(B_{s_{1}}\left(\xi_{12}\right) \backslash B\right) \times[0, \infty) \times \mathcal{S}\right)$ whose time coordinate is minimal. We claim that if $\left|\eta_{1}-\xi_{12}\right| \leq s_{1} / 2$ and $\Phi \cap\left(B_{s_{1}}\left(\xi_{12}\right) \times\left(\sigma_{1},\left(\sigma_{1}+s_{1}\right)\right) \times \mathcal{S}\right)=\emptyset$, then the ball at $y_{1}$ attaches to the set $B$. Indeed, part (ii) of Lemma 20 shows that the radius of the ball (i.e., the growth duration) at $y_{1}$ is at most $s_{1} / 4$, so that the ball is contained in $B_{3 s_{1} / 4}\left(\xi_{12}\right)$. Parts (i) and (ii) of Lemma 20 show that no balls corresponding to germs in $\left(\mathbb{R}^{d} \backslash B_{s_{1}}\left(\xi_{12}\right)\right) \times[0, \infty)$ that are different from $x_{1}$ and $x_{2}$ intersect $B_{3 s_{1} / 4}\left(\xi_{12}\right)$. Finally, using $\Phi \cap\left(B_{s_{1}}\left(\xi_{12}\right) \times\left(\sigma_{1},\left(\sigma_{1}+s_{1}\right)\right) \times \mathcal{S}\right)=\emptyset$ shows that the ball at $y_{1}$ can only be in contact with the ball at $x_{1}$ or the ball at $x_{2}$.

Now, proceed recursively as follows. For $i \geq 2$ choose any $s_{i} \in\left(0, s_{i-1}\right)$ such that $\Phi \cap\left(B_{s_{i}}\left(\xi_{12}\right) \times\right.$ $\left.\left[0, \sigma_{i-1}+s_{i-1}\right] \times \mathcal{S}\right)=\emptyset$. Let $y_{i}=\left(\eta_{i}, \sigma_{i}, \ell_{i}^{\prime}\right)$ be chosen as the germ in $\Phi \cap\left(\left(B_{s_{i}}\left(\xi_{12}\right) \backslash B\right) \times[0, \infty) \times \mathcal{S}\right)$ whose time coordinate is minimal. Then, again Lemma 20 can be used to see that if $\left|\eta_{i}-\xi_{12}\right| \leq s_{i} / 2$ and $\Phi \cap\left(B_{s_{i}}\left(\xi_{12}\right) \times\left(\sigma_{i}, \sigma_{i}+s_{i}\right) \times \mathcal{S}\right)=\emptyset$, then the ball at $y_{i}$ attaches to the set $B$.

Hence, we conclude that conditioned on $\Phi \cap \mathbb{R}^{d, \sigma_{i-1}+s_{i-1} \times \mathcal{S}}$ the probability that the ball at $y_{i}$ attaches to the set $B$ is at least

$$
\frac{\nu_{d}\left(B_{s_{i} / 2}\left(\xi_{12}\right) \backslash B\right)}{\nu_{d}\left(B_{s_{i}}\left(\xi_{12}\right) \backslash B\right)} \exp \left(-\lambda \nu_{d}\left(B_{s_{i}}\left(\xi_{12}\right) \backslash B\right) s_{i}\right)
$$

The second factor is at least $\exp \left(-\lambda \kappa_{d}\right)$, whereas elementary geometry shows that the first can be bounded from below by a constant depending only on the radii $r_{1}$ and $r_{2}$. Therefore, with probability 1 , for infinitely many $i \geq 1$ the ball at $y_{i}$ attaches to the set $B$.

### 6.3 Uniqueness of stopping neighbors

In the previous subsection, we have seen that the question of uniqueness of stopping neighbors is important for proving the absence of percolation. This uniqueness property has already been investigated in literature. If there exists some $t_{0}>0$ such that $\Phi \subset \mathbb{R}^{d, t_{0}, \mathcal{S}}$, then the latter property has been considered in 9 . It should be possible to adapt the arguments presented in that paper to the case $\Phi \subset \mathbb{R}^{d,+, \mathcal{S}}$, but to keep our presentation self-contained we provide a different proof for the Poisson case with non-rotated grains. First, we discuss the effects on the growth-stopping times when removing one germ from the process. To state this result precisely, we need to introduce some notation.

Let $\varphi \in \mathbb{N}^{*}$ be locally finite and assume that the marks of $\varphi$ are constant and given by the unit ball $A$ with respect to a certain norm on $\mathbb{R}^{d}$. Let $f: \varphi \rightarrow[0, \infty)$ a family of $\varphi$-growth-stopping times. Furthermore, let $x_{0}=\left(\xi_{0}, \tau_{0}, \ell_{0}\right) \in \varphi$ be such that $\xi_{0} \notin \bigcup_{x \in \varphi \backslash\left\{x_{0}\right\}} \operatorname{int} g\left(x, \min \left\{\tau_{0}, f(x)\right\}\right)$ and put $t_{0}=f\left(x_{0}\right)<\infty$. Also put $\varphi^{\prime}=\varphi \cap \mathbb{R}^{d, t_{0}, \mathcal{S}} \backslash\left\{x_{0}\right\}$ and define $\psi=\varphi^{\prime} \cap \varphi_{x_{0}}$ as the set of all $x=(\xi, \tau, \ell) \in \varphi^{\prime}$ with $\xi \in \operatorname{int} g\left(x_{0}, \tau\right)$. Finally, let $f^{\prime}: \psi \rightarrow\left[0, t_{0}\right]$ be a family of $\left(t_{0}, \psi\right)$-growthstopping times and define the function $f^{\prime \prime}: \varphi^{\prime} \rightarrow\left[0, t_{0}\right]$ by $f^{\prime \prime}(x)=\min \left\{f(x), t_{0}\right\}$ if $x \in \varphi^{\prime} \backslash \psi$ and $f^{\prime \prime}(x)=f^{\prime}(x)$ if $x \in \psi$.

Lemma 21 The function $f^{\prime \prime}$ defines a family of $\left(t_{0}, \varphi^{\prime}\right)$-growth-stopping times.
Proof. First, observe that if $x=(\xi, \tau, \ell) \in \varphi^{\prime}$ is such that $\xi \in g\left(x_{0}, \tau\right)$, then $g(x, t) \subset g\left(x_{0}, t\right)$ for all $t \geq \tau$. Indeed, for any $\zeta \in \mathbb{R}^{d}$ we note that $\zeta \in g(\xi, t)$ is equivalent to $\zeta-\xi \in(t-\tau) A$. From $\xi-\xi_{0} \in\left(\tau-\tau_{0}\right) A$ and from the assumption that $A$ describes the unit ball with respect to a certain norm on $\mathbb{R}^{d}$, we conclude $\zeta-\xi_{0} \in\left(t-\tau_{0}\right) A$, i.e., $\zeta \in g\left(\xi_{0}, t\right)$. To check the hard-core property of $f^{\prime \prime}$ we have to verify that for every $x \in \varphi^{\prime}$ and $y \in \varphi^{\prime} \backslash \varphi_{x}^{\prime}$,

$$
\begin{equation*}
\left(\operatorname{int} g\left(x, f^{\prime \prime}(x)\right)\right) \cap g\left(y, \min \left\{f^{\prime \prime}(x), f^{\prime \prime}(y)\right\}\right)=\emptyset \tag{16}
\end{equation*}
$$

The cases where $x, y \in \psi$ or $x, y \in \varphi^{\prime} \backslash \psi$ are immediate. Next, assume $x \in \psi$ and $y \in \varphi^{\prime} \backslash \psi$. Then, the argument at the beginning of the proof yields $g\left(x, f^{\prime}(x)\right) \subset g\left(x_{0}, t_{0}\right)$. Since $f$ defines a family of $\varphi$-growth-stopping times, we have $\left(\operatorname{int} g\left(x_{0}, t_{0}\right)\right) \cap g(y, f(y))=\emptyset$. Finally, assume that $x \in \varphi^{\prime} \backslash \psi$ and $y \in \psi$. Then,

$$
\left(\operatorname{int} g\left(x, \min \left\{f(x), t_{0}\right\}\right)\right) \cap g\left(y, f^{\prime}(y)\right) \subset\left(\operatorname{int} g\left(x, \min \left\{f(x), t_{0}\right\}\right)\right) \cap g\left(x_{0}, t_{0}\right)
$$

where the right-hand side is empty, since $f$ defines a family of $\varphi$-growth-stopping times. This completes the proof of 16 .

It remains to verify existence of stopping neighbors. If $x \in \psi$ is such that $f^{\prime}(x)<t_{0}$ and $y \in \psi$ forms a stopping neighbor of $x$ with respect to $f^{\prime}$, then it is easy to see that $y$ also forms a stopping neighbor of $x$ with respect to $f^{\prime \prime}$. So let $x \in \varphi^{\prime} \backslash \psi$ be such that $f^{\prime \prime}(x)<t_{0}$ and let $y \in \varphi$ denote a stopping neighbor of $x$ with respect to $f$. Then, we distinguish two cases and first assume that $\xi \in \operatorname{int} g(y, \min \{\tau, f(y)\})$. In particular, our assumption $x \in \varphi^{\prime} \backslash \psi$ implies $y \in \varphi^{\prime} \backslash \psi$, so that $y$ is also a stopping neighbor of $x$ with respect to $f^{\prime \prime}$. Now assume $\xi \notin \operatorname{int} g(y, \min \{\tau, f(y)\})$. Then, $f(y) \leq f(x)<t_{0}$ so that it suffices to show $y \notin \psi$. However, $y \in \psi$ would imply

$$
\emptyset \neq g(x, f(x)) \cap g(y, f(y)) \subset g(x, f(x)) \cap g\left(x_{0}, f(y)\right)
$$

Since the right-hand side is contained in (int $\left.g\left(x_{0}, t_{0}\right)\right) \cap g(x, f(x))=\emptyset$, this completes the proof of Lemma 21.

In Lemma 22, we assume additionally that $A$ is strictly convex, i.e., $A$ is convex and the topological boundary of $A$ does not contain any line segments of positive length.
Lemma 22 Let $\alpha>-1, \lambda>0$ and $\Phi$ be an independently marked Poisson point process in $\mathbb{R}^{d,+, \mathcal{S}}$ whose marks are constant equal to $A$ and whose underlying Poisson point process has a spatially constant intensity function $\lambda: \mathbb{R}^{d,+} \rightarrow[0, \infty)$. Then, with probability 1 , for each $x \in \Phi$ there exists precisely one stopping neighbor $y \in \Phi$ with respect to $f_{\Phi}$.

Proof. Since $\lambda$ is spatially constant, we can write $\lambda(\tau)$ instead of $\lambda(\xi, \tau)$. The statement is easy if there exists $y \in \Phi$ such that $\xi \in \operatorname{int} g\left(y, f_{\Phi}(y)\right)$. Therefore, from now on we assume $\xi \notin$ $\bigcup_{y \in \Phi \backslash\{x\}}$ int $g\left(y, f_{\Phi}(y)\right)$. To show that with probability 1 , any such $x$ does not admit two distinct stopping neighbors $y_{1}, y_{2} \in \Phi$, we distinguish several cases.
Case 1: $\max \left\{f_{\Phi}\left(y_{1}\right), f_{\Phi}\left(y_{2}\right)\right\}<f_{\Phi}(x)$. Before we begin with the proof, it is convenient to recall some geometric notions from [16]. For $x \in \mathbb{R}^{d}$ and $B \subset \mathbb{R}^{d}$ a strictly convex body with $o \in$ int $B$ we write $h_{B}^{\prime}(x)=\min \{r \geq 0: x \in r B\}$. Furthermore, if additionally $K \subset \mathbb{R}^{d}$ is closed, put $\Pi_{B}(K, x)=\left\{y \in K: d^{B}(K, x)=h_{B}^{\prime}(y-x)\right\}$ and define the exoskeleton $\operatorname{exo}_{B}(K)$ of $K$ with respect to $B$ by $\operatorname{exo}_{B}(K)=\left\{x \in \mathbb{R}^{d} \backslash K: \# \Pi_{B}(K, x) \geq 2\right\}$ (recall from Section 3.2 that we write $\left.d^{B}(K, x)=\min \{r \geq 0:(x+r B) \cap K \neq \emptyset\}\right)$. It is shown in [16, Lemma 2.1] that $\nu_{d}\left(\operatorname{exo}_{B}(K)\right)=0$ if $B \subset \mathbb{R}^{d}$ is strictly convex and $K \subset \mathbb{R}^{d}$ defines a closed subset of $\mathbb{R}^{d}$. Note that if $x \in \Phi$ is such that $x$ admits two different stopping neighbors $y_{1}, y_{2} \in \Phi$, then

$$
\xi \in \operatorname{exo}_{A}\left(g\left(y_{1}, f_{\Phi}\left(y_{1}\right)\right) \cup g\left(y_{2}, f_{\Phi}\left(y_{2}\right)\right)\right)
$$

By Lemma 21, we compute

$$
f_{\Phi}\left(y_{i}\right)=\min \left\{f_{\Phi}(x), f_{\Phi}\left(y_{i}\right)\right\}=\min \left\{f_{\Phi}(x), f_{\Phi \backslash\{x\}}\left(y_{i}\right)\right\}=f_{\Phi \backslash\{x\}}\left(y_{i}\right)
$$

for all $i \in\{1,2\}$, so that

$$
\begin{aligned}
\mathbb{E} & \sum_{\substack{x, y_{1}, y_{2} \in \Phi \\
x, y_{1}, y_{2} \text { pw. disjoint }}} 1_{\xi \in \operatorname{exo}_{A}\left(g\left(y_{1}, f_{\Phi \backslash\{x\}}\left(y_{1}\right)\right) \cup g\left(y_{2}, f_{\Phi \backslash\{x\}}\left(y_{2}\right)\right)\right)} \\
= & \int_{\mathbb{R}^{d,+, s}} \lambda\left(\sigma_{1}\right) \int_{\mathbb{R}^{d,+, s}} \lambda\left(\sigma_{2}\right) \int_{0}^{\infty} \lambda(\tau) \\
& \mathbb{E} \nu_{d}\left(\operatorname{exo}_{A}\left(g\left(y_{1}, f_{\Phi \cup\left\{y_{1}, y_{2}\right\}}\left(y_{1}\right)\right) \cup g\left(y_{2}, f_{\Phi \cup\left\{y_{1}, y_{2}\right\}}\left(y_{2}\right)\right)\right)\right) \mathrm{d} \tau \mathrm{~d} y_{2} \mathrm{~d} y_{1} \\
= & 0
\end{aligned}
$$

Case 2: $f_{\Phi}(x)=f_{\Phi}\left(y_{1}\right)>f_{\Phi}\left(y_{2}\right)$. As before, we may use Lemma 21 to deduce that

$$
f_{\Phi \backslash\left\{y_{1}\right\}}\left(y_{2}\right)=\min \left\{f_{\Phi}\left(y_{2}\right), f_{\Phi}\left(y_{1}\right)\right\}=f_{\Phi}\left(y_{2}\right)
$$

Hence, by the hard-core property, $f_{\Phi \backslash\left\{y_{1}\right\}}(x) \leq f_{\Phi}(x)$ and another application of Lemma 21 yields

$$
f_{\Phi \backslash\left\{y_{1}\right\}}(x)=\min \left\{f_{\Phi}(x), f_{\Phi}\left(y_{1}\right)\right\}=f_{\Phi}(x)
$$

In particular, we conclude that

$$
\sigma_{1}=f_{\Phi}\left(y_{1}\right)-d^{A}\left(g\left(x, f_{\Phi}(x)\right), \eta_{1}\right)=f_{\Phi \backslash\left\{y_{1}\right\}}(x)-d^{A}\left(g\left(x, f_{\Phi \backslash\left\{y_{1}\right\}}(x)\right), \eta_{1}\right)
$$

where $y_{1}=\left(\eta_{1}, \sigma_{1}, \ell_{1}^{\prime}\right)$. Furthermore,

$$
\begin{aligned}
& \mathbb{E} \sum_{\substack{x, y_{1} \in \Phi \\
x, y_{1} \text { pw. disjoint }}} 1_{\sigma_{1}=f_{\Phi \backslash\left\{y_{1}\right\}}(x)-d^{A}\left(g\left(x, f_{\Phi \backslash\left\{y_{1}\right\}}(x)\right), \eta_{1}\right)} \\
& =\int_{\mathbb{R}^{d},+, \mathcal{S}} \lambda(\tau) \int_{\mathbb{R}^{d}} \mathbb{E} \int_{0}^{\infty} \lambda\left(\sigma_{1}\right) 1_{\sigma_{1}=f_{\Phi \cup\{x\}}(x)-d^{A}\left(g\left(x, f_{\Phi \cup\{x\}}(x)\right), \eta_{1}\right)} \mathrm{d} \sigma_{1} \mathrm{~d} \eta_{1} \mathrm{~d} x \\
& =0
\end{aligned}
$$

Case 3: $f_{\Phi}(x)=f_{\Phi}\left(y_{1}\right)=f_{\Phi}\left(y_{2}\right)$. As $x$ and $y_{2}$ are mutual stopping neighbors, we obtain that

$$
\sigma_{1}=f_{\Phi}\left(y_{1}\right)-d^{A}\left(g\left(x, f_{\Phi}(x)\right), \eta_{1}\right)=f_{\left\{x, y_{2}\right\}}(x)-d^{A}\left(g\left(x, f_{\left\{x, y_{2}\right\}}(x)\right), \eta_{1}\right)
$$

where $y_{1}=\left(\eta_{1}, \sigma_{1}, \ell_{1}^{\prime}\right)$, so that as in Case 2,

$$
\begin{aligned}
\mathbb{E} & \sum_{\substack{x, y_{1}, y_{2} \in \Phi \\
x, y_{1}, y_{2} \text { pw. disjoint }}} 1_{\sigma_{1}=f_{\left\{x, y_{2}\right\}}(x)-d^{A}\left(g\left(x, f_{\left\{x, y_{2}\right\}}(x)\right), \eta_{1}\right)} \\
= & \int_{\mathbb{R}^{d,+, \mathcal{S}}} \lambda(\tau) \int_{\mathbb{R}^{d,+, \mathcal{S}}} \lambda\left(\sigma_{2}\right) \int_{\mathbb{R}^{d}} \int_{0}^{\infty} \lambda\left(\sigma_{1}\right) 1_{\sigma_{1}=f_{\left\{x, y_{2}\right\}}(x)-d^{A}\left(g\left(x, f_{\left\{x, y_{2}\right\}}(x)\right), \eta_{1}\right)} \mathrm{d} \sigma_{1} \mathrm{~d} \eta_{1} \mathrm{~d} y_{2} \mathrm{~d} x \\
= & 0 .
\end{aligned}
$$

## 7 Open problems and topics of further research

In the present paper, we provided the basis for a rigorous mathematical treatment of random Apollonian packings and rotational random Apollonian packings which constitute popular grain packing models in physics. The results discussed in Sections 3 to 6 provide a hint to the rich mathematical structure of stationary Apollonian packings and we conclude this paper by advertising two conjectures as starting points for future research. Let $\lambda>0, \alpha>-1, A \in \mathcal{S}$ and $\Phi$ be an independently marked Poisson point process in $\mathbb{R}^{d,+, \mathcal{S}}$ that is constructed as in Theorem 7 We have seen in Section 5 that $\operatorname{AP}(\Phi)$ is space-filling in the sense that $\nu_{d}\left(\mathbb{R}^{d} \backslash \operatorname{AP}(\Phi)\right)=0$ a.s. More precisely, the random closed set $\mathbb{R}^{d} \backslash \mathrm{AP}(\Phi)$ is expected to be of fractal nature in the following sense.

Conjecture. $\mathbb{P}\left(d-1<\operatorname{dim}_{\text {Hausdorff }} \mathbb{R}^{d} \backslash \mathrm{AP}(\Phi)<d\right)=1$.
The problem of determining the Hausdorff dimension has already been considered for deterministic Apollonian packings of disks in dimension $d=2$. In [2] it has been shown that the latter dimension is at least 1.1 and at most 1.4. To estimate the Hausdorff dimension for planar random Apollonian packings with $\alpha=0$ and disk-shaped grains, we performed Monte Carlo simulations whose results support the conjecture that the Hausdorff dimension is not an integer. To be more precise, Figure 5 shows a plot of $\log \delta$ versus $\log a_{\delta}$, where $a_{\delta}=\mathbb{P}\left(Q_{\delta}(o) \cap \operatorname{AP}(\Phi) \neq \emptyset\right)$. Estimating the slope based on the last two data points suggests that, approximately, $a_{\delta} \in O\left(\delta^{0.43}\right)$, regardless of the value of $\alpha$. It is well-known that $a_{\delta} \in O\left(\delta^{0.43}\right)$ implies $\operatorname{dim}_{\text {Hausdorff }} \mathrm{AP}(\Phi) \leq 2-0.43$ with probability 1, see e.g. [18, 22]. Hence, we obtain sound evidence that the Hausdorff dimension is strictly smaller than 2 .


Fig. 5: $-\log _{2} \delta$ versus $\log _{2} a_{\delta}$ for $\alpha=-1 / 2$ (black), $\alpha=0$ (green), $\alpha=1$ (red), $\alpha=2$ (blue) and $\alpha=3$ (orange).

A second research question concerns connectivity in rotational random Apollonian packings. In Section 3.3, we provided an existence result for stationary approximations to rotational Apollonian packings and we conjecture that in distribution these approximations converge to a limiting object whose configurations inside a bounded sampling window would be similar to ordinary rotational random Apollonian packings, as illustrated in Figure 6. Moreover, we conjecture that in contrast to the results obtained in Section 6 the analogue of the graph $G^{\prime}(\Phi)$ for stationary rotational Apollonian packings consists of a single connected component with probability 1.


Fig. 6: Realization of a rotational random Apollonian packing

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