Applied Probability Trust (12 May 2009)

SCALING LIMITS FOR SHORTEST PATH LENGTHS ALONG THE EDGES OF STATIONARY TESSELLATIONS

FLORIAN VOSS,* *Ulm University*

CATHERINE GLOAGUEN,** *Orange Labs*

VOLKER SCHMIDT,* *Ulm University*

Abstract

We consider spatial stochastic models, which can be applied e.g. to telecommunication networks with two hierarchy levels. In particular, we consider Cox processes X_L and X_H concentrated on the edge set $T^{(1)}$ of a random tessellation T, where the points $X_{L,n}$ and $X_{H,n}$ of X_L and X_H can describe the locations of low–level and high–level network components, respectively, and $T^{(1)}$ the underlying infrastructure of the network, like road systems, railways, etc. Furthermore, each point $X_{L,n}$ of X_L is marked with the shortest path along the edges of T to the nearest (in the Euclidean sense) point of X_H. We investigate the typical shortest path length C^* of the resulting marked point process, which is an important characteristic e.g. in performance analysis and planning of telecommunication networks. In particular, we show that the distribution of C^* converges to simple parametric limit distributions if a scaling factor κ converges to zero and infinity, respectively. This can be used to approximate the density of C^* by analytical formulae for a wide range of κ.

Keywords: Stochastic Geometry, Random Geometric Graph, Cox Process, Palm Distribution, Poisson Approximation, Uniform Integrability, Subadditive Ergodic Theorem, Blaschke-Petkantschin Formula, Telecommunication Network

2000 Mathematics Subject Classification: Primary 60D05

Secondary 60G55,60F99,90B15

* Postal address: Institute of Stochastics, Ulm University, Helmholtzstr. 18, 89069 Ulm, Germany

** Postal address: Orange Labs, 38-40, rue du Général Leclerc, 92794 Issy-les-Moulineaux, France

1. Introduction

Asymptotic properties of spatial stochastic models are considered, which can be applied e.g. in the analysis and planning of telecommunication networks. More precisely, we consider stochastic models for networks with two hierarchy levels, i.e., there are network components of two different kinds: low-level components (LLC) and high-level components (HLC). The locations of both HLC and LLC are represented by points in the Euclidean plane $\mathbb{R}^2$. We then associate with each HLC a certain subset of $\mathbb{R}^2$ which is called its serving zone. This is done in such a way that the serving zones of the HLC are disjoint convex polygons which cover the whole $\mathbb{R}^2$. Each LLC is linked to the HLC in whose serving zone the LLC is located. In particular, we assume that the serving zones are constructed as the cells of the Voronoi tessellation with respect to the locations of HLC. This is equivalent to link each LLC to its nearest HLC, where "nearest" means with respect to the Euclidean distance. Furthermore, we assume that the HLC and LLC are located on the edges of a random geometric graph, where the link from a LLC to its nearest HLC is assumed to be the shortest path along the edges of that graph. In the case of telecommunication networks the edges of the random geometric graph represent the underlying infrastructure, e.g. an inner-city street system.

Thus, we study a class of stochastic network models which has been introduced in [10] as the Stochastic Subscriber Line Model (SSLM) for urban access networks. Note that the SSLM is a model from stochastic geometry which provides tools for the description of geometric features of the network. Based on this model, stochastic econometrical analysis can be done for real telecommunication networks, e.g. connection costs for access networks can be determined, see [13, 14, 34, 36], where we focus on the case that the infrastructure of the network is modeled by the edge set of a stationary random tessellation and both the HLC and LLC are modeled by Cox processes concentrated on this edge set. Then we are especially interested in the shortest path length along the edge set between LLC and HLC, which is an important performance characteristic in cost and risk analysis as well as in strategic planning of wired telecommunication. In order to define an appropriately chosen (global) distribution of the shortest path length we regard the so-called typical shortest path length C^*. It can be interpreted as the length of the shortest path from a location of LLC, which is chosen at random among all locations of LLC, and its nearest HLC. We are then interested in the asymptotic behaviour of the distribution of C^* for two extreme cases of model parameters. In particular, we show that the distribution of C^* converges to simple parametric limit distributions if a scaling factor κ converges to zero and infinity, respectively. This can be used to approximate the density of C^* by analytical formulae for a wide range of κ which is a great advantage e.g. for the econometrical analysis of real telecommunication networks, see [14]. The mathematical techniques, which we exploit in order to derive our main results presented in Theorems 3.1 and 3.2, include Palm calculus and Poisson approximation for stationary point processes, Kingman's subadditive ergodic theorem, and the generalized Blaschke-Petkantschin formula from geometric measure theory.

The paper is organized as follows. In Section 2 we give a short description of the particular stochastic network model considered in the present paper. Then, in Section 3, we present the main results stated in Theorems 3.1 and 3.2. The proof of Theorem 3.2 is given in Section 4, where some details are postponed to the Appendix. In Section 5, it is shown that the mixing and integrability conditions of Theorems 3.1 and 3.2 are fulfilled for various examples of random tessellations. Some extensions of our results to other performance characteristics, more general classes of random geometric graphs, and more general connection rules are discussed in Section 5.4. Finally, Section 6 concludes the paper and gives an outlook to possible future research.

2. Stochastic modelling of hierarchical networks

To begin with we give a short description of the particular stochastic network model considered in the present paper. For more details on this model see also [13]. Moreover, we briefly explain the mathematical background and introduce the notation we are using. For further details on spatial point processes and random tessellations, see e.g. [8, 29, 30, 31]. Surveys on applications of tools from stochastic geometry to spatial stochastic modelling of telecommunication networks can be found e.g. in [16, 40].

2.1. Marked point processes

First we recall some basic notions and results regarding marked point processes in $\mathbb{R}^2$. They can be used to model locations of customers or equipments in telecommunication networks. Let $\mathcal{B}^2$ denote the family of Borel sets of $\mathbb{R}^2$ and N the family of all simple and locally finite counting measures on $\mathcal{B}^2$. Note that each $\varphi \in N$ can be represented by the sequence $\{x_n\}$ of its atoms, i.e. $\varphi = \sum_n \delta_{x_n}$, where δ_x is the Dirac measure with $\delta_x(B) = 1$ if $x \in B$ and $\delta_x(B) = 0$ if $x \notin B$. Let $\mathcal{N}$ denote the σ-algebra of subsets of N generated by the sets $\{\varphi \in N : \varphi(B) = j\}$ for $j \in \mathbb{N}$ and $B \in \mathcal{B}^2$. The shift operator $t_x : N \mapsto N$ is defined by $t_x\varphi(B) = \varphi(B + x)$ for $x \in \mathbb{R}^2$ and $B \in \mathcal{B}^2$, where $B + x = \{x + y : y \in B\}$. Then a point process X is a random element of the measurable space $(N, \mathcal{N})$, where we identify X with the sequence $\{X_n\}$ of its (random) atoms, writing $X = \{X_n\}$ for brevity.

Let $\mathbb{M}$ be a Polish space with its Borel σ-algebra $\mathcal{B}_{\mathbb{M}}$. Then we use the notation $N_{\mathbb{M}}$ for the family of all counting measures on $\mathcal{B}^2 \otimes \mathcal{B}_{\mathbb{M}}$ which are simple and locally finite in the first component. Note that the atoms (x_n, m_n) of the counting measure $\psi = \sum_n \delta_{(x_n,m_n)} \in N_{\mathbb{M}}$ have two components: the location $x_n \in \mathbb{R}^2$ and the mark $m_n \in \mathbb{M}$. The σ-algebra $\mathcal{N}_{\mathbb{M}}$ is defined in the same way as above and the shift operator $t_x : N_{\mathbb{M}} \mapsto N_{\mathbb{M}}$ translates the first component of the atoms of $\psi \in N_{\mathbb{M}}$ by $-x$, i.e. $t_x(\psi) = \sum_n \delta_{(x_n - x, m_n)}$. A random element $X = \{(X_n, M_n)\}$ of $(N_{\mathbb{M}}, \mathcal{N}_{\mathbb{M}})$ is then called a marked point process.

2.2. Palm distributions

Stationarity and isotropy of (marked) point processes are defined in the usual way, i.e., assuming the invariance of their distributions with respect to arbitrary translations and rotations around the origin, respectively. By $\lambda > 0$ we denote the intensity of a stationary marked point process $X = \{(X_n, M_n)\}$, i.e. $\lambda = \mathbb{E}\#\{n : X_n \in [0,1]^2\}$, and the Palm mark distribution $\mathbb{P}^o_X : \mathcal{B}_{\mathbb{M}} \to [0,1]$ of X is given by

$$\mathbb{P}^o_X(G) = \frac{\mathbb{E}\#\{n : X_n \in [0,1)^2, M_n \in G\}}{\lambda}, \qquad G \in \mathcal{B}_{\mathbb{M}}. \tag{2.1}$$

A random variable M^* distributed according to $\mathbb{P}^o_X$ is called the typical mark of X.

Furthermore, two jointly stationary marked point processes $X^{(1)} = \{(X_n^{(1)}, M_n^{(1)})\}$ and $X^{(2)} = \{(X_n^{(2)}, M_n^{(2)})\}$ with intensities λ_1 and λ_2 and mark spaces $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively, will be considered as random element $Y = (X^{(1)}, X^{(2)})$ of the product space $N_{\mathbb{M}_1,\mathbb{M}_2} = N_{\mathbb{M}_1} \times N_{\mathbb{M}_2}$. The Palm distribution $\mathbb{P}^*_{X^{(i)}}$ of Y with respect to the i-th component, $i = 1, 2$, is then defined on $\mathcal{N}_{\mathbb{M}_1} \otimes \mathcal{N}_{\mathbb{M}_2} \otimes \mathcal{B}_{\mathbb{M}_i}$ by

$$\mathbb{P}^*_{X^{(i)}}(A \times G) = \frac{\mathbb{E}\#\{n : X_n^{(i)} \in [0,1)^2, M_n^{(i)} \in G, t_{X_n^{(i)}} Y \in A\}}{\lambda_i}, \tag{2.2}$$

where $A \in \mathcal{N}_{\mathbb{M}_1} \otimes \mathcal{N}_{\mathbb{M}_2}$ and $G \in \mathcal{B}_{\mathbb{M}_i}$. Note that the Palm mark distribution $\mathbb{P}^o_{X^{(i)}}$ of $X^{(i)}$ can be obtained from $\mathbb{P}^*_{X^{(i)}}$ as a marginal distribution.

2.3. Random tessellations

As a model for the underlying random geometric graph we consider the edge set of random tessellations of $\mathbb{R}^2$. Note that a random tessellation T is a locally finite partition $\{\Xi_n\}$ of $\mathbb{R}^2$ into random (compact and convex) polygons Ξ_n, which are called the cells of T. We can also regard T as a marked point process $\{(\alpha(\Xi_n), \Xi_n^o)\}$, where the shifted cells $\Xi_n^o = \Xi_n - \alpha(\Xi_n)$ contain the origin. The points $\alpha(\Xi_n) \in \Xi_n \subset \mathbb{R}^2$ are then called the nuclei of the cells Ξ_n of T. Furthermore, we can identify T with the edge set $T^{(1)} = \bigcup_n \partial\Xi_n$ of T. Note that $T^{(1)}$ is a random closed set in $\mathbb{R}^2$, i.e., $T^{(1)}$ is a random element of $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$, where $\mathcal{F}$ denotes the family of all closed subsets of $\mathbb{R}^2$ and $\mathcal{B}(\mathcal{F})$ is the smallest σ-algebra of subsets of $\mathcal{F}$ which contains the ,,hitting sets" $\mathcal{F}_C = \{B \in \mathcal{F} : B \cap C \neq \emptyset\}$ for all compact $C \in \mathcal{B}^2$.

If T is stationary, i.e., $T^{(1)} \stackrel{\mathrm{d}}{=} T^{(1)} + x$ for each $x \in \mathbb{R}^2$, then the intensity γ of T is defined as $\gamma = \mathbb{E}\nu_1(T^{(1)} \cap [0,1]^2)$, i.e. the mean length of $T^{(1)}$ per unit area, where ν_1 denotes the 1-dimensional Hausdorff measure. In the following we always assume that T is a (normalized) stationary tessellation with $\mathbb{E}\nu_1(T^{(1)} \cap [0,1]^2) = 1$. Furthermore, for each $\gamma > 0$ we consider the scaled tessellation T_γ with intensity γ which is defined by $T_\gamma = T/\gamma$, i.e., we scale the edge set $T^{(1)}$ with $1/\gamma$ getting $T_\gamma^{(1)}$ such that $\mathbb{E}\nu_1(T_\gamma^{(1)} \cap [0,1]^2) = \gamma$.

A random tessellation T is called isotropic if the distribution of $T^{(1)}$ is invariant with respect to rotations around the origin. Furthermore, a stationary tessellation T is called mixing if

$$\lim_{|x| \longrightarrow \infty} \mathbb{P}(T^{(1)} \in A, T^{(1)} + x \in A') = \mathbb{P}(T^{(1)} \in A)\, \mathbb{P}(T^{(1)} \in A')$$

for any $A, A' \in \mathcal{B}(\mathcal{F})$. Note that for any T which is mixing it holds that

$$\mathbb{P}(T^{(1)} \in A) = 1 \quad \text{or} \quad \mathbb{P}(T^{(1)} \in A) = 0 \qquad \text{for each } A \in \mathcal{I}(\mathcal{F}), \tag{2.3}$$

where $\mathcal{I}(\mathcal{F})$ denotes the sub-σ-algebra of invariant sets of $\mathcal{B}(\mathcal{F})$, i.e. $A + x = A$ for all $A \in \mathcal{I}(\mathcal{F})$ and $x \in \mathbb{R}^2$. A stationary tessellation T which satisfies condition (2.3) is said to be ergodic.

2.4. Cox processes on edge sets

For any $\gamma > 0$, we consider Cox point processes $X_H = \{X_{H,n}\}$ and $X_L = \{X_{L,n}\}$ concentrated on $T_\gamma^{(1)}$, in order to model the locations of HLC and LLC, respectively. In particular, we assume that X_H is a Cox process on $T_\gamma^{(1)}$ with linear intensity λ_ℓ which is constructed by placing homogeneous Poisson processes on the edges of T_γ with linear intensity λ_ℓ. The random driving measure $\Lambda_{X_H} : \mathcal{B}^2 \longrightarrow [0, \infty]$ of X_H is then given by

$$\Lambda_{X_H}(B) = \lambda_\ell \nu_1(B \cap T_\gamma^{(1)}), \qquad B \in \mathcal{B}^2 \,. \tag{2.4}$$

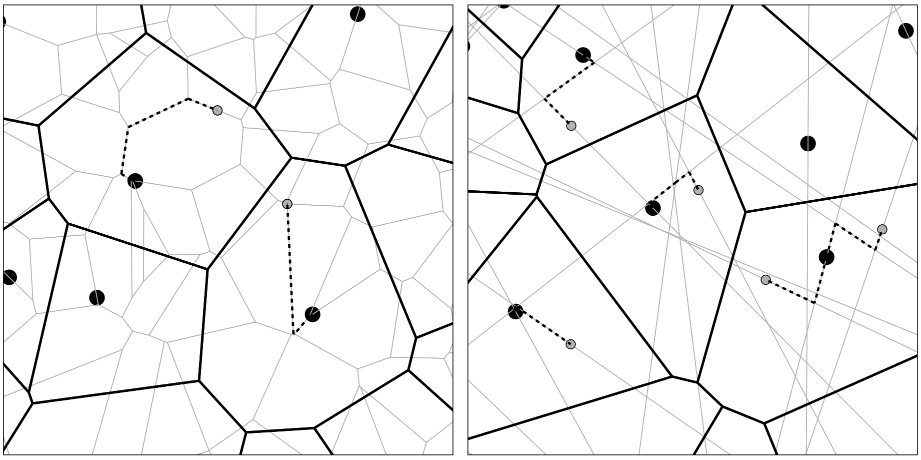
Analogously, X_L is a Cox process on $T_\gamma^{(1)}$ with linear intensity λ'_ℓ which is constructed in the same way, i.e., by placing Poisson processes on the edges of T_γ with linear intensity λ'_ℓ. Thus, X_H and X_L are Cox processes concentrated on the same edge set $T_\gamma^{(1)}$, where we assume that X_H and X_L are conditionally independent given T_γ. Furthermore, note that X_H and X_L are stationary, isotropic, and ergodic if T is stationary, isotropic, and ergodic, respectively. The planar intensities λ and λ' of X_H and X_L are given by $\lambda = \lambda_\ell \gamma$ and $\lambda' = \lambda'_\ell \gamma$.

2.5. Serving zones and shortest paths

Let $T_H = \{\Xi_{H,n}\}$ denote the Voronoi tessellation induced by the points $X_{H,n}$ of the Cox process $X_H = \{X_{H,n}\}$, i.e.

$$\Xi_{H,n} = \{x \in \mathbb{R}^2 : \, |x - X_{H,n}| \le |x - X_{H,m}| \, \text{for all } m \ne n\} \,.$$

The cells $\Xi_{H,n}$ of T_H are considered to be the serving zones of HLC. By means of the four modelling components T_γ, X_H, X_L and T_H we can construct the marked point process $X_{L,C} = \{(X_{L,n}, C_n)\}$, where the mark C_n is the length of the shortest path from $X_{L,n}$ to $X_{H,j}$ along the edge set $T_\gamma^{(1)}$ of T_γ provided that $X_{L,n} \in \Xi_{H,j}$. It is not



(a) PVT as infrastructure model (b) PLT as infrastructure model

FIGURE 1: Higher-level components with their serving zones (black) and lower-level components (grey with black boundary) with shortest paths (dashed) along the edge set (grey).

difficult to show that $X_{L,C}$ is a stationary and isotropic marked point process if T_γ is stationary and isotropic, respectively. Realizations of service zones and shortest paths are displayed in Figure 1(a) and (b) for T_γ being a Poisson-Voronoi tessellation (PVT) and a Poisson line tessellation (PLT), respectively.

The model characteristic we are mainly interested in is the distribution of the typical mark C^* of $X_{L,C}$. Thus, we are interested in the Palm mark distribution $\mathbb{P}^o_{X_{L,C}}$ of $X_{L,C}$, i.e., the distribution of the typical shortest path length.

Note that the realizations of $X_{L,C}$ can be constructed from the corresponding realizations of X_L and $X_{H,S}$, where $X_{H,S} = \{(X_{H,n}, S^o_{H,n})\}$ is a stationary marked point process with marks $S^o_{H,n} = (T^{(1)}_\gamma \cap \Xi_{H,n}) - X_{H,n}$. Thus, instead of $X_{L,C}$, we can consider the vector $Y = (X_L, X_{H,S})$ and the Palm distribution $\mathbb{P}^*_{X_L}$ of Y with respect to X_L, which has been introduced in (2.2). Let $(X^*_L, \widetilde{X}_{H,S})$ be distributed according to $\mathbb{P}^*_{X_L}$, where we use the notation $\widetilde{X}_{H,S} = \{(\widetilde{X}_{H,n}, \widetilde{S}^o_{H,n})\}$ and

$$\widetilde{T}^{(1)}_\gamma = \bigcup_{n\geq 1} \big(\widetilde{S}^o_{H,n} + \widetilde{X}_{H,n}\big)\,. \tag{2.5}$$

Note that $\widetilde{X}_H = \{\widetilde{X}_{H,n}\}$ is a (non-stationary) Cox process on $\widetilde{T}^{(1)}_\gamma$ with linear intensity λ_ℓ. Moreover, by $\widetilde{X}_{H,0}$ we denote the closest point (in the Euclidean sense) of $\{\widetilde{X}_{H,n}\}$

to the origin. Then, the typical shortest path length C^* can be given by $C^* = c(\widetilde{X}_{H,0})$, where $c(\widetilde{X}_{H,0})$ denotes the length of shortest path from the origin to $\widetilde{X}_{H,0}$, along the edges of $\widetilde{T}_\gamma^{(1)}$. In the following we always assume that the joint distribution of $C^*, \widetilde{X}_H$ and $\widetilde{T}_\gamma$ is given by $\mathbb{P}^*_{X_L}$.

3. Limit theorems for the typical shortest path length

We investigate the asymptotic behavior of the distribution of C^* for two different cases: $\gamma \to 0$ and $\gamma \to \infty$, i.e., unboundedly sparse edge sets and unboundedly dense edge sets, respectively. For $\gamma \to 0$, we show in Theorem 3.1 that the distribution of C^* converges weakly to an exponential distribution, where no specific assumption on the underlying stationary tessellation T is needed. Furthermore, for $\gamma \to \infty$ and T being a stationary and isotropic random tessellation which is mixing, we get in Theorem 3.2 that the distribution of C^* converges weakly to a Weibull distribution.

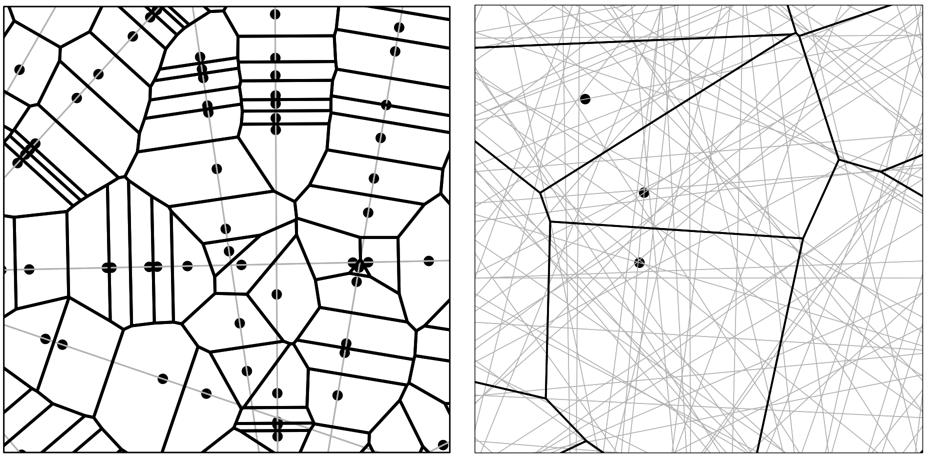
3.1. Scaling invariance property

Recall that the stochastic network model introduced in Section 2 and, in particular, the distribution of C^* is fully specified by T, γ, λ_ℓ and λ'_ℓ. Moreover, it can be shown (see e.g. [13, 34]) that the distribution of C^* does not depend on λ'_ℓ. Therefore, we only regard the parameters γ and λ_ℓ in the following. Sometimes we use the notation $C^* = C^*(\gamma, \lambda_\ell)$ to emphasize that the distribution of C^* depends on γ and λ_ℓ.

Furthermore, a scaling invariance property holds for this model. If the value of the quotient $\kappa = \gamma/\lambda_\ell$ is constant, then the structure of $X_{H,S}$ is fixed, but on different scales for different parameter vectors $(\gamma, \lambda_\ell) = (\kappa\lambda_\ell, \lambda_\ell)$. We are interested in the limiting behavior of the distribution of C^* for $\kappa \to 0$ with λ_ℓ fixed and for $\kappa \to \infty$ with $\lambda = \lambda_\ell\gamma$ fixed. In Figure 2 realizations of $X_{H,S}$ are shown for two (extremely small and large) values of κ, where the realization of T is sampled from a PLT. One can see that for small values of κ the segment systems within the serving zones mainly consist of one single segment only, whereas for large values of κ the networks inside the serving zones become rather dense.

3.2. Asymptotic exponential distribution for $\kappa \to 0$

First we regard the case that $\kappa = \gamma/\lambda_\ell \to 0$ with λ_ℓ fixed, i.e., $\gamma \to 0$.



(a) $\kappa = 0.5$ (b) $\kappa = 1000$

FIGURE 2: Realizations of $X_{H,S} = \{(X_{H,n}, S^o_{H,n})\}$ for extremal values of κ

Theorem 3.1. *Let T be an arbitrary stationary tessellation. Then, for any fixed $\lambda_\ell >\ 0$, it holds that*

$$C^*(\gamma, \lambda_\ell) \xrightarrow{\mathrm{d}} Z \qquad as \quad \gamma \to 0\,, \tag{3.1}$$

where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution and $Z \sim \mathrm{Exp}(2\lambda_\ell)$, i.e., the random variable Z is exponentially distributed with expectation $(2\lambda_\ell)^{-1}$.

Proof. Let $R_\gamma = \max\{r > 0 :\ B(o,r) \cap \widetilde{L}^o_\gamma = B(o,r) \cap \widetilde{T}^{(1)}_\gamma\}$, where $B(o,r)$ denotes the ball centered at the origin with radius r and $\widetilde{L}^o_\gamma$ is the segment containing the origin of the random edge set $\widetilde{T}^{(1)}_\gamma$ introduced in (2.5). It is not difficult to see that

$$\lim_{\gamma\to 0} R_\gamma = \infty \qquad \text{a.s.} \tag{3.2}$$

Recall that $C^* = c(\widetilde{X}_{H,0})$, where $\widetilde{X}_{H,0}$ is the closest point to the origin of the point process $\widetilde{X}_H = \{\widetilde{X}_{H,n}\}$ of HLC under $\mathbb{P}^*_{X_L}$, and note that the values of the distribution function $F_{C^*} : (0,\infty) \to (0,1)$ of C^* can be written as

$$\begin{aligned} F_{C^*}(x) \quad &= \quad \mathbb{P}(\widetilde{X}_{H,0} \in B(o,R_\gamma))\, \mathbb{P}(C^* \le x \mid \widetilde{X}_{H,0} \in B(o,R_\gamma)) \\ &\qquad + \mathbb{P}(\widetilde{X}_{H,0} \notin B(o,R_\gamma))\, \mathbb{P}(C^* \le x \mid \widetilde{X}_{H,0} \notin B(o,R_\gamma)) \end{aligned}$$

for each $x \ge 0$. It can be shown (see e.g. [9]) that $\widetilde{X}_H$ is a Cox process which consists of homogeneous Poisson processes with linear intensity λ_ℓ on the edges of $\widetilde{T}^{(1)}_\gamma$. This implies that

$$\mathbb{P}(C^* \le x \mid \widetilde{X}_{H,0} \in B(o,R_\gamma)) = \frac{\mathbb{P}(\min\{Z_1,Z_2\} \le x, \min\{Z_1,Z_2\} \le R_\gamma)}{\mathbb{P}(\widetilde{X}_{H,0} \in B(o,R_\gamma))}$$

for each $x > 0$, where the random variables Z_1 and Z_2 are independent, exponentially distributed with parameter λ_ℓ and independent of R_γ. Furthermore, we get that

$$\mathbb{P}(\widetilde{X}_{H,0} \notin B(o,R_\gamma)) = \mathbb{P}(\min\{Z_1,Z_2\} > R_\gamma) = \mathbb{E}\exp(-2\lambda_\ell R_\gamma)\,,$$

since $\min\{Z_1,Z_2\}$ is exponentially distributed with parameter $2\lambda_\ell$ and independent of R_γ. Thus, using (3.2), it follows that

$$\lim_{\gamma\to 0}\mathbb{P}(\widetilde{X}_{H,0} \notin B(o,R_\gamma)) = 0 \qquad \text{and} \qquad \lim_{\gamma\to 0}\mathbb{P}(\widetilde{X}_{H,0} \in B(o,R_\gamma)) = 1$$

and, consequently, $\lim_{\gamma\to 0} F_{C^*}(x) = \mathbb{P}(\min\{Z_1,Z_2\} \le x) = 1-\exp(-2\lambda_\ell x)$ for each $x \ge 0$.

Note that the case $\kappa = \gamma/\lambda_\ell \to 0$ with γ fixed and $\lambda_\ell \to \infty$ can be treated in the following way. Due to the scaling invariance property mentioned in Section 3.1 we have

$$\frac{1}{\lambda_\ell}\; C^*(\gamma,\lambda_\ell) \overset{\mathrm{d}}{=} C^*(\gamma/\lambda_\ell, 1)$$

for any $\gamma, \lambda_\ell > 0$. Thus, Theorem 3.1 yields that

$$\frac{1}{\lambda_\ell}\; C^*(\gamma,\lambda_\ell) \xrightarrow{\mathrm{d}} Z \qquad \text{as} \quad \lambda_\ell \to \infty\,,$$

where $Z \sim Exp(2)$.

3.3. Asymptotic Weibull distribution for $\kappa \to \infty$

In this section we assume that T is a stationary and isotropic random tessellation which is mixing. Furthermore, we assume that

$$\mathbb{E}\,\nu_1^2(\partial\Xi^*) < \infty\,, \tag{3.3}$$

where $\nu_1(\partial\Xi^*)$ denotes the circumference of the typical cell Ξ^* of T.

We investigate the asymptotic behavior of the distribution of $C^* = C^*(\gamma,\lambda_\ell)$ for $\kappa \to \infty$, where $\gamma \to \infty$ and $\lambda_\ell \to 0$ such that $\lambda_\ell\gamma = \lambda$ is fixed. In particular, we show that C^* converges in distribution to ξZ, where $\xi \ge 1$ is a certain constant which is multiplied by the (random) Euclidean distance Z from the origin to the nearest point of a stationary Poisson process of intensity λ. Then, it is easy to see that Z as well as ξZ have Weibull distributions.

Theorem 3.2. *Let* $Z \sim \mathrm{Wei}(\lambda\pi, 2)$ *for some* $\lambda > 0$. *Then there exists a constant* $\xi \geq 1$ *such that*

$$C^*(\gamma, \lambda_\ell) \xrightarrow{\mathrm{d}} \xi Z \qquad \textit{as} \quad \kappa \to \infty \tag{3.4}$$

provided that $\gamma \to \infty$ *and* $\lambda_\ell \to 0$ *with* $\lambda_\ell \gamma = \lambda$, *where* $\xi Z \sim Wei(\lambda\pi/\xi^2, 2)$.

The *proof* of Theorem 3.2 is split into several steps. We first show in Lemma 4.2 that under the Palm probability measure $\mathbb{P}^*_{X_L}$, the Euclidean distance $|\widetilde{X}_{H,0}|$ from the origin to the nearest point $\widetilde{X}_{H,0}$ of the point process $\widetilde{X}_H = \{\widetilde{X}_{H,n}\}$ of HLC converges in distribution to the corresponding characteristic of a stationary Poisson process with intensity λ. Furthermore, in Lemma 4.4, we show that for some constant $\xi \geq 1$ the difference between $\xi|\widetilde{X}_{H,0}|$ and the shortest path length $C^* = C^*(\gamma, \lambda_\ell)$ from the origin to $\widetilde{X}_{H,0}$ along the edge set $\widetilde{T}_\gamma^{(1)}$ converges in probability to zero. Then, combining the results of Lemmas 4.2 and 4.4, the assertion of Theorem 3.2 follows.

4. Proof of Theorem 3.2

4.1. Some auxiliary results on convergence of point processes

In the proofs of Lemmas 4.1 and 4.2 which will be given below, we use two classic results regarding the convergence in distribution of point processes, see e.g. [8, 19, 25]. Note that a sequence of point processes $X^{(1)}, X^{(2)}, \ldots$ in $\mathbb{R}^2$ is said to converge in distribution to a point process X in $\mathbb{R}^2$ if

$$\lim_{m\to\infty} \mathbb{P}(X^{(m)}(B_1) = i_1, \ldots, X^{(m)}(B_k) = i_k) = \mathbb{P}(X(B_1) = i_1, \ldots, X(B_k) = i_k)$$

for any $k \geq 1$, $i_1, \ldots, i_k \geq 0$ and for all finite sequences of bounded sets $B_1, \ldots, B_k \in \mathcal{B}^2$ which satisfy the condition $\mathbb{P}(X(\partial B_j) > 0) = 0$ for each $j = 1, \ldots, k$. In this case we shortly write $X^{(m)} \Longrightarrow X$.

Let $X = \{X_n\}$ be an arbitrary ergodic point process in $\mathbb{R}^2$ with $\mathbb{P}(X(\mathbb{R}^2) = 0) = 0$, and let $\lambda \in (0, \infty)$ denote the intensity of X. Then, the following limit theorem for independently thinned and appropriately re-scaled versions of X is true. For each

$c \in (0,1)$, let $X^{(c)}$ denote a point process which arises from X by independent thinning, where each atom X_n of X is deleted with probability $1-c$ (and „survives" with probability c). Furthermore, let $Y^{(c)}$ be a re-scaled version of $X^{(c)}$, where $Y^{(c)}(B) = X^{(c)}(B/\sqrt{c})$ for each $B \in \mathcal{B}^2$. Then, for each $c \in (0,1)$, the point process $Y^{(c)}$ is stationary with the same intensity λ as X, and

$$Y^{(c)} \Longrightarrow Y \qquad \text{if } c \to 0, \tag{4.1}$$

where Y is a stationary Poisson process in $\mathbb{R}^2$ with intensity λ, see e.g. Section 11.3 of [8] or Theorem 7.3.1 in [25]. Moreover, the following continuity property of Palm distributions holds. Let $X, X^{(1)}, X^{(2)}, \ldots$ be stationary point processes in $\mathbb{R}^2$ such that $\mathbb{P}(X(\mathbb{R}^2) = 0) = \mathbb{P}(X^{(m)}(\mathbb{R}^2) = 0) = 0$ for each $m \geq 1$ and let $\lambda, \lambda^{(1)}, \lambda^{(2)}, \ldots$ denote the intensity of $X, X^{(1)}, X^{(2)}, \ldots$, respectively. If $\lambda_m = \lambda$ for each $m \geq 1$ and $X^{(m)} \Longrightarrow X$ as $m \to \infty$, then

$$Y^{(m)} \Longrightarrow Y \qquad \text{as } m \to \infty, \tag{4.2}$$

where $Y, Y^{(1)}, Y^{(2)}, \ldots$ are point processes in $\mathbb{R}^2$ whose distribution is equal to the Palm distribution of $X, X^{(1)}, X^{(2)}, \ldots$, respectively, see e.g. Proposition 10.3.6 in [25].

4.2. Euclidean distance from the typical LLC to its closest HLC

Throughout this section we assume that the underlying tessellation T is ergodic. In order to prove that the Euclidean distance $|\widetilde{X}_{H,0}|$ from the typical LLC to its closest HLC is asymptotically Weibull distributed, we first show that the (stationary) Cox process X_H converges in distribution to a homogeneous Poisson process if $\kappa \to \infty$ provided that $\lambda_\ell \gamma = \lambda$ is constant.

Lemma 4.1. *If $\kappa \to \infty$, where $\lambda_\ell \gamma = \lambda$ for some constant $\lambda \in (0,\infty)$, then $X_H \Longrightarrow Y$, where Y is a stationary Poisson process with intensity λ.*

Proof. For each $\gamma > 1$, let $X_H = X_H(\gamma)$ denote the Cox process of HLC with parameters γ and λ_ℓ, where $\lambda_\ell = \lambda/\gamma$ for some constant $\lambda \in (0,\infty)$. Note that the Cox process $X_H(\gamma)$ can be obtained from $X_H(1)$ by independent thinning with survival probability $c = 1/\gamma$ and by subsequent re-scaling with scaling factor $\sqrt{1/\gamma}$. Furthermore, the Cox process $X_H(1)$ is ergodic, since T is ergodic. Thus, using (4.1), we get that $X_H(\gamma) \Longrightarrow Y$ as $\gamma \to \infty$.

Lemma 4.2. *Let $Z \sim \mathrm{Wei}(\lambda\pi, 2)$ for some $\lambda > 0$. Then $|\widetilde{X}_{H,0}| \xrightarrow{\mathrm{d}} Z$ as $\kappa \to \infty$ provided that $\gamma \to \infty$ and $\lambda_\ell \to 0$ such that $\lambda_\ell \gamma = \lambda$.*

Proof. Let $X_H^*(\gamma)$ be a point process in $\mathbb{R}^2$ whose distribution is equal to the Palm distribution of $X_H = X_H(\gamma)$. Furthermore, let Y be a stationary Poisson process with intensity λ. Note that the distribution of $Y + \delta_o$ is then equal to the Palm distribution of Y, see e.g. Proposition 13.1.VII in [8]. Thus, using (4.2), Lemma 4.1 gives that

$$X_H^*(\gamma) \Longrightarrow Y + \delta_o \tag{4.3}$$

as $\gamma \to \infty$ and $\lambda_\ell \to 0$, where $\lambda_\ell \gamma = \lambda$. Since X_L and X_H are Cox processes concentrated on $T_\gamma^{(1)}$ which are conditionally independent given $T_\gamma^{(1)}$, we get that $\widetilde{X}_H + \delta_0$ and the Palm version X_H^* of X_H have the same distributions. This is an easy consequence of the representation formula for the Palm distribution of stationary Cox processes, see e.g. Section 5.2 in [31]. In particular, this gives that for each $r > 0$

$$\begin{aligned}
\lim_{\gamma\to\infty} \mathbb{P}(|\widetilde{X}_{H,0}| > r) &= \lim_{\gamma\to\infty} \mathbb{P}(\widetilde{X}_H(B(o,r)) = 0) \\
&= \lim_{\gamma\to\infty} \mathbb{P}((\widetilde{X}_H + \delta_o)(B(o,r)) = 1) \\
&= \lim_{\gamma\to\infty} \mathbb{P}(X_H^*(B(o,r)) = 1) \\
&= \mathbb{P}((Y + \delta_0)(B(o,r)) = 1) \\
&= \mathbb{P}(Y(B(o,r)) = 0)\,,
\end{aligned}$$

where we used (4.3) in the last but one equality. Thus, for each $r > 0$,

$$\lim_{\gamma\to\infty} \mathbb{P}(|\widetilde{X}_{H,0}| > r) \;=\; \mathbb{P}(Y(B(o,r)) = 0) \;=\; \exp(-\lambda\pi r^2)\,,$$

which means that $|\widetilde{X}_{H,0}| \xrightarrow{\mathrm{d}} Z \sim \mathrm{Wei}(\lambda\pi, 2)$.

4.3. Shortest path length vs. scaled Euclidean distance

In this section we assume that T is a stationary and isotropic random tessellation which is mixing. Furthermore, we assume that the integrability condition (3.3) is satisfied. Then, we can show that for some constant $\xi \geq 1$ the difference between $\xi|\widetilde{X}_{H,0}|$ and the shortest path length $C^* = C^*(\gamma, \lambda_\ell)$ from the origin to $\widetilde{X}_{H,0}$ along the edge set $\widetilde{T}_\gamma^{(1)}$ converges in probability to zero. In order to show this we need the following auxiliary result.

Lemma 4.3. *Let* $\widetilde{T}^{(1)}_{\gamma,\varepsilon} = \big\{u \in \widetilde{T}^{(1)}_\gamma : \big|c(u) - \xi|u|\big| < \varepsilon\big\}$*, where* $\xi \geq 1$ *is some constant and* $c(u)$ *denotes the length of the shortest path from* u *to the origin along the edges of* $\widetilde{T}^{(1)}_\gamma$*. If* $\gamma \to \infty$ *and* $\lambda_\ell \to 0$*, where* $\lambda_\ell \gamma = \lambda$ *is fixed, then there exists* $\xi \geq 1$ *such that for each* $\varepsilon > 0$ *and* $r > 0$

$$\lim_{\gamma\to\infty} \mathbb{E} \exp\Big(- \frac{\lambda}{\gamma}\, \nu_1\big(\widetilde{T}^{(1)}_\gamma \backslash \widetilde{T}^{(1)}_{\gamma,\varepsilon} \cap B(o,r)\big)\Big) = 1\,. \tag{4.4}$$

The *proof* of this lemma is postponed to the Appendix. Now, using Lemma 4.3, we are able to complete the proof of Theorem 3.2 by showing that the following is true.

Lemma 4.4. *If* $\gamma \to \infty$ *and* $\lambda_\ell \to 0$ *such that* $\lambda_\ell\gamma = \lambda$*, then there is a constant* $\xi \geq 1$ *with* $C^*(\gamma, \lambda_\ell) - \xi|\widetilde{X}_{H,0}| \overset{\mathrm{P}}{\to} 0$*, where* $\overset{\mathrm{P}}{\to}$ *denotes convergence in probability.*

Proof. We have to show that there exists a constant $\xi \geq 1$ such that for any $\varepsilon > 0$ and $\delta > 0$ we can choose $\gamma_0 > 0$ with

$$\mathbb{P}\big(\big|C^* - \xi|\widetilde{X}_{H,0}|\big| > \varepsilon\big) \leq \delta$$

for all $\gamma > \gamma_0$. Note that

$$\begin{aligned}
&\mathbb{P}\big(\big|C^* - \xi|\widetilde{X}_{H,0}|\big| > \varepsilon\big) \\
&\qquad = \mathbb{P}\big(\big|C^* - \xi|\widetilde{X}_{H,0}|\big| > \varepsilon, |\widetilde{X}_{H,0}| \leq r\big) + \mathbb{P}\big(\big|C^* - \xi|\widetilde{X}_{H,0}|\big| > \varepsilon, |\widetilde{X}_{H,0}| > r\big)\,,
\end{aligned}$$

where $r > 0$ is an arbitrary fixed number. Since

$$\mathbb{P}\big(|\widetilde{X}_{H,0}| > r\big) \longrightarrow e^{-\lambda\pi r^2} \quad \text{as} \quad \gamma \longrightarrow \infty\,,$$

see Lemma 4.2, we can choose $r > 0$ such that $\mathbb{P}\big(|\widetilde{X}_{H,0}| > r\big) < \delta/2$ for all $\gamma > 0$ sufficiently large. Thus, it is enough to show that there exists $\gamma_0 > 0$ such that $\mathbb{P}\big(\big|C^* - \xi|\widetilde{X}_{H,0}|\big| > \varepsilon, |\widetilde{X}_{H,0}| \leq r\big) \leq \delta/2$ for all $\gamma > \gamma_0$. Let $\widetilde{N} = \widetilde{X}_H(B(o,r))$ denote the number of points of $\widetilde{X}_H$ in $B(o,r)$. Then we have

$$\begin{aligned}
&\mathbb{P}\big(\big|C^* - \xi|\widetilde{X}_{H,0}|\big| > \varepsilon, |\widetilde{X}_{H,0}| \leq r\big) \\
&\qquad \leq \mathbb{E}\left(\sum_{k=1}^{\infty} \mathbb{P}(\widetilde{N} = k \mid \widetilde{T}_\gamma)\, \mathbb{P}\Big(\max_{i=1,\ldots,k} \big(\big|c(Y_i) - \xi|Y_i|\big|\big) > \varepsilon \;\Big|\; \widetilde{T}_\gamma, \widetilde{N} = k\Big)\right) \\
&\qquad = \mathbb{E}\left(\sum_{k=1}^{\infty} \mathbb{P}(\widetilde{N} = k \mid \widetilde{T}_\gamma)\, \Big(1 - \mathbb{P}\big(\big|c(Y_1) - \xi|Y_1|\big| \leq \varepsilon \mid \widetilde{T}_\gamma\big)^k\Big)\right)\,,
\end{aligned}$$

where the points $Y_1, \ldots, Y_k$ are conditionally independent and identically distributed according to $\nu_1\big(\,\cdot\,\cap \widetilde{T}_\gamma^{(1)} \cap B(o,r)\big)/\nu_1\big(\widetilde{T}_\gamma^{(1)} \cap B(o,r)\big)$ for given $\widetilde{T}_\gamma$ and $\widetilde{N} = k$. In particular, for the conditional probability in the latter expression, we have

$$\begin{aligned}
\mathbb{P}\big(\big|c(Y_1) - \xi|Y_1|\big| \le \varepsilon \mid \widetilde{T}_\gamma\big) &= \frac{\displaystyle\int_{\widetilde{T}_\gamma^{(1)} \cap B(o,r)} \mathbb{I}_{[-\varepsilon,\varepsilon]}(c(u) - \xi|u|)\, \nu_1(du)}{\nu_1(\widetilde{T}_\gamma^{(1)} \cap B(o,r))} \\
&= \frac{\nu_1\big(\widetilde{T}_{\gamma,\varepsilon}^{(1)} \cap B(o,r)\big)}{\nu_1\big(\widetilde{T}_\gamma^{(1)} \cap B(o,r)\big)}\,.
\end{aligned}$$

Using that $\widetilde{N} \sim Poi(\widetilde{\lambda})$ with $\widetilde{\lambda} = \lambda_\ell \nu_1\big(\widetilde{T}_\gamma^{(1)} \cap B(o,r)\big)$ given $\widetilde{T}_\gamma$, we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathbb{P}(\widetilde{N} = k \mid \widetilde{T}_\gamma) \Big(1 - \mathbb{P}\big(\big|c(Y_1) - \xi|Y_1|\big| \le \varepsilon \mid \widetilde{T}_\gamma\big)^k\Big) \\
&\qquad = 1 - \sum_{k=0}^{\infty} e^{-\widetilde{\lambda}} \frac{\widetilde{\lambda}^k}{k!} \Big(\frac{\lambda_\ell \nu_1\big(\widetilde{T}_{\gamma,\varepsilon}^{(1)} \cap B(o,r)\big)}{\widetilde{\lambda}}\Big)^k \\
&\qquad = 1 - \sum_{k=0}^{\infty} e^{-\widetilde{\lambda}} \frac{1}{k!} \Big(\lambda_\ell \nu_1\big(\widetilde{T}_{\gamma,\varepsilon}^{(1)} \cap B(o,r)\big)\Big)^k \\
&\qquad = 1 - e^{-\lambda_\ell \big(\nu_1(\widetilde{T}_\gamma^{(1)} \cap B(o,r)) - \nu_1(\widetilde{T}_{\gamma,\varepsilon}^{(1)} \cap B(o,r))\big)}\,.
\end{aligned}$$

Thus we have

$$\lim_{\gamma\to\infty} \mathbb{P}\big(\big|C^* - |\widetilde{X}_{H,0}|\big| > \varepsilon, |\widetilde{X}_{H,0}| \le r\big) \;\le\; 1 - \lim_{\gamma\to\infty} \mathbb{E} \exp\Big(-\frac{\lambda}{\gamma}\,\nu_1\big(\widetilde{T}_\gamma^{(1)} \setminus \widetilde{T}_{\gamma,\varepsilon}^{(1)} \cap B(o,r)\big)\Big)\,.$$

Using Lemma 4.3 this gives that $\lim_{\gamma\to\infty} \mathbb{P}\big(\big|C^* - |\widetilde{X}_{H,0}|\big| > \varepsilon, |\widetilde{X}_{H,0}| \le r\big) = 0$, which completes the proof.

5. Examples

Recall that in Theorem 3.2 we assumed that the underlying tessellation T is stationary and isotropic. The examples of tessellations discussed in the present section obviously possess these properties. Furthermore, we assumed in Theorem 3.2 that T is mixing and fulfills the integrability condition (3.3). We first show that the mixing condition is satisfied for a wide class of tessellations. Moreover, we also show that (3.3) is true for these tessellations.

The tessellation models considered in the literature focus mainly on PLT and PVT as well as on Poisson-Delaunay tessellations (PDT) and on iterated tessellations constructed from these basic tessellations of Poisson type, see e.g. [1]–[5], [9]–[14] and [34]–[38]. Here, we assume that an iterated tessellation is either a T_I/T_{II}-superposition or a T_I/T_{II}-nesting of tessellations T_I and T_{II} as defined e.g. in [2, 24, 38]. Note that the edge set of a T_I/T_{II}-superposition is given by the union $T_I^{(1)} \cup T_{II}^{(1)}$, where T_I and T_{II} are independent. Furthermore, a T_I/T_{II} nesting is constructed by subdividing each cell of T_I by independent copies of T_{II}. We show that for these important models Theorem 3.2 can be applied. Furthermore, if T is a PLT or a T_I/T_{II}-superposition/nesting with T_I being a PLT, then we can even calculate the constant ξ explicitly that appears in Theorem 3.2. On the other hand, if T is a PDT, we get an upper bound for ξ.

5.1. Mixing tessellations

In order to apply Theorem 3.2 we have to show that the underlying tessellation T is mixing, where we will use the following criterion to show that a stationary random closed set is mixing.

Lemma 5.1. *A stationary random closed set Ξ in $\mathbb{R}^2$ is mixing if and only if*

$$\lim_{|x|\to\infty} \mathbb{P}(\Xi \cap C_1 = \emptyset, \Xi \cap (C_2 + x) = \emptyset) = \mathbb{P}(\Xi \cap C_1 = \emptyset)\, \mathbb{P}(\Xi \cap C_2 = \emptyset) \tag{5.1}$$

for all $C_1, C_2 \in \mathcal{R}$, where $\mathcal{R}$ is the family of all subsets of $\mathbb{R}^2$ which are finite unions of closed balls with rational radii and centres with rational coordinates.

Note that the statement of Lemma 5.1 is essentially Lemma 4 in [17], see also Theorem 9.3.2 in [30], where the (stronger) condition is considered that (5.1) holds for all compact sets $C_1, C_2 \subset \mathbb{R}^2$. However, it is easy to see that it suffices to assume that (5.1) holds for the separating class $\mathcal{R}$; see also Section 1.4 of [28]. To make this clear, we only have to show that $\mathcal{E} = \{\mathcal{F}^{C_0}_{C_1,\ldots,C_k} : C_0, \ldots, C_k \in \mathcal{R}', k \geq 0\}$ is a semi-algebra which generates $\mathcal{B}(\mathcal{F})$, where $\mathcal{R}' = \mathcal{R} \cup \emptyset$ and

$$\mathcal{F}^{C_0}_{C_1,\ldots,C_k} = \{F \in \mathcal{F} : F \cap C_0 = \emptyset, F \cap C_1 \neq \emptyset, \ldots, F \cap C_1 \neq \emptyset\}\,.$$

Note that the family $\mathcal{R}'$ is union-stable. Thus, by Lemma 2.2.2 in [30], we get that $\mathcal{E}$ is a semi-algebra. Moreover, let $G \subset \mathbb{R}^2$ denote an open set, then $G = \bigcup_{i=1}^\infty C_i$ for some $C_1, C_2, \ldots \in \mathcal{R}'$ and $\mathcal{F}_G = \{F \in \mathcal{F} : F \cap G \neq \emptyset\} = \bigcup_{n=1}^\infty \mathcal{F}_{\bigcup_{i=1}^n C_i}$, thus $\mathcal{F}_G \in \sigma(\mathcal{E})$. Since $\{\mathcal{F}_G : G \subset \mathbb{R}^2 \text{ open}\}$ generates $\mathcal{B}(\mathcal{F})$, we get that $\sigma(\mathcal{E}) = \mathcal{B}(\mathcal{F})$.

Now the statement of Lemma 5.1 can be proven by exactly the same arguments used in the proof of Lemma 4 in [17].

It is well known that T is mixing if T is a PDT, PVT and PLT, respectively, see e.g. Chapter 10.5 in [30]. Furthermore, using Lemma 5.1, we can show that T is mixing if T is an iterated tessellation constructed from these basic tessellations of Poisson type.

Lemma 5.2. *The tessellation T is mixing if T is a T_I/T_{II}-superposition of two mixing tessellations T_I and T_{II}, or a T_I/T_{II}-nesting of a mixing initial tessellation T_I and any stationary component tessellation T_{II}.*

Proof. Suppose first that T is a T_I/T_{II}-superposition. Then, for any $C_1, C_2 \in \mathcal{R}$

$$\begin{aligned}
&\mathbb{P}(T^{(1)} \cap C_1 = \emptyset,\, T^{(1)} \cap (C_2 + x) = \emptyset) \\
&\quad = \mathbb{P}(T_I^{(1)} \cap C_1 = \emptyset,\, T_I^{(1)} \cap (C_2 + x) = \emptyset,\, T_{II}^{(1)} \cap C_1 = \emptyset,\, T_{II}^{(1)} \cap (C_2 + x) = \emptyset) \\
&\quad = \mathbb{P}(T_I^{(1)} \cap C_1 = \emptyset,\, T_I^{(1)} \cap (C_2 + x) = \emptyset)\, \mathbb{P}(T_{II}^{(1)} \cap C_1 = \emptyset,\, T_{II}^{(1)} \cap (C_2 + x) = \emptyset)\,,
\end{aligned}$$

since T_I and T_{II} are independent. Thus, using Lemma 5.1, we get that T is mixing if T_I and T_{II} are mixing. Let now T be a T_I/T_{II}-nesting and assume that $C_1 = \cup_{j=1}^{n} B_j, C_2 = \cup_{j=n+1}^{n+m} B_j$ for closed balls $B_1, \ldots, B_{n+m} \subset \mathbb{R}^2$ with rational radii and centres with rational coordinates. Let $\Xi_1, \Xi_2, \ldots$ be the cells of the initial tessellation $T_I = \{\Xi_n\}$, let D denote the family of all decompositions of the index set $\{1, \ldots, n{+}m\}$ into nonempty subsets, and for $J = \{J_1, \ldots, J_k\} \in D$ consider the set

$$A_J(x) = \{\cup_{j \in J_i}(B_j + x\mathbb{1}_{\{j>n\}}) \subset \operatorname{int} \Xi_{j_i},\ i = 1, \ldots, k,\ \Xi_{j_i} \neq \Xi_{j_l} \text{ for } j_i \neq j_l\}\,, \quad (5.2)$$

i.e., each of the sets $\cup_{j \in J_i}(B_j + x\mathbb{1}_{\{j>n\}})$ is contained in a different cell of T_I. Using this notation we get

$$\begin{aligned}
&\lim_{|x|\to\infty} \mathbb{P}(T^{(1)} \cap C_1 = \emptyset,\, T^{(1)} \cap (C_2 + x) = \emptyset) \\
&\quad = \sum_{J \in D} \lim_{|x|\to\infty} \mathbb{P}(T^{(1)} \cap C_1 = \emptyset,\, T^{(1)} \cap (C_2 + x) = \emptyset,\, A_J(x))\,.
\end{aligned}$$

Since the cells $\Xi_1, \Xi_2, \ldots$ of T_I are finite with probability 1, we have

$$\lim_{|x|\to\infty} \mathbb{P}(T^{(1)} \cap C_1 = \emptyset,\, T^{(1)} \cap (C_2 + x) = \emptyset,\, A_J(x)) \quad = \quad 0$$

if there are $i \leq n$ and $j > n$ with $i, j \in J_l$ for some $l \in \{1, \ldots, k\}$. On the other hand, suppose that $J = \{J_1, \ldots, J_k\}$ is a decomposition of $\{1 \ldots, n{+}m\}$ with $J_i \subset \{1, \ldots, n\}$

for $i = 1, \ldots, l$ and $J_i \subset \{n+1, \ldots, n+m\}$ for $i = l+1, \ldots, k$. Then we get that

$$\begin{aligned}
&\mathbb{P}(T^{(1)} \cap C_1 = \emptyset,\, T^{(1)} \cap (C_2 + x) = \emptyset,\, A_J(x)) \\
&\qquad = \mathbb{P}(A_J(x),\, B_{J_i} \cap T^{(1)}_{II,i} = \emptyset,\, i = 1, \ldots, l,\, B_{J_i} + x \cap T^{(1)}_{II,i} = \emptyset,\, i = l+1, \ldots, k)\,,
\end{aligned}$$

where $B_{J_i} = \cup_{j \in J_i} B_j$ and $T_{II,1}, \ldots, T_{II,k}$ are independent copies of T_{II} which are independent of T_I. Thus we have

$$\begin{aligned}
&\mathbb{P}(A_J(x),\, B_{J_i} \cap T^{(1)}_{II,i} = \emptyset,\, i = 1, \ldots, l,\, B_{J_i} + x \cap T^{(1)}_{II,i} = \emptyset,\, i = l+1, \ldots, k) \\
&\qquad = \mathbb{P}(A_J(x))\, \mathbb{P}(B_{J_i} \cap T^{(1)}_{II,i} = \emptyset,\, i = 1, \ldots, l)\, \mathbb{P}(B_{J_i} \cap T^{(1)}_{II,i} = \emptyset,\, i = l+1, \ldots, k)\,.
\end{aligned}$$

Moreover, since T_I is mixing, we get

$$\lim_{|x| \to \infty} \mathbb{P}(A_J(x)) \quad = \quad \mathbb{P}(A_{J'}(o))\, \mathbb{P}(A_{J''}(o))\,,$$

where $J' = \{J_1, \ldots, J_l\}$ and $J'' = \{J_{l+1}, \ldots, J_k\}$ are the decompositions of $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$, respectively, induced by J, and $A_{J'}(o)$ resp. $A_{J''}(o)$ are defined analogously to (5.2). Summarizing the above considerations, we get

$$\begin{aligned}
&\lim_{|x| \to \infty} \mathbb{P}(T^{(1)} \cap C_1 = \emptyset,\, T^{(1)} \cap (C_2 + x) = \emptyset,\, A_J(x)) \\
&\quad = \quad \mathbb{P}(A_{J'}(o),\, B_{J_i} \cap T^{(1)}_{II,i} = \emptyset,\, i = 1, \ldots, l) \\
&\qquad\qquad\qquad\qquad \times\; \mathbb{P}(A_{J''}(o),\, B_{J_i} \cap T^{(1)}_{II,i} = \emptyset,\, i = l+1, \ldots, k) \\
&\quad = \quad \mathbb{P}(T^{(1)} \cap C_1 = \emptyset, A_{J'}(o))\, \mathbb{P}(T^{(1)} \cap C_2 = \emptyset, A_{J''}(o))\,,
\end{aligned}$$

which yields

$$\begin{aligned}
&\lim_{|x| \to \infty} \mathbb{P}(T^{(1)} \cap C_1 = \emptyset,\, T^{(1)} \cap (C_2 + x) = \emptyset) \\
&\quad = \quad \sum_{J \in D} \lim_{|x| \to \infty} \mathbb{P}(T^{(1)} \cap C_1 = \emptyset,\, T^{(1)} \cap (C_2 + x) = \emptyset,\, A_J(x)) \\
&\quad = \quad \sum_{J' \in D'} \sum_{J'' \in D''} \mathbb{P}(T^{(1)} \cap C_1 = \emptyset, A_{J'}(o))\, \mathbb{P}(T^{(1)} \cap C_2 = \emptyset, A_{J''}(o)) \\
&\quad = \quad \mathbb{P}(T^{(1)} \cap C_1 = \emptyset)\, \mathbb{P}(T^{(1)} \cap C_2 = \emptyset)\,,
\end{aligned}$$

where D', D'' is the family of all decompositions of $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$, respectively. Thus, by Lemma 5.1, the nested tessellation T is mixing.

5.2. Integrability condition (3.3)

The next result provides several classes of stationary tessellations such that the second moment of the circumference of their typical cell is finite, where $R(\Xi)$ denotes the radius of the minimal ball containing the random convex polygon Ξ.

Lemma 5.3. *If T is a PVT, PDT and PLT, respectively, then $\mathbb{E}R^2(\Xi^*) < \infty$ and, consequently,*

$$\mathbb{E}\nu_1^2(\partial\Xi^*) < \infty\,. \tag{5.3}$$

Moreover, (5.3) *holds if T is a a T_I/T_{II}-superposition/nesting such that*

$$\max\{\mathbb{E}R^2(\Xi_I^*), \mathbb{E}R^2(\Xi_{II}^*)\} < \infty\,, \tag{5.4}$$

where Ξ_I^* and Ξ_{II}^* is the typical cell of T_I and T_{II}, respectively.

Proof. Note that

$$\mathbb{E}\nu_1^2(\partial\Xi^*) \le \pi^2\mathbb{E}R^2(\Xi^*) \tag{5.5}$$

holds for the typical cell Ξ^* of any stationary tessellation T. Furthermore, if T is a PDT, then it is well-known that $\mathbb{E}R^2(\Xi^*) < \infty$. This result goes back to [26], see also Theorem 7.5 in [27] and Theorem 10.4.4 in [30]. Similarly, it is well known that $\mathbb{E}R^2(\Xi^*) < \infty$ holds if T is a PVT or PLT, see e.g. [7]. If $T = T_I/T_{II}$ is an iterated tessellation with cell intensity λ_T, then we can use Proposition 3.1 in [23] and Campbell's theorem in order to get

$$\begin{aligned}
\mathbb{E}\nu_1^2(\partial\Xi^*) \quad &= \quad \frac{\lambda_I}{\lambda_T}\,\mathbb{E}\Big(\sum_{\Xi_i\in T_{II}}\nu_1^2(\partial(\Xi_i\cap\Xi_I^*))\,\mathbb{I}_{\{\Xi_i\cap\Xi_I^*\neq\emptyset\}}\Big)\\
&= \quad \frac{\lambda_I\lambda_{II}}{\lambda_T}\,\mathbb{E}\int_{\mathbb{R}^2}\nu_1^2(\partial(\Xi_{II}^*+x\cap\Xi_I^*))\,\mathbb{I}_{\{\Xi_{II}^*+x\cap\Xi_I^*\neq\emptyset\}}\,\nu_2(dx)\,,
\end{aligned}$$

where λ_I, λ_{II} and Ξ_I^*, Ξ_{II}^* denote the cell intensities and the typical cells, respectively, of T_I and T_{II}. Note that we can assume that Ξ_I^* and Ξ_{II}^* are independent random convex bodies. Since $\nu_1^2(\partial(\Xi_{II}^*+x\cap\Xi_I^*)) \le \min\{\nu_1^2(\partial\Xi_I^*), \nu_1^2(\partial\Xi_{II}^*)\}$ we get

$$\begin{aligned}
\mathbb{E}\nu_1^2(\partial\Xi^*) \quad &\le \quad \frac{\lambda_I\lambda_{II}}{\lambda_T}\,\mathbb{E}\Big(\min\{\nu_1^2(\partial\Xi_I^*), \nu_1^2(\partial\Xi_{II}^*)\}\nu_2(\check{\Xi}_{II}^*\oplus\Xi_I^*)\Big)\\
&\le \quad \frac{4\pi\,\lambda_I\lambda_{II}}{\lambda_T}\,\mathbb{E}\Big(\min\{\nu_1^2(\partial\Xi_I^*), \nu_1^2(\partial\Xi_{II}^*)\}\;\max\{R^2(\Xi_I^*), R^2(\Xi_{II}^*)\}\Big)\,,
\end{aligned}$$

where in the latter inequality we used that

$$\nu_2(\check{\Xi}_{II}^*\oplus\Xi_I^*) \;\le\; \pi R^2(\check{\Xi}_I^*\oplus\Xi_{II}^*) \;\le\; 4\pi\max\{R^2(\Xi_I^*), R^2(\Xi_{II}^*)\}\,.$$

Using (5.5) and the independence of Ξ_I^* and Ξ_{II}^*, this gives

$$\begin{aligned}\mathbb{E}\nu_1^2(\partial\Xi^*) &\le \frac{4\pi^3\,\lambda_I\lambda_{II}}{\lambda_T}\,\mathbb{E}\Big(\min\{R^2(\Xi_I^*),R^2(\Xi_{II}^*)\}\ \max\{R^2(\Xi_I^*),R^2(\Xi_{II}^*)\}\Big)\\ &\le \frac{4\pi^3\,\lambda_I\lambda_{II}}{\lambda_T}\,\mathbb{E}R^2(\Xi_I^*)\,\mathbb{E}R^2(\Xi_{II}^*)\ <\ \infty\,,\end{aligned}$$

provided that (5.4) holds.

5.3. Asymptotic Weibull distribution of shortest path lengths

In Sections 5.1 and 5.2 we showed that the assumptions of Theorem 3.2 are fulfilled for several classes of random tessellations T. Thus, we are now able to apply Theorem 3.2 to these tessellations.

Corollary 5.1. *Let $Z \sim Wei(\lambda\pi, 2)$ and let T be a PDT, PVT or PLT, or an iterated tessellation $T = T_I/T_{II}$ such that condition* (5.4) *is fulfilled, where T is either*

1. *a superposition of two mixing tessellations T_I and T_{II}, or*
2. *a nesting of a mixing initial tessellation T_I and any stationary component tessellation T_{II}.*

Then $C^* \xrightarrow{\mathrm{d}} \xi Z$ for some constant $\xi \ge 1$ provided that $\gamma \to \infty$ and $\lambda_\ell \to 0$ such that $\lambda_\ell\gamma = \lambda$. Furthermore, if T is a PLT or a T_I/T_{II}-superposition/nesting, where T_I is a PLT, then $\xi = 1$. If T is a PDT, then $\xi \le 4/\pi \approx 1.27$.

Proof. The first part of the assertion follows from Theorem 3.2 if the results of Lemmas 5.2 and 5.3 as well as the comments immediately before Lemma 5.2 are taken into account. Now we consider the cases that T is a PLT, a T_I/T_{II}-superposition/nesting with a PLT T_I, or a PDT. To begin with, let T be a PLT with intensity 1. Then, the edge set $\widetilde{T}_\gamma^{(1)}$ of the tessellation $\widetilde{T}_\gamma$ introduced in Section 2.5 is generated by a random sequence of lines $L_0, L_1, \ldots$, where $L_1, L_2, \ldots$ form the edge set $T_\gamma^{(1)}$ of the (stationary and isotropic) PLT T_γ and L_0 is an isotropic line through the origin o, which is independent of T_γ. Thus we have

$$\frac{1}{\gamma}\,\nu_1(\widetilde{T}_\gamma^{(1)}\backslash\widetilde{T}_{\gamma,\varepsilon}^{(1)}\cap B(o,r)) \quad\le\quad \frac{1}{\gamma}\,\nu_1(T_\gamma^{(1)}\cap B(o,r)) + \frac{2r}{\gamma}\,.$$

Using Theorem A.1, together with Lemma A.1, this yields that the family of random variables $\{X_\gamma,\,\gamma>0\}$ with $X_\gamma = \nu_1(\widetilde{T}_\gamma^{(1)}\backslash\widetilde{T}_{\gamma,\varepsilon}^{(1)}\cap B(o,r))/\gamma$ is uniformly integrable since $\nu_1(T^{(1)}_\gamma \cap B(o,r))/\gamma = \pi r^2 \nu_1(T^{(1)} \cap B(o,r\gamma))/\nu_2(B(o,r\gamma))$ converges to $r^2\pi$ in L^1 due to the fact the PLT T is mixing and, therefore, ergodic ([8], Theorem 12.2.IV). Furthermore, in Lemma 4.3 we showed that the Laplace transform of X_γ converges to 1, which implies that $X_\gamma \xrightarrow{\mathrm{P}} 0$ ([20], Theorem 5.3). Thus, applying Theorem A.1 again, we get that

$$\lim_{\gamma\to\infty} \mathbb{E}\, X_\gamma = 0\,. \tag{5.6}$$

This result can be used to show that $\xi = 1$. Suppose that $\xi > 1$ and let $r > 2 > \varepsilon > 0$ with $\xi > 1+\varepsilon$. If the line L_i intersects the segment $L_{0,\varepsilon}$, where $L_{0,\varepsilon} = L_0 \cap B(o,\varepsilon/2)$, then for each $y \in L_i$ it holds that $0 \le c(y) - |y| \le \varepsilon$ since the path from y to o via the intersection point $L_i \cap L_{0,\varepsilon}$ is not longer than $|y|+\varepsilon$. Thus, if $|y| > 2$, then $(\xi-1)|y| \ge \varepsilon$ and, consequently,

$$\big|c(y) - \xi|y|\big| \;=\; \big|c(y) - |y| - (\xi-1)|y|\big| \;\ge\; \varepsilon(|y|-1) \ge \varepsilon\,,$$

which means that $y \in \widetilde{T}^{(1)}_\gamma \backslash \widetilde{T}^{(1)}_{\gamma,\varepsilon}$. Furthermore, if $L_i \cap L_{0,\varepsilon} \ne \emptyset$, it is not difficult to see that $L_i \cap B(o,r)\backslash B(o,2) \ge a$ for some constant $a > 0$. These two observations lead to

$$\begin{aligned} X_\gamma \;=\; \frac{1}{\gamma}\nu_1(\widetilde{T}^{(1)}_\gamma \backslash \widetilde{T}^{(1)}_{\gamma,\varepsilon} \cap B(o,r)) \;&\ge\; \frac{1}{\gamma}\,\nu_1\Big(\bigcup_{i: L_i \cap L_{0,\varepsilon} \ne \emptyset} \{L_i \cap B(o,r)\backslash B(o,2)\}\Big) \\ &\ge\; \frac{a}{\gamma}\,\#\{L_i : L_i \cap L_{0,\varepsilon} \ne \emptyset\} \end{aligned}$$

and, since $\#\{L_i : L_i \cap L_{0,\varepsilon} \ne \emptyset\} \sim Poi(2\,\varepsilon\gamma/\pi)$,

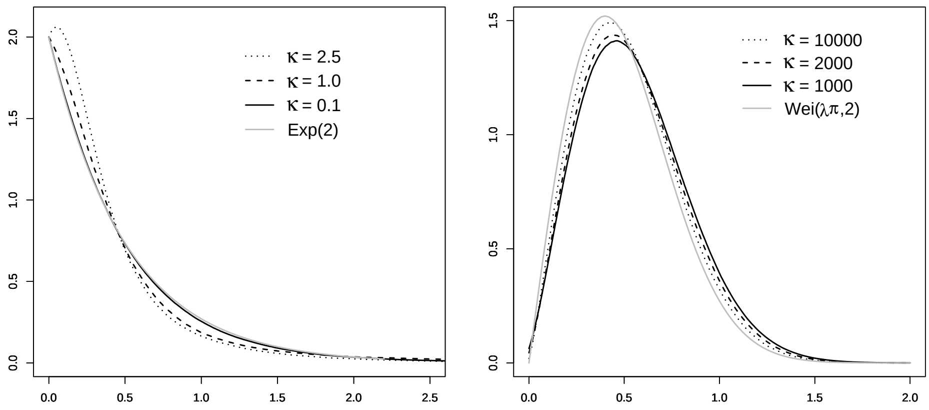
$$\liminf_{\gamma\to\infty} \mathbb{E}X_\gamma \;\ge\; \lim_{\gamma\to\infty} \frac{a}{\gamma}\,\mathbb{E}\#\{L_i : L_i \cap L_{0,\varepsilon} \ne \emptyset\} \;=\; \frac{2\,\varepsilon a}{\pi} > 0\,,$$

which is a contradiction to (5.6). Thus, $\xi = 1$ holds. If the tessellation $T = T_I/T_{II}$ is a superposition/nesting such that T_I is a PLT, then

$$\frac{1}{\gamma}\,\nu_1(\widetilde{T}^{(1)}_\gamma \backslash \widetilde{T}^{(1)}_{\gamma,\varepsilon} \cap B(o,r)) \;\ge\; 1\!\!1_{\{o \in \widetilde{T}^{(1)}_{I,\gamma}\}}\,\frac{1}{\gamma}\,\nu_1(\widetilde{T}^{(1)}_{I,\gamma} \backslash \widetilde{T}^{(1)}_{I,\gamma,\varepsilon} \cap B(o,r))\,,$$

where $\widetilde{T}^{(1)}_{I,\gamma}$ denotes the part of $\widetilde{T}^{(1)}_\gamma$ which corresponds to T_I. Since T_I is assumed to be a PLT, the same arguments as above can be applied to show that $\xi = 1$. Finally, let T be a PDT and let $N(y)$ denote that node of T which is closest to $y \in \mathbb{R}^2$. It has been shown in [4] that for any $t > 0$ and $y \in \partial B(o,1)$, there is a path $P(ty)$ from $N(o)$ to $N(ty)$ on $T^{(1)}$ with length $c(P(ty))$ such that almost surely

$$\lim_{t\longrightarrow\infty} \frac{c(P(ty))}{t} = \frac{4}{\pi}\,. \tag{5.7}$$



(a) $\kappa \to 0$ (where $\lambda_\ell = 1$) (b) $\kappa \to \infty$ (where $\gamma\lambda_\ell = 1$)

FIGURE 3: Densities of C^* if T is a PLT (together with corresponding limit distributions)

Consider the stationary point process $T^{(1)} \cap L$ of intersection points $\{X_i\}$, where $L = \{sy : s \in \mathbb{R}\}$ and $\cdots < X_{-1} < X_0 \le 0 < X_1 < \cdots$, and denote by $c(X_i, X_j)$ the shortest path length from X_i to X_j on $T^{(1)}$. Furthermore, consider the stationary marked point process $\{(X_i, c(X_i, N(X_i))\}$ and denote its typical mark by c_N^*. For each $i > 0$ we then have

$$\begin{aligned}\frac{c(X_0, X_i)}{|X_i - X_0|} &\le \frac{c(N(o), N(X_i))}{|X_i - X_0|} + \frac{c(X_0, N(o))}{|X_i - X_0|} + \frac{c(X_i, N(X_i))}{|X_i - X_0|} \\ &\le \frac{c(P(X_i))}{|X_i|} + \frac{c(X_0, N(o))}{|X_i|} + \frac{c(X_i, N(X_i))}{|X_i|} \,.\end{aligned}$$

Clearly, the second summand of the latter expression tends to 0 as $i \to \infty$. The same is true for the third summand, because $\{(X_i, c(X_i, N(X_i))\}$ is ergodic and $\mathbb{E}\, c_N^* < \infty$. Thus, by (5.7), we get that

$$\limsup_{i\to\infty} \frac{c(X_0, X_i)}{|X_i - X_0|} \le \frac{4}{\pi} \,. \tag{5.8}$$

On the other hand, we have $\mathbb{P}(\lim_{i\to\infty} c(X_0, X_i)/|X_i - X_0| = \xi) = 1$ if and only if $\mathbb{P}(\lim_{i\to\infty} c(X_0^*, X_i^*)/|X_i^* - X_0^*| = \xi) = 1$, where $\{X_i^*\}$ is the Palm version of $\{X_i\}$. Now, using (5.8) and (B.7), it follows that $\xi \le 4/\pi$.

5.4. Some extensions

Note that the setting of Theorem 3.2 can be generalized in different ways. For example, the statement of this theorem remains valid if instead of C^* the typical subscriber line length S^* is considered, where S^* is the shortest path length from the origin to the nearest point $X_{H,0}$ of X_H, which is defined as the sum of the distance from the origin to the nearest point of the edge set $T^{(1)}$ and the shortest path length on $T^{(1)}$ from this point to $X_{H,0}$ ([13]). Note that in this case the auxiliary results corresponding to Lemmas 4.3 and 4.4 can be proved basically in the same way.

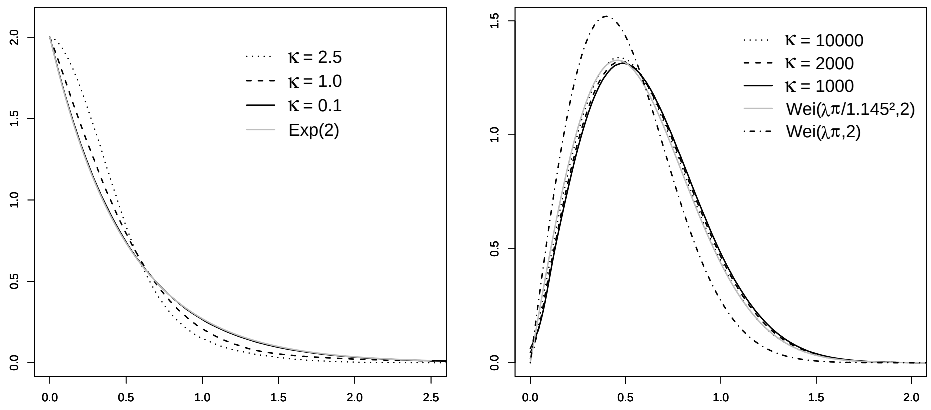
Furthermore, in the proof of Theorem 3.2 it is not necessary to assume that T is a random tessellation. But it is possible to consider an arbitrary stationary and isotropic segment process in $\mathbb{R}^d$ which is mixing and such that there is only one single cluster with probability 1. This means in particular that Theorem 3.2 can be extended to random geometric graphs.

Another kind of extensions can be obtained by relaxing the assumption that $X_{L,n}$ is connected to the nearest point of X_H, i.e., T_H is a Voronoi tessellation. For instance, $X_{L,n}$ can be connected to its k-th nearest neighbour of X_H for any $k \geq 1$. Then, in Theorem 3.2 we only have to replace Z by the distance from the origin to the k-th nearest point of a Poisson process which is distributed according to a generalized Gamma distribution ([15, 39]). Further possible extensions include that T_H is a certain Cox-Laguerre tessellation ([22]) or an aggregated tessellation ([2, 32]).

6. Conclusion and Outlook

We consider the typical shortest path length C^* of stochastic network models with two hierarchy levels, where the locations of network components are modelled by Cox processes on the edges of random tessellations. It is shown that the distribution of C^* converges to known limit distributions for extreme cases of the model parameters, i.e., if a certain scaling factor κ tends to zero or infinity.

The results of the present paper have applications in the analysis of telecommunication access networks since the distribution of C^* is closely related to cost and risk analysis of such networks ([14]). Using the fitting techniques introduced in [12], an optimal tessellation model can be chosen for a given set of road data. Moreover, the



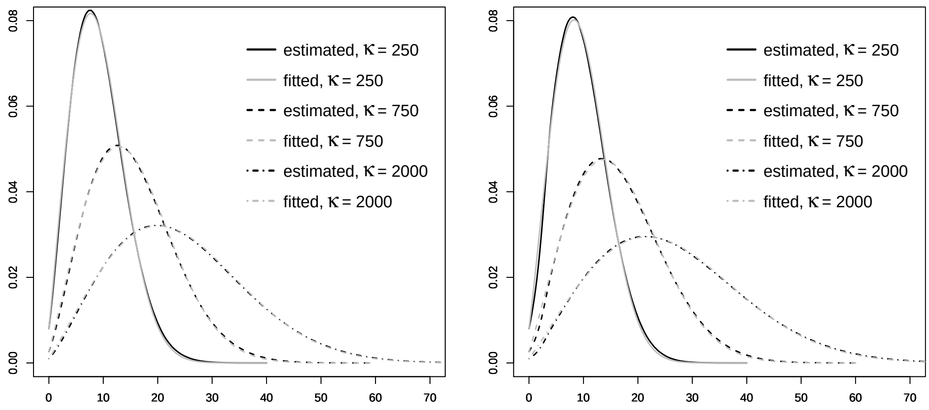
(a) $\kappa \to 0$ (where $\lambda_\ell = 1$) (b) $\kappa \to \infty$ (where $\gamma\lambda_\ell = 1$)

FIGURE 4: Densities of C^* if T is a PVT (together with corresponding limit distributions)

scaling factor κ can be estimated. Then, on the one hand, for small values of κ the limit distribution of C^* is directly available and it does not depend on the type of the optimal tessellation model. On the other hand, for large values of κ the limit distribution of C^* and an upper bound for this distribution is directly available if the optimal model is PLT or PLT-superposition/nesting and PDT, respectively.

In order to get an idea how small or large the scaling factor κ should be (to replace the distribution of C^* by the corresponding limit distribution) and how to calculate the constant ξ appearing in the limit distribution for C^* as $\kappa \to \infty$, the density of C^* can be estimated by Monte Carlo simulation of the typical serving zone ([34]). This can be done for PVT, PLT and PDT as well as for superpositions and nestings built from these basic tessellation models, using simulation algorithms of the typical serving zone introduced in [9, 11, 35, 37]. In Figures 3 and 4 estimated densities for different values of κ are shown together with the corresponding limit distributions if the tessellation model chosen for the underlying road system is a PLT and PVT, respectively. As can be seen in Figure 4 (b), the density of the $Wei(\lambda\pi/1.145^2, 2)$-distribution approximates the density of C^* very well for T being a PVT and $\kappa \geq 1000$. This suggests that in this case the constant ξ appearing in Theorem 3.2 and Corollary 5.1, respectively, is approximately 1.145.

Furthermore, the limiting distributions derived in the present paper can be used to



(a) PLT

(b) PVT

FIGURE 5: Densities of C^* (together with fitted parametric densities)

choose parametric densities which can be fitted to the estimated density of C^* for a large range of κ. Parametric families which include both exponential distributions and Weibull distributions turned out to be good choices, see [14]. In Figure 5 estimated densities for different values of κ are shown together with fitted truncated Weibull distributions. Note that these truncated Weibull distributions have two parameters and there is a quite good fit for both tessellation models considered in Figure 5 and for a large range of values of κ.

Acknowledgements

This work was supported by Orange Labs through Research grant No. 46 14 37 14.

References

[1] ALDOUS, D. AND KENDALL, W.S. (2008). Short-length routes in low-cost networks via Poisson line patterns. *Advances in Applied Probability* **40,** 1–21.

[2] BACCELLI, F., GLOAGUEN, C. AND ZUYEV, S. (2002). Superposition of planar Voronoi tessellations. *Communications in Statistics, Series Stochastic Models* **16,** 69–98.

[3] Baccelli, F., Klein, M., Lebourges, M. and Zuyev, S. (1997). Stochastic geometry and the architecture of communication networks. *Telecommunication Systems* **7,** 353–377.

[4] Baccelli, F., Tchoumatchenko, K. and Zuyev, S. (2000). Markov paths on the Poisson-Delaunay graph with applications to routeing in mobile networks. *Advances in Applied Probability* **32,** 1–18.

[5] Baccelli, F. and Zuyev, S. (1996). Poisson-Voronoi spanning trees with applications to the optimization of communication networks. *Operations Research* **47,** 619–631.

[6] Bauer, H. (1981). *Probability Theory and Elements of Measure Theory,* 2nd ed. Academic Press, London.

[7] Calka, P. (2002). The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane. *Advances in Applied Probability* **34,** 702–717.

[8] Daley, D. and Vere-Jones, D. (2003/2008). *An Introduction to the Theory of Point Processes,* 2nd ed., vol. I and II. Springer, New York.

[9] Fleischer, F., Gloaguen, C., Schmidt, V. and Voss, F. (2009). Simulation of the typical Poisson-Voronoi-Cox-Voronoi cell. *Journal of Statistical Computation and Simulation* (to appear).

[10] Gloaguen, C., Coupé, P., Maier, R. and Schmidt, V. (2002). Stochastic modelling of urban access networks. In *Proc. 10th Internat. Telecommun. Network Strategy Planning Symp.* VDE, Berlin, Munich. pp. 99–104.

[11] Gloaguen, C., Fleischer, F., Schmidt, H. and Schmidt, V. (2005). Simulation of typical Cox-Voronoi cells, with a special regard to implementation tests. *Mathematical Methods of Operations Research* **62,** 357–373.

[12] Gloaguen, C., Fleischer, F., Schmidt, H. and Schmidt, V. (2006). Fitting of stochastic telecommunication network models, via distance measures and Monte-Carlo tests. *Telecommunication Systems* **31,** 353–377.

[13] Gloaguen, C., Fleischer, F., Schmidt, H. and Schmidt, V. (2009). Analysis of shortest paths and subscriber line lengths in telecommunication access networks. *Networks and Spatial Economics* (to appear).

[14] Gloaguen, C., Voss, F. and Schmidt, V. (2009). Parametric distance distributions for fixed access network analysis and planning. *Proceedings of the 21st International Teletraffic Congress,* Paris, September 2009 (to appear).

[15] Haenggi, M. (2005). On Distances in Uniformly Random Networks. *IEEE Trans. on Information Theory* **51,** 3584–3586.

[16] Haenggi, M., Andrews, J.G., Baccelli, F., Dousse, O. and Franceschetti, M. (2009). Stochastic geometry and random graphs for the analysis and design of wireless networks. *IEEE Journal on Selected Areas in Communication* (to appear).

[17] Heinrich, L. (1992). On existence and mixing properties of germ-grain models. *Statistics* **23,** 271–286.

[18] Jensen, E. B. V. (1998). *Local Stereology.* World Scientific Publ. Co., Singapore.

[19] Kallenberg, O. (1986). *Random Measures,* 4th ed. Akademie-Verlag, Berlin.

[20] Kallenberg, O. (2002). *Foundations of Modern Probability,* 2nd ed. Springer, New York.

[21] Kingman, J. F. C. (1973). Subadditive ergodic theory. *Annals of Probability* **1,** 883–909.

[22] Lautensack, C. and Zuyev, S. (2008). Random Laguerre tessellations. *Advances in Applied Probability* **40,** 630–650.

[23] Maier, R., Mayer, J. and Schmidt, V. (2004). Distributional properties of the typical cell of stationary iterated tessellations. *Mathematical Methods of Operations Research* **59,** 287–302.

[24] Maier, R. and Schmidt, V. (2003). Stationary iterated random tessellations. *Advances in Applied Probability* **35,** 337–353.

[25] Matthes, K., Kerstan, J. and Mecke, J. (1978). *Infinitely Divisible Point Processes.* J. Wiley & Sons, Chichester.

[26] Miles, R.E. (1974). A synopsis of ‚Poisson flats in Euclidean spaces'. In: *Harding, E.F., Kendall, D.G. (eds.) Stochastic Geometry.* J. Wiley & Sons, New York, 202–227.

[27] Møller, J. (1989). Random tessellations in $\mathbb{R}^d$. *Advances in Applied Probability* **21,** 37–73.

[28] Molchanov, I. (2005). *Theory of Random Sets.* Springer, London.

[29] Okabe, A., Boots, B., Sugihara, K. and Chiu, S. N. (2000). *Spatial Tessellations,* 2nd ed. J. Wiley & Sons, Chichester.

[30] Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry.* Springer, Berlin.

[31] Stoyan, D., Kendall, W. S. and Mecke, J. (1995). *Stochastic Geometry and its Applications,* 2nd ed. J. Wiley & Sons, Chichester.

[32] Tchoumatchenko, K. and Zuyev, S. (2001). Aggregate and fractal tessellations. *Probability Theory and Related Fields* **121,** 198–218.

[33] Thorisson, H. (2000). *Coupling, Stationarity and Regeneration.* Springer, New York.

[34] Voss, F., Gloaguen, C., Fleischer, F. and Schmidt, V. (2009). Density estimation of typical shortest path lengths in telecommunication networks. Working Paper.

[35] Voss, F., Gloaguen, C., Fleischer, F. and Schmidt, V. (2009). Distributional properties of Euclidean distances in wireless networks involving road systems. *IEEE Journal on Selected Areas in Communication* (to appear).

[36] Voss, F., Gloaguen, C. and Schmidt, V. (2009). Capacity distributions in spatial stochastic models for telecommunication networks. *Proceedings of the 10th European Congress on Stereology and Image Analysis,* Milan, June 2009 (to appear).

[37] Voss, F., Gloaguen, C. and Schmidt V. (2009) Palm calculus for stationary Cox processes on iterated random tessellations. *Proceedings of the 7th International Symposium on Modeling and Optimization of Mobile, Ad Hoc and Wireless Networks,* Seoul, June 2009 (to appear).

[38] Weiss, R. and Nagel, W. (1999). Interdependences of directional quantities of planar tessellations. *Advances in Applied Probability* **31,** 664–678.

[39] Zuyev, S. (1999). Stopping sets: Gamma-type results and hitting properties. *Advances in Applied Probability* **31,** 355–366.

[40] Zuyev, S. (2009). Stochastic geometry and telecommunication networks. In: *Kendall, W.S. and Molchanov, I. (eds.) Stochastic Geometry: Highlights, Interactions and New Perspectives.* (to appear)

Appendix A. Some mathematical background

In the proof of Lemma 4.3 given below we make use of some well-known results from measure theory, the theory of subadditive processes, and geometric measure theory which are briefly summarized. We start with the definition of convergence in measure and uniform integrability which can be used to characterize L^1-convergence. A family of measurable functions $\{f_\gamma, \gamma \geq 1\}$ defined on a measurable space $(\Omega, \mathcal{A}, \mu)$ and taking values in $\mathbb{R}$ converges locally in μ-measure to a measurable function $f : \Omega \to \mathbb{R}$ if

$$\lim_{\gamma \to \infty} \mu(\{|f_\gamma - f| \geq \varepsilon\} \cap A) = 0 \tag{A.1}$$

for all $\varepsilon > 0$ and $A \in \mathcal{A}$ with $\mu(A) < \infty$, where μ is assumed to be a σ-finite measure. If μ is a probability measure such that (A.1) holds for each $\varepsilon > 0$ and $A = \Omega$, then one says that f_γ converges in probability to f. Furthermore, if for each $\varepsilon > 0$ there is a μ-integrable function g such that

$$\int_{\{|f_\gamma| \geq g\}} |f_\gamma(\omega)| \mu(d\omega) \leq \varepsilon \quad \text{for all } \gamma \geq 1\,, \tag{A.2}$$

then the family $\{f_\gamma, \gamma \geq 1\}$ is said to be uniformly μ-integrable. With the above definitions it is possible to characterize the L^1-convergence as follows; see Theorem 2.12.4 in [6].

Theorem A.1. *A sequence of μ-integrable functions $f_1, f_2, \dots : \Omega \to \mathbb{R}$ converges in L^1 to a μ-integrable function $f : \Omega \to \mathbb{R}$ if and only if* (i) *f_n converges locally in μ-measure to f and* (ii) *$\{f_n\}$ is uniformly μ-integrable.*

We still mention an elementary but useful result which immediately follows from the definition of uniform integrability.

Lemma A.1. *Let $\{f_\gamma, \gamma \geq 1\}$ and $\{g_\gamma, \gamma \geq 1\}$ be two families of measurable functions on $(\Omega, \mathcal{A}, \mu)$ which satisfy that $|f_\gamma| \leq |g_\gamma|$ for all $\gamma \geq 1$. Then $\{f_\gamma, \gamma \geq 1\}$ is uniformly μ-integrable if $\{g_\gamma, \gamma \geq 1\}$ is uniformly μ-integrable.*

Another useful tool is the notion of subadditivity. Let $\mathbf{Y}$ be a family of real-valued random variables $\mathbf{Y} = \{Y_{ij}, i, j \geq 1, i < j\}$ defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Note that $\mathbf{Y}$ can be seen as a random element of some measurable space $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ of double-indexed sequences, where $\mathcal{B}(\mathcal{S})$ is the Borel-σ-algebra of $\mathcal{S}$. Then $\mathbf{Y}$ is called a subadditive process if

1. $Y_{ik} \leq Y_{ij} + Y_{jk}$ for all $i < j < k$,
2. $\mathbf{Y} = \{Y_{ij}\} \stackrel{\mathrm{d}}{=} \mathbf{Y}' = \{Y_{i+1,j+1}\}$,
3. $\mathbb{E}Y_{01}^+ < \infty$, where $Y_{01}^+ = \max\{0, Y_{01}\}$.

The following result is due to Kingman ([21], Theorem 1). It is called the subadditive ergodic theorem; see also Theorem 10.22 in [20].

Theorem A.2. *Let $\mathbf{Y}$ be a subadditive process. Then the limit*

$$\zeta = \lim_{j \to \infty} \frac{1}{j} Y_{0j} \tag{A.3}$$

exists and is finite with probability one, where $\mathbb{E}\zeta = \inf_{j \to \infty} \mathbb{E}Y_{0j}/j$. If $\mathbb{E}\zeta > -\infty$, then the convergence in (A.3) also holds in the L^1-norm. Moreover, let $\mathcal{I}_\mathcal{S} \subset \mathcal{B}(\mathcal{S})$ be the σ-algebra of subsets of $\mathcal{S}$ which are invariant under the shift $\mathbf{Y} \mapsto \mathbf{Y}'$, where $Y'_{ij} = Y_{i+1,j+1}$, and let $\mathcal{I} = \mathbf{Y}^{-1}\mathcal{I}_\mathcal{S} \subset \mathcal{A}$ be the corresponding sub-σ-algebra of events. Then,

$$\zeta = \lim_{j \to \infty} \frac{1}{j} \, \mathbb{E}\big(Y_{0j} \mid \mathcal{I}\big)\,. \tag{A.4}$$

Note that a subadditive process $\mathbf{Y}$ is called ergodic if $\mathbb{P}(\mathbf{Y} \in A) = 0$ or $\mathbb{P}(\mathbf{Y} \in A) = 1$ for each $A \in \mathcal{I}_{\mathcal{S}}$. Thus, in the ergodic case, the limit ζ considered in (A.3) and (A.4), respectively, is almost surely constant.

Finally, we use a decomposition of the Hausdorff measure ν_1 which is a special case of the generalized Blaschke-Petkantschin formula ([18], Proposition 5.4).

Theorem A.3. *Let $C \subset \mathbb{R}^2$ be a differentiable curve and assume that*

$$\nu_1(\{x \in C : \ \mathit{Tan}[C,x] = \mathit{span}\{x\}\}) = 0\,, \tag{A.5}$$

where $\mathit{Tan}[C,x]$ is the tangent at x to C and $\mathit{span}\{x\} = \{cx : c \in \mathbb{R}\}$ is the line which goes through the origin $o \in \mathbb{R}^2$ and the point $x \in C$. Then, for any measurable $g : C \to [0,\infty)$ it holds that

$$\int_C g(x)\, \nu_1(dx) = \int_0^{2\pi} \sum_{x_i \in C \cap L_\Phi^+} \frac{|x_i|}{\sin \alpha_i}\, g(x_i)\; d\Phi\,, \tag{A.6}$$

where L_Φ^+ is the half line of direction $\Phi \in [0, 2\pi)$ emanating from o and α_i is the angle between $\mathit{Tan}[C, x_i]$ and $\mathit{span}\{x_i\}$.

Appendix B. Proof of Lemma 4.3

With the help of Theorems A.1 – A.3 stated above we are now able to prove Lemma 4.3. Obviously, $\limsup_{\gamma\to\infty} \mathbb{E}\exp\Big(- \frac{\lambda}{\gamma}\, \nu_1\big(\widetilde{T}^{(1)}_\gamma \setminus \widetilde{T}^{(1)}_{\gamma,\varepsilon} \cap B(o,r)\big)\Big) \le 1$. Thus it is sufficient to show that

$$\liminf_{\gamma\to\infty} \mathbb{E}\exp\Big(- \frac{\lambda}{\gamma}\, \nu_1\big(\widetilde{T}^{(1)}_\gamma \setminus \widetilde{T}^{(1)}_{\gamma,\varepsilon} \cap B(o,r)\big)\Big) \ge 1\,. \tag{B.1}$$

Proof of (B.1). First recall that we can identify $\widetilde{T}^{(1)}_\gamma$ with the Palm version $\Lambda^*_{T^{(1)}_\gamma}$ of the stationary random measure $\Lambda_{T^{(1)}_\gamma}$ given by $\Lambda_{T^{(1)}_\gamma}(B) = \nu_1(B \cap T^{(1)}_\gamma)$ for $B \in \mathcal{B}^2$ since $\Lambda_{T^{(1)}_\gamma}$ is the random driving measure of the Cox process X_L, see [31], p. 156. Then, using the abbreviation

$$h(\tau^{(1)}) = \exp\Big(-\frac{\lambda}{\gamma}\, \nu_1(\tau^{(1)} \setminus \tau^{(1)}_\varepsilon \cap B(o,r))\Big)\,,$$

where $\tau^{(1)}_\varepsilon = \big\{u \in \tau^{(1)} : \ \big|c(u) - \xi|u|\big| < \varepsilon\big\}$ and $c(u)$ denotes the length of the shortest path from u to the origin along the edge set $\tau^{(1)}$ of a tessellation τ with $o \in \tau^{(1)}$, we get from the Campbell theorem for stationary random measures ([8], Proposition 13.2.V) that

$$\begin{aligned}\mathbb{E}h(\widetilde{T}_\gamma^{(1)}) &= \frac{1}{\gamma\nu_2(B(o,1/\gamma))}\mathbb{E}\Big(\int_{T_\gamma^{(1)}\cap B(o,1/\gamma)} h(T_\gamma^{(1)}-x)\nu_1(dx)\Big)\\ &= \frac{1}{\pi}\mathbb{E}\Big(\int_{T^{(1)}\cap B(o,1)} h\big(T_\gamma^{(1)}-\frac{z}{\gamma}\big)\nu_1(dz)\Big),\end{aligned}$$

where we used the substitution $z=\gamma x$ in the last expression bearing in mind that $(1/\gamma)T^{(1)}=T_\gamma^{(1)}$. Furthermore, we put $T^{(1)}_{\gamma,\varepsilon,z}=\{y\in T_\gamma^{(1)}:|c(y,z/\gamma)-\xi|y-z/\gamma||<\varepsilon\}$, where $c(y,z/\gamma)$ denotes the length of the shortest path from y to z/γ along the edges of the considered graph. Then, for each $\gamma\ge 1$, we get that

$$\begin{aligned}\mathbb{E}h\big(\widetilde{T}_\gamma^{(1)}\big) &= \frac{1}{\pi}\,\mathbb{E}\Big(\int_{T^{(1)}\cap B(o,1)} \exp\Big(-\frac{\lambda}{\gamma}\,\nu_1\big(T_\gamma^{(1)}\setminus T^{(1)}_{\gamma,\varepsilon,z}\cap B(z/\gamma,r)\big)\Big)\nu_1(dz)\Big)\\ &\ge \frac{1}{\pi}\,\mathbb{E}\Big(\nu_1\big(T^{(1)}\cap B(o,1)\big)\inf_{z\in T^{(1)}\cap B(o,1)}\exp\Big(-\frac{\lambda}{\gamma}\,\nu_1\big(T_\gamma^{(1)}\setminus T^{(1)}_{\gamma,\varepsilon,z}\cap B(z/\gamma,r)\big)\Big)\Big)\\ &= \frac{1}{\pi}\,\mathbb{E}\Big(\nu_1\big(T^{(1)}\cap B(o,1)\big)\exp\Big(-\sup_{z\in T^{(1)}\cap B(o,1)}\frac{\lambda}{\gamma}\,\nu_1\big(T_\gamma^{(1)}\setminus T^{(1)}_{\gamma,\varepsilon,z}\cap B(z/\gamma,r)\big)\Big)\Big)\\ &\ge \frac{1}{\pi}\,\mathbb{E}\Big(\nu_1\big(T^{(1)}\cap B(o,1)\big)\exp\Big(-\sup_{z\in T^{(1)}\cap B(o,1)}\frac{\lambda}{\gamma}\,\nu_1\big(T_\gamma^{(1)}\setminus T^{(1)}_{\gamma,\varepsilon,z}\cap B(o,r+1)\big)\Big)\Big).\end{aligned}$$

Now, in order to prove (B.1), it is sufficient to show that

$$X_\gamma \xrightarrow{L^1} 0 \qquad \text{for } \gamma\to\infty, \tag{B.2}$$

where $X_\gamma=\sup_{z\in T^{(1)}\cap B(o,1)}\frac{1}{\gamma}\,\nu_1\Big(T_\gamma^{(1)}\setminus T^{(1)}_{\gamma,\varepsilon,z}\cap B(o,r+1)\Big)$. To see this, note first that (B.2) implies that X_γ converges in probability to 0. Thus, the random variable $Y_\gamma=\exp(-\lambda X_\gamma)\nu_1(T^{(1)}\cap B(o,1))$ converges in probability to $\nu_1\big(T^{(1)}\cap B(o,1)\big)$ if (B.2) holds. Moreover, $Y_\gamma\le\nu_1(T^{(1)}\cap B(o,1))$ for all $\gamma\ge1$ and $\mathbb{E}\nu_1(T^{(1)}\cap B(o,1))=\pi<\infty$, which means that $\{Y_\gamma,\,\gamma\ge1\}$ is uniformly integrable. Hence, Theorem A.1 yields that Y_γ converges in L^1 to $\nu_1\big(T^{(1)}\cap B(o,1)\big)$ and, in particular, $\lim_{\gamma\to\infty}1/\pi\mathbb{E}Y_\gamma = 1/\pi\mathbb{E}\nu_1\big(T^{(1)}\cap B(o,1)\big)=1$ if (B.2) holds. Thus, (B.1) follows if we can show that (B.2) is true.

Proof of (B.2). Since $X_\gamma\ge0$ it suffices to show that $\mathbb{E}X_\gamma\to0$. Furthermore, note that with probability 1 the segments of the segment system $T_\gamma^{(1)}\cap B(o,r+1)$ fulfill the conditions of Theorem A.3, since none of these segments ,,points" to the origin. Thus, using Theorem A.3 we get that

$$
\begin{aligned}
\mathbb{E}X_\gamma &= \mathbb{E}\Big(\sup_{z\in T^{(1)}\cap B(o,1)} \frac{1}{\gamma} \int_{T_\gamma^{(1)}\cap B(o,r+1)} \mathbb{1}_{[\varepsilon,\infty)}\Big(\big|c(y,\frac{z}{\gamma}) - \xi|y-\frac{z}{\gamma}|\big|\Big)\,\nu_1(dy)\Big) \\
&= \mathbb{E}\Big(\frac{1}{\gamma}\sup_{z\in T^{(1)}\cap B(o,1)} \int_0^{2\pi} \sum_{\substack{X_i\in T_\gamma^{(1)}\cap L_\Phi^+:\\ |X_i|\le r+1}} \frac{|X_i|}{\sin\alpha_i}\mathbb{1}_{[\varepsilon,\infty)}\Big(\big|c(X_i,\frac{z}{\gamma}) - \xi|X_i-\frac{z}{\gamma}|\big|\Big)\,d\Phi\Big) \\
&\le \frac{r+1}{\gamma}\mathbb{E}\Big(\int_0^{2\pi} \sup_{z\in T^{(1)}\cap B(o,1)} \sum_{\substack{X_i\in T_\gamma^{(1)}\cap L_\Phi^+:\\ |X_i|\le r+1}} \frac{1}{\sin\alpha_i}\mathbb{1}_{[\varepsilon,\infty)}\Big(\big|c(X_i,\frac{z}{\gamma}) - \xi|X_i-\frac{z}{\gamma}|\big|\Big)\,d\Phi\Big) \\
&= \frac{2\pi(r+1)}{\gamma}\mathbb{E}\Big(\sup_{z\in T^{(1)}\cap B(o,1)} \sum_{\substack{X_i\in T_\gamma^{(1)}\cap L^+:\\ |X_i|\le r+1}} \frac{1}{\sin\alpha_i}\mathbb{1}_{[\varepsilon,\infty)}\Big(\big|c(X_i,\frac{z}{\gamma}) - \xi|X_i-\frac{z}{\gamma}|\big|\Big)\Big) \\
&= 2\pi(r+1)\mathbb{E}g_\gamma\big(T^{(1)}\big)\,,
\end{aligned}
$$

where in the last but one line we used Fubini's theorem and the isotropy of $T_\gamma^{(1)}$, denoting by $L^+ = L_0^+$ the half line with direction $\Phi = 0$, and in the last expression we used the abbreviation

$$
g_\gamma\big(T^{(1)}\big) = \frac{1}{\gamma}\sup_{z\in T^{(1)}\cap B(o,1)} \sum_{\substack{X_i\in T_\gamma^{(1)}\cap L^+:\\ |X_i|\le r+1}} \frac{1}{\sin\alpha_i}\mathbb{1}_{[\varepsilon,\infty)}\Big(\big|c(X_i,\frac{z}{\gamma}) - \xi|X_i-\frac{z}{\gamma}|\big|\Big)\,. \tag{B.3}
$$

Since the point process $T^{(1)}\cap\mathbb{R}$ is stationary with intensity $2/\pi$ ([30], Theorem 4.5.3) we can apply the inversion formula for Palm distributions of stationary point processes on $\mathbb{R}$; see Proposition 11.3 (iii) in [20]. Thus, if $T^{(1)*}$ denotes the Palm version of $T^{(1)}$ with respect to the point process $T^{(1)}\cap\mathbb{R}$, we get that

$$
\mathbb{E}g_\gamma\big(T^{(1)}\big) \;=\; \frac{2}{\pi}\,\mathbb{E}\Big(\int_0^\infty \mathbb{1}_{[0,X_1^*]}(x)\,g_\gamma\big(T^{(1)*}-x\big)\,dx\Big)\,,
$$

where the points of $\{X_i^*\} = T^{(1)*}\cap\mathbb{R}$ are numbered in ascending order such that $\ldots < X_{-1}^* < X_0^* = 0 < X_1^* < X_2^* < \ldots$. Thus, in order to prove (B.2) it suffices to show that

$$
\lim_{\gamma\to\infty}\mathbb{E}\Big(\int_0^\infty \mathbb{1}_{[0,X_1^*]}(x)\,g_\gamma\big(T^{(1)*}-x\big)\,dx\Big) = 0\,, \tag{B.4}
$$

where the function $g_\gamma : \mathcal{F}\to[0,\infty)$ is given in (B.3). The proof of (B.4) is subdivided into two main steps. First, we show that

$$
\lim_{\gamma\to\infty}\widetilde{g}_\gamma\big(x,T^{(1)*}\big) = 0 \tag{B.5}
$$

almost everywhere with respect to the product measure $\nu_1 \otimes \mathbb{P}^*$, where we used the abbreviating notation $\widetilde{g}_\gamma(x, T^{(1)*}) = \mathbb{1}_{[0,X_1^*]}(x)\, g_\gamma(T^{(1)*} - x)$ and $\mathbb{P}^*$ denotes the distribution of $T^{(1)*}$. Then, we show that $\{\widetilde{g}_\gamma,\ \gamma > 0\}$ is uniformly $(\nu_1 \otimes \mathbb{P}^*)$-integrable. By means of Theorem A.1, this implies that (B.4) holds.

Proof of (B.5). Note that for each $x \in [0, X_1^*]$ we get

$$
\begin{aligned}
& g_\gamma\big(T^{(1)*} - x\big) \\
&\quad \le \frac{1}{\gamma} \sup_{z \in (T^{(1)*} - x) \cap B(o,1)} \sum_{\substack{X_i \in (T^{(1)*}_\gamma - \frac{x}{\gamma}) \cap L^+: \\ |X_i| \le r+1}} \frac{1}{\sin \alpha_i} \mathbb{1}_{[\varepsilon,\infty)}\Big(\big|c(X_i, \frac{z}{\gamma}) - \xi |X_i - \frac{z}{\gamma}|\big|\Big) \\
&\quad = \frac{1}{\gamma} \sup_{z \in T^{(1)*} \cap B(x,1)} \sum_{\substack{X_i^* \in T^{(1)*} \cap (L^+ + x): \\ X_i^* \in B(x,(r+1)\gamma)}} \frac{1}{\sin \alpha_i} \mathbb{1}_{[\varepsilon,\infty)}\Big(\frac{1}{\gamma}\big|c(X_i^*, z) - \xi |X_i^* - z|\big|\Big) \\
&\quad \le \frac{1}{\gamma} \sum_{\substack{X_i^* \in T^{(1)*} \cap (L^+ + x): \\ X_i^* \in B(x,(r+1)\gamma)}} \frac{1}{\sin \alpha_i} \sup_{z \in T^{(1)*} \cap B(x,1)} \mathbb{1}_{[\varepsilon,\infty)}\Big(\frac{1}{\gamma}\big|c(X_i^*, z) - \xi |X_i^* - z|\big|\Big) .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& g_\gamma\big(T^{(1)*} - x\big) \\
&\quad \le \frac{1}{\gamma} \sum_{\substack{X_i^* \in T^{(1)*} \cap L^+: \\ |X_i^*| \le (r+a)\gamma}} \frac{1}{\sin \alpha_i} \sup_{z \in T^{(1)*} \cap B(o,a)} \mathbb{1}_{[\varepsilon,\infty)}\Big(\frac{1}{\gamma}\big|c(X_i^*, z) - \xi |X_i^* - z|\big|\Big) \\
&\quad = \frac{1}{\gamma} \sum_{\substack{X_i^* \in T^{(1)*} \cap L^+: \\ |X_i^*| \le (r+a)\gamma}} \frac{1}{\sin \alpha_i} \mathbb{1}_{[\varepsilon,\infty)}\Big(\frac{1}{\gamma} \sup_{z \in T^{(1)*} \cap B(o,a)} \big|c(X_i^*, z) - \xi |X_i^* - z|\big|\Big) ,
\end{aligned}
$$

where $a = 1 + X_1^*$. Furthermore, we have

$$
\begin{aligned}
\frac{1}{\gamma} \sup_{z \in T^{(1)*} \cap B(o,a)} \big|c(X_i^*, z) - \xi |X_i^* - z|\big| \quad &\le \quad \frac{1}{\gamma} \big(c(o, X_i^*) - \xi |X_i^*|\big) \\
&\quad + \quad \frac{1}{\gamma} \left(\sup_{z \in T^{(1)*} \cap B(o,a)} c(z, o) + \xi a \right),
\end{aligned}
$$

since $c(X_i^*, o) - c(o, z) \le c(X_i^*, z) \le c(X_i^*, o) + c(o, z)$ and $\xi |X_i^*| - \xi a \le \xi |X_i^* - z| \le \xi |X_i^*| + \xi a$ for all $i \ge 1$ and $z \in T^{(1)*} \cap B(o, a)$. Clearly, the second term of this upper bound tends to zero $\mathbb{P}^*$-almost surely as $\gamma \to \infty$. Thus in order to show that (B.5) holds, it suffices to prove that $\mathbb{P}^*$-almost surely

$$
\frac{1}{\gamma} \big(c(o, X_i^*) - \xi X_i^*\big) \in \Big(-\frac{\varepsilon}{2}\, ,\, \frac{\varepsilon}{2}\Big) \tag{B.6}
$$

for all sufficiently large $i \geq 1$ such that $X_i^* \leq (r+a)\gamma$.

Proof of (B.6). Note that $\mathbf{X} = \{|X_i^* - X_j^*|, i, j \geq 1, i < j\}$ is an additive process, because $|X_i^* - X_k^*| = |X_i^* - X_j^*| + |X_j^* - X_k^*|$ for $i < j < k$. Since $T^{(1)*} \cap \mathbb{R}$ is cycle-stationary (see e.g. [33]), we have that $\{|X_i^* - X_j^*|\} \stackrel{\mathrm{d}}{=} \{|X_{i+1}^* - X_{j+1}^*|\}$, where $0 < \mathbb{E}X_1^* < \infty$. Thus, by Theorem A.2 we get that the finite limit $\lim_{i\to\infty} X_i^*/i = \zeta_{\mathbf{X}}$ exists $\mathbb{P}^*$-almost surely. Furthermore, consider the family $\mathbf{Y} = \{Y_{ij}, i, j \geq 1, i < j\}$ of non-negative random variables with $Y_{ij} = c(X_i^*, X_j^*)$, where $c(X_i^*, X_j^*)$ denotes the shortest path length from X_i^* to X_j^* on $T^{(1)*}$. Then, it is easy to see that $Y_{ik} \leq Y_{ij} + Y_{jk}$ for $i < j < k$. By the cycle-stationarity of $T^{(1)*} \cap \mathbb{R}$, we have that $\{Y_{ij}\} \stackrel{\mathrm{d}}{=} \{Y_{i+1,j+1}\}$, where $\mathbb{E}Y_{01} = \mathbb{E}c(X_0^*, X_1^*) < \infty$ holds by condition (3.3); see the next paragraph below. Thus $\mathbf{Y}$ is a subadditive process and we can again apply Theorem A.2 to get that the finite limit $\lim_{j\to\infty} c(X_0^*, X_j^*)/j = \zeta_{\mathbf{Y}}$ exists $\mathbb{P}^*$-almost surely. Since $\mathbf{X}$ and $\mathbf{Y}$ are ergodic (see the paragraphs below), the limits $\zeta_{\mathbf{X}}$ and $\zeta_{\mathbf{Y}}$ are constant. Noticing that $0 < \mathbb{E}X_1^* = \zeta_{\mathbf{X}} \leq \zeta_{\mathbf{Y}} < \infty$, this gives that

$$\lim_{j\to\infty} \frac{c(o, X_j^*)}{X_j^*} = \lim_{j\to\infty} \frac{j}{X_j^*} \, \frac{c(X_0^*, X_j^*)}{j} = \xi \,, \tag{B.7}$$

where $\xi = \zeta_{\mathbf{Y}}/\zeta_{\mathbf{X}} \in [1, \infty)$. Now let $\widetilde{\varepsilon} > 0$ such that $\widetilde{\varepsilon}(r+a) < \varepsilon/2$. Then (B.7) implies that with probability 1

$$\frac{c(o, X_i^*)}{X_i^*} - \xi \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$$

for all i sufficiently large and, therefore,

$$\frac{1}{\gamma}\left(c(o, X_i^*) - \xi X_i^*\right) \in \left(-\frac{\varepsilon}{2}\,,\,\frac{\varepsilon}{2}\right)$$

if i is sufficiently large and $X_i^*/\gamma \leq r + a$.

Proof of $\mathbb{E}c(X_0^*, X_1^*) < \infty$. Consider the stationary marked point process $\{(X_n, \Xi_n^+)\}$, where $\{X_n\} = T^{(1)} \cap \mathbb{R}$ is the point process of intersection points of the edge set $T^{(1)}$ with the line $\mathbb{R}$, and Ξ_n^+ the cell of T on the right of X_n. Let λ^+ denote the intensity of the marked point process $\{(X_n, \Xi_n^+)\}$, and Ξ^{+*} its typical mark. Then, by the definition of the Palm mark distribution (see e.g. Section 2.2), we get that

$$\begin{aligned}\mathbb{E}\, c(X_0^*, X_1^*) &\leq \mathbb{E}\,\nu_1(\partial\Xi^{+*}) = \frac{1}{\lambda^+}\,\mathbb{E}\sum_{X_i\in T^{(1)}\cap[0,1)} \nu_1(\partial\Xi_i^+)\\ &= \frac{1}{\lambda^+}\,\mathbb{E}\sum_{\Xi_i\in T} \mathbb{I}_{\{\partial^+\Xi_i\cap[0,1)\neq\emptyset\}}\nu_1(\partial\Xi_i)\,,\end{aligned}$$

where $\partial^+\Xi$ denotes that part of the boundary of Ξ with outer unit normal vector in $[\pi/2, 3\pi/2)$. Thus, applying Campbell's theorem to the latter expression, we have

$$\begin{aligned}\mathbb{E}c(X_0^*, X_1^*) &\leq \frac{\lambda_T}{\lambda^+}\,\mathbb{E}\nu_1(\partial\Xi^*)\int_{\mathbb{R}^2} \mathbb{I}_{\{\partial^+\Xi^*+x\cap[0,1)\neq\emptyset\}}\nu_2(dx)\\ &= \frac{\lambda_T}{\lambda^+}\,\mathbb{E}\nu_1(\partial\Xi^*)\nu_2([0,1)\oplus\partial^+\Xi^*)\,,\end{aligned}$$

where $\lambda_T = 1/\mathbb{E}\,\nu_2(\Xi^*)$. Since $\nu_2([0,1)\oplus\partial^+\Xi^*) \leq a\,\nu_1(\partial\Xi^*)$ for some constant $a < \infty$, this implies that $\mathbb{E}\,c(X_0^*, X_1^*) \leq (a\lambda_T/\lambda^+)\,\mathbb{E}\nu_1^2(\partial\Xi^*)$. Thus, the assertion is shown.

Ergodicity. We only prove that $\mathbf{X}$ is ergodic, because the ergodicity of $\mathbf{Y}$ can be shown in the same way. Recall that by $\mathcal{I}_\mathcal{S} \subset \mathcal{B}(\mathcal{S})$ we denote the σ-algebra of those subsets of the space $\mathcal{S}$ of double-indexed sequences, which are invariant under the shift $\{|X_i^* - X_j^*|\} \longmapsto \{|X_{i+1}^* - X_{j+1}^*|\}$. Furthermore, note that $\mathbf{X} = h(T_\gamma^{(1)*})$ for some measurable function $h : \mathcal{F} \to \mathcal{S}$, where for any tessellation τ in $\mathbb{R}^2$ and $A \in \mathcal{I}_\mathcal{S}$, we have $h(\tau^{(1)}) \in A$ if and only if $h(\tau^{(1)} - x) \in A$ for all $x \in [0, \infty)$. Thus, from the definition of the Palm distribution of the stationary point process $\{X_i\} = T^{(1)} \cap \mathbb{R}$ with intensity $2/\pi$, we get for any $A \in \mathcal{I}_\mathcal{S}$ that

$$\begin{aligned}\mathbb{P}(\mathbf{X}\in A) &= \mathbb{P}(h(T^{(1)*})\in A)\\ &= \frac{\pi}{2}\,\mathbb{E}\sum_{X_i\in T^{(1)}\cap B(o,1)\cap L^+} \mathbb{I}_A(h(T^{(1)}-X_i))\\ &= \frac{\pi}{2}\,\mathbb{E}\big(\mathbb{I}_A(h(T^{(1)}))\,\#\{X_i\in T^{(1)}\cap B(o,1)\cap L^+\}\big)\\ &= \frac{\pi}{2}\,\mathbb{E}\big(\mathbb{I}_{h^{-1}(A)}(T^{(1)})\,\#\{X_i\in T^{(1)}\cap B(o,1)\cap L^+\}\big)\,.\end{aligned}$$

On the other hand, since T_1 is mixing and $h^{-1}(A) = h^{-1}(A) + x$ for any $A \in \mathcal{I}_\mathcal{S}$ and $x \in L^+$, we have

$$\begin{aligned}\mathbb{P}\big(T^{(1)}\in h^{-1}(A)\big) &= \lim_{|x|\to\infty, x\in L^+} \mathbb{P}\big(T^{(1)}\in h^{-1}(A), T^{(1)}-x\in h^{-1}(A)\big)\\ &= \mathbb{P}\big(T^{(1)}\in h^{-1}(A)\big)^2\,,\end{aligned}$$

which implies that $\mathbb{P}\big(T^{(1)}\in h^{-1}(A)\big) = 0$ or $\mathbb{P}\big(T^{(1)}\in h^{-1}(A)\big) = 1$. Thus, altogether, we have

$$\begin{aligned}\mathbb{P}(\mathbf{X} \in A) &= \mathbb{P}\big(T^{(1)} \in h^{-1}(A)\big)\, \frac{\pi}{2}\, \mathbb{E}\, \#\{X_i \in T^{(1)} \cap B(o,1) \cap L^+\} \\ &= \mathbb{P}\big(T^{(1)} \in h^{-1}(A)\big)\end{aligned}$$

and, consequently, $\mathbb{P}(\mathbf{X} \in A) = 0$ or $\mathbb{P}(\mathbf{X} \in A) = 1$ for any $A \in \mathcal{I}_{\mathcal{S}}$, which means that $\mathbf{X}$ is ergodic.

Uniform integrability. Finally, we show that the family $\{\widetilde{g}_\gamma,\, \gamma > 0\}$ considered in (B.5) is uniformly $(\nu_1 \otimes \mathbb{P}^*)$-integrable. From the ergodic theorem for stationary marked point processes ([8], Theorem 12.2.IV), we get that

$$\begin{aligned}\lim_{\gamma\to\infty} \frac{1}{\gamma} \sum_{\substack{X_i \in T^{(1)}\cap L^+:\\ |X_i| \le (r+1)\gamma}} \frac{1}{\sin\alpha_i} &= (r+1) \lim_{\gamma\to\infty} \frac{1}{(r+1)\gamma} \sum_{\substack{X_i \in T^{(1)}\cap L^+:\\ |X_i| \le (r+1)\gamma}} \frac{1}{\sin\alpha_i} \\ &= (r+1)\, \mathbb{E}(\sin\alpha^*)^{-1}\end{aligned}$$

almost surely and in L^1 since the point process $T^{(1)} \cap L$ marked with the intersection angles is ergodic, which can be shown in the same way as the ergodicity of $\mathbf{X}$. Here α^* denotes the typical intersection angle which is distributed according to the density $f_{\alpha^*}(\alpha) = \sin(\alpha)/2$ for $0 \le \alpha < \pi$, see e.g. [31], p. 288. This yields $\mathbb{E}(\sin\alpha^*)^{-1} = \pi/2 < \infty$. Thus

$$\begin{aligned}0 &= \lim_{\gamma\to\infty} \mathbb{E} \left| \frac{1}{\gamma} \sum_{\substack{X_i \in T^{(1)}\cap L^+:\\ |X_i| \le (r+1)\gamma}} \frac{1}{\sin\alpha_i} - (r+1)\, \mathbb{E}(\sin\alpha^*)^{-1} \right| \\ &= \lim_{\gamma\to\infty} \frac{2}{\pi}\, \mathbb{E} \int_{\mathbb{R}} \mathbb{1}_{[0,X_1^*]}(x) \left| \frac{1}{\gamma} \sum_{\substack{X_i \in (T^{(1)*}-x)\cap L^+:\\ |X_i| \le (r+1)\gamma}} \frac{1}{\sin\alpha_i} - (r+1)\, \mathbb{E}(\sin\alpha^*)^{-1} \right| dx\,,\end{aligned}$$

where in the last equality we used the inversion formula for Palm distributions of stationary marked point processes on $\mathbb{R}$; see Proposition 11.3 (iii) in [20]. In other words, we showed that

$$\mathbb{1}_{[0,X_1^*]}(x)\, \frac{1}{\gamma} \sum_{\substack{X_i \in (T^{(1)*}-x)\cap L^+:\\ |X_i| \le (r+1)\gamma}} \frac{1}{\sin\alpha_i} \longrightarrow (r+1)\mathbb{1}_{[0,X_1^*]}(x)\, \mathbb{E}(\sin\alpha^*)^{-1}$$

in $L^1(\nu_1 \otimes \mathbb{P}^*)$ as $\gamma \to \infty$. This means in particular that the family $\{h_\gamma,\, \gamma > 0\}$ with

$$h_\gamma(x, T^{(1)*}) = 1\!\!1_{[0,X_1^*]}(x)\, \frac{1}{\gamma} \sum_{\substack{X_i \in (T^{(1)*}-x)\cap L^+:\\ |X_i| \le (r+1)\gamma}} \frac{1}{\sin \alpha_i}$$

is uniformly $(\nu_1 \otimes \mathbb{P}^*)$-integrable; see Theorem A.1. Furthermore, we have that

$$1\!\!1_{[0,X_1^*]}(x) g_\gamma(T^{(1)*} - x) \le 1\!\!1_{[0,X_1^*]}(x)\, \frac{1}{\gamma} \sum_{\substack{X_i \in (T^{(1)*}-x)\cap L^+:\\ |X_i| \le (r+1)\gamma}} \frac{1}{\sin \alpha_i}\,.$$

Thus Lemma A.1 yields that the family $\{\widetilde{g}_\gamma,\, \gamma > 0\}$ considered in (B.5) is uniformly $(\nu_1 \otimes \mathbb{P}^*)$-integrable.