On the estimation of integrated covariance functions of stationary random fields

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Running Title: Estimation of integrated covariance functions

For stationary vector–valued random fields on $\mathbb{R}^d$ the asymptotic covariance matrix for estimators of the mean vector can be given by integrated covariance functions. In order to construct asymptotic confidence intervals and significance tests for the mean vector, nonparametric estimators of these integrated covariance functions are required. Integrability conditions are derived under which the estimators of the covariance matrix are mean-square consistent. For random fields induced by stationary Boolean models with convex grains, these conditions are expressed by sufficient assumptions on the grain distribution. Performance issues are discussed by means of numerical examples for Gaussian random fields and the intrinsic volume densities of planar Boolean models with uniformly bounded grains.

Keywords: asymptotic unbiasedness; Boolean model; consistency; covariance matrix; empirical covariance; intrinsic volumes; nonparametric estimation; stationary random field.

1 Introduction

In various fields of practical application, such as medicine, biology, geology or material sciences, huge amounts of spatial data sets have to be categorized by means of certain characteristics of the underlying material. Assuming that these characteristics can be expressed by the mean vector of some stationary random field $Y = \{Y(x), x \in \mathbb{R}^d\}$, where $Y(x) = (Y_1(x), \ldots, Y_m(x))$ for $m \in \mathbb{N}$, we are interested in (asymptotic) confidence intervals and significance tests for the mean vector $\mu = \mathbb{E}\, Y(x)$. On a sequence of expanding observation windows W_n, $n \in \mathbb{N}$ mean–square consistent estimators of μ are given by $\widehat{\mu}_n = (\widehat{\mu}_{n1}, \ldots, \widehat{\mu}_{nm})$, $n \geq 1$ with $\widehat{\mu}_{ni} = \int_{W_n} Y_i(x) G_i(W_n, x)\, dx$ for weighting functions G_i, $i = 1, .., m$. Under appropriate assumptions, it holds that $\sqrt{|W_n|}(\widehat{\mu}_n - \mu)$ converges in distribution to a Gaussian random vector with mean vector zero and covariance matrix Σ as W_n tends to $\mathbb{R}^d$, where

$$\Sigma = \Big(\theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}\big(Y_i(o), Y_j(x)\big)\, dx\Big)_{i,j=1,..,m}$$

for certain constants $\theta_{ij} \in (0, \infty)$ (with $o = (0, \ldots, 0)^\top$); see, e.g., the paper Pantle *et al.* (2006) for a general class of stationary random fields induced by germ–grain models. In order to perform asymptotic significance tests for the mean vector μ (e.g., with the aim of an automated classification of the underlying material), the matrix of integrated covariance functions Σ has to be estimated consistently, since it is in general unknown or too complicated to be evaluated explicitly. Notice that due to this estimation, the rate of convergence in the above central limit theorem (which is usually $O(n^{-1/2})$ by the well–known result of Berry–Esseen) slows down. In Bulinski & Kryzhanovskaya (2006), an empirical covariance estimator similar to $\widetilde{\Sigma}_n$ of Section 5 below is considered, for which it is shown that the rate of convergence in the corresponding limit theorem is $O(n^{-d/(6(2d+1))})$, $d > 6$ (cf. Corollary 1 in Bulinski & Kryzhanovskaya (2006)).

A new nonparametric estimator $\widehat{\Sigma}_n$ of the asymptotic covariance matrix Σ is proposed in Section 3, where

$$\widehat{\Sigma}_n = \Big(\int_{\mathbb{R}^d} \widehat{\mathrm{Cov}}_{nij}(x)\, \gamma_{ij}(W_n, x)\, dx \Big)_{i,j=1,..,m}, \qquad n \geq 1$$

for some weighting function $\gamma_{ij}(W_n, x)$ and a consistent estimator $\widehat{\mathrm{Cov}}_{nij}(x)$ of $\mathrm{Cov}(Y_i(o), Y_j(x))$ for fixed $x \in \mathbb{R}^d$. The construction principle employed is similar to the techniques used, e.g., in Böhm *et al.* (2004, Section 3) and Schmidt & Spodarev (2005, Section 3.5). Sufficient conditions for the asymptotic unbiasedness and for the mean-square consistency of $\widehat{\Sigma}_n$ are given in Lemma 1 and Theorem 1, respectively. It is shown in Section 3.3 that they can be easily verified for a special class of stationary Gaussian random fields with exponential (cross)covariance structure. In Section 4, random fields induced by special random sets called *Boolean models* are discussed. For this case, sufficient conditions on the volume of the typical grain M_0 enlarged by some test set K can be formulated so that the assumptions of Theorem 1 are satisfied. Section 5 deals with an estimator of Σ derived from the empirical covariance of observations of $\widehat{\mu}_n$ on disjoint subwindows of W_n. A numerical comparison of the different estimates of the asymptotic covariance matrix of random fields considered in Sections 3.3 and 4 is given in Section 6. The paper closes with a summary of results and discussion thereof. The proof of Theorem 2 from Section 4.3 is given in the Appendix.

2 Preliminaries

In the present section, we introduce some notation used throughout this paper and recall basic facts from stochastic geometry. Further details can be found for example in Adler & Taylor (2007), Schneider & Weil (2008) or Stoyan *et al.* (1995). In the second part, a class of estimators of the mean value of stationary random fields and their asymptotic properties are considered (compare, in particular, Ivanov & Leonenko (1989) and Pantle *et al.* (2006)).

2.1 Basic notations

Let $d \geq 1$ be an arbitrary fixed integer and let the d-dimensional Euclidean space $\mathbb{R}^d$ be equipped with the Borel σ-algebra $\mathcal{B}^d$. Denote the set of bounded Borel sets by $\mathcal{B}_0^d$ and write $o \in \mathbb{R}^d$ for the origin in $\mathbb{R}^d$. The

Euclidean norm of a vector $x \in \mathbb{R}^d$ is denoted by $|x|$, whereas $|B|$ stands for the d-dimensional Lebesgue measure (or volume) of a set $B \in \mathcal{B}^d$. By ∂B we denote the boundary of a Borel set and by $\mathrm{int}(B)$ its interior. Furthermore, let $B_r(x) = \{y \in \mathbb{R}^d : |y-x| \le r\}$ be the closed ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with radius $r > 0$. The *Minkowski sum* of two sets $B, B' \subset \mathbb{R}^d$ is given by $B \oplus B' = \{x + y : x \in B, y \in B'\}$, where we write $B + x$ instead of $B \oplus \{x\}$ for the *translation* of B by a vector $x \in \mathbb{R}^d$. Besides this, consider the *reflection* $\check{B} = \{-x : x \in B\}$ of B at the origin and denote the *Minkowski difference* of B and B' by $B \ominus B' = \{x : \check{B}' + x \subseteq B\}$.

Consider an arbitrary probability space $(\Omega, \mathcal{A}, P)$ and some $\mathcal{B}^d \otimes \mathcal{A}$-measurable mapping $Y : \mathbb{R}^d \times \Omega \to \mathbb{R}$ with mean function $\mu(x) = \mathbb{E}\, Y(x)$ and covariance function $\mathrm{Cov}(x, y) = \mathrm{Cov}(Y(x), Y(y))$ assuming that $\mathbb{E}\, Y^2(x) < \infty$ for any $x \in \mathbb{R}^d$. For simplicity, we use the notation $Y(x)$ instead of $Y(x, \cdot)$ and denote the random field $\{Y(x), x \in \mathbb{R}^d\}$ by Y as well. A random field Y is called *stationary* if its finite dimensional distributions are invariant with respect to translations, that is to say, the distribution of the random vectors $(Y(x_1 + h), \dots, Y(x_m + h))$ and $(Y(x_1), \dots, Y(x_m))$ coincide for all $x_1, \dots, x_m, h \in \mathbb{R}^d$ and $m \in \mathbb{N}$. Stationarity implies, in particular, that $\mu(x)$ is constant and $\mathrm{Cov}(x, y)$ is a function depending on the difference $y - x$ only. Hence, set $\mu = \mathbb{E}\, Y(x)$ and $\mathrm{Cov}(x) = \mathrm{Cov}(y, y + x)$ for any $x, y \in \mathbb{R}^d$.

In Section 4 we will consider the case, where Y is defined as a functional of a *random closed set* (RACS) Ξ in $\mathbb{R}^d$. A random closed set is a $(\mathcal{A}, \sigma_{\mathcal{F}})$-measurable mapping from $(\Omega, \mathcal{A})$ into $(\mathcal{F}, \sigma_{\mathcal{F}})$, where $\mathcal{F} \subset \mathcal{B}^d$ denotes the family of all closed sets in $\mathbb{R}^d$, and $\sigma_{\mathcal{F}}$ is the σ-algebra generated by $\{F \in \mathcal{F} : F \cap C \neq \emptyset\}$ for arbitrary compact $C \subset \mathbb{R}^d$. Finally, let $\mathcal{K} \subset \mathcal{F}$ be the family of all convex and compact sets in $\mathbb{R}^d$, and define the σ-algebra $\sigma_{\mathcal{K}} = \{B \cap \mathcal{K} : B \in \sigma_{\mathcal{F}}\}$ on $\mathcal{K}$.

2.2 Estimating the mean of vector–valued random fields

Let $Y_1 = \{Y_1(x), x \in \mathbb{R}^d\}, \dots, Y_m = \{Y_m(x), x \in \mathbb{R}^d\}$ be a set of $m \in \mathbb{N}$ random fields on the same probability space such that the finite dimensional distributions of the vector–valued random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ with $Y(x) = (Y_1(x), \dots, Y_m(x))^\top$ are invariant with respect to translations. The random fields $Y_1, \dots, Y_m$ are then called *jointly stationary.* For $i, j = 1, \dots, m$, set $\mu_i = \mathbb{E}\, Y_i(o)$ and $\mathrm{Cov}_{ij}(x) = \mathrm{Cov}(Y_i(o), Y_j(x))$, $x \in \mathbb{R}^d$, where we assume that $\mathbb{E}\, Y_i^2(o) < \infty$, but μ_i and $\mathrm{Cov}_{ij}(x)$ are unknown. With regard to the estimation of $\mu = (\mu_1, \dots, \mu_m)^\top$, consider a sequence $\{W_n\}$ of monotonously increasing, bounded Borel sets $W_n \subset \mathbb{R}^d, n \ge 1$ such that

$$\lim_{n\to\infty} |W_n| = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{|\partial W_n \oplus B_r(o)|}{|W_n|} = 0 \tag{1}$$

for any $r > 0$. Furthermore, suppose that the random fields $Y_i = \{Y_i(x), x \in \mathbb{R}^d\}$ are observable on sub-windows $W_{ni} = W_n \ominus \check{K}_i$ for some $K_i \in \mathcal{K}$, respectively. An unbiased estimator of the mean vector μ is then given by $\widehat{\mu}_n = (\widehat{\mu}_{n1}, \dots, \widehat{\mu}_{nm})^\top, n \ge 1$ with

$$\widehat{\mu}_{ni} = \int_{W_n} Y_i(x)\, G_i(W_n, x)\, dx \tag{2}$$

for functionals $G_i : \mathcal{B}_0^d \otimes \mathbb{R}^d \to [0, \infty), i = 1, \dots, m$, which are $\mathcal{B}^d$-measurable in the second component and satisfy

$$G_i(W, x) = 0 \quad \text{if } x \in \mathbb{R}^d \setminus W \ominus \check{K}_i, \qquad \text{and} \qquad \int_{\mathbb{R}^d} G_i(W, x)\, dx = 1 \tag{3}$$

for any $W \in \mathcal{B}_0^d$. Put $\Gamma_{nij}(x) = \int_{\mathbb{R}^d} G_i(W_n, y) G_j(W_n, y+x)\, dy$ for $i, j = 1, \ldots, m$. Note that $\Gamma_{nij}(x) = 0$ if $x \notin W_{ni} \oplus \check{W}_{nj}$. For any $n \geq 1$, it holds that

$$\mathrm{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj}) = \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x) \Gamma_{nij}(x)\, dx\,. \tag{4}$$

To study the asymptotic behaviour of $\widehat{\mu}_n$, we assume that there exist constants $c_1, \theta_{ij} \in (0, \infty)$ for all $i, j = 1, \ldots, m$ such that

$$\sup_{x \in \mathbb{R}^d} G_i(W_n, x) \leq \frac{c_1}{|W_n|} \ \text{ for any } n \geq 1 \qquad \text{and} \qquad \lim_{n \to \infty} |W_n| \Gamma_{nij}(x) = \theta_{ij} \ \text{ for any } x \in \mathbb{R}^d\,. \tag{5}$$

Both conditions (3) and (5) hold, for instance, if $G_i(W_n, x) = \mathbb{I}(x \in W_n \ominus \check{K}_i) \,/\, |W_n \ominus \check{K}_i|, i = 1, \ldots, m$ for any $n \geq 1$ and $x \in \mathbb{R}^d$, where $\mathbb{I}(B)$ denotes the indicator function of the set B.

Under appropriate mixing and integrability conditions one can show that

$$\begin{pmatrix} \sqrt{|W_n|}\, (\widehat{\mu}_{n1} - \mu_1) \\ \vdots \\ \sqrt{|W_n|}\, (\widehat{\mu}_{nm} - \mu_m) \end{pmatrix} \Longrightarrow \mathcal{N}_m(o, \Sigma)\,, \qquad n \to \infty\,,$$

where $\Longrightarrow$ denotes convergence in distribution and $\mathcal{N}_m(o, \Sigma)$ is an m-dimensional Gaussian random variable with mean zero and covariance matrix $\Sigma = (\sigma_{ij})$

$$\Sigma = \begin{pmatrix} \theta_{11} \int_{\mathbb{R}^d} \mathrm{Cov}_{11}(x)\, dx & \ldots & \theta_{1m} \int_{\mathbb{R}^d} \mathrm{Cov}_{1m}(x)\, dx \\ \vdots & \ddots & \vdots \\ \theta_{m1} \int_{\mathbb{R}^d} \mathrm{Cov}_{m1}(x)\, dx & \ldots & \theta_{mm} \int_{\mathbb{R}^d} \mathrm{Cov}_{mm}(x)\, dx \end{pmatrix}, \tag{6}$$

confer for example Ivanov & Leonenko (1989, Section 1.7), Pantle *et al.* (2006, Sections 4 and 5) and references therein. Regarding the symmetry of the covariance matrix Σ, note that $\mathrm{Cov}_{ij}(x) = \mathrm{Cov}_{ji}(-x)$ for all $x \in \mathbb{R}^d$ as well as $\Gamma_{nij}(x) = \Gamma_{nji}(-x)$, and consequently $\theta_{ij} = \theta_{ji}$. Since explicit formulae for σ_{ij}, $i, j = 1, \ldots, m$ are in general unknown, we are interested in the estimation of the integrated covariance functions $\int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\, dx$. At this, we assume that

$$\int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, dx < \infty\,, \qquad i, j = 1, \ldots, m\,. \tag{7}$$

3 A weighted covariance estimator

3.1 Definition and growth conditions

Throughout the following, let $Y_1, \ldots, Y_m$ be a set of *jointly stationary* random fields with finite fourth moments and $\widehat{\mu}_n, n \geq 1$ the estimator of $\mathbb{E}\, Y(o)$ as defined in (2). The aim of this section is to establish a consistent estimator $\widehat{\Sigma}_n$ of the asymptotic covariance matrix Σ given in (6).

For any pair $i \leq j$, choose an increasing sequence $\{U_{nij}\}$ of compact Borel sets with

- $U_{nij} \subseteq W_{ni} \oplus \check{W}_{nj}$ and $o \in U_{nij}$.
- Denote by $\mathrm{supp}(\mathrm{Cov}_{ij})$ the support of Cov_{ij} and assume that $\mathrm{supp}(\mathrm{Cov}_{ij}) \subseteq \lim_{n\to\infty} U_{nij}$.
- In addition, suppose that

$$\lim_{n\to\infty} \sup_{x\in U_{nij}} \mid \theta_{ij} - |W_n|\Gamma_{nij}(x) \mid = 0\,, \qquad \forall\, i,j = 1,\ldots,m. \tag{8}$$

- and

$$\lim_{n\to\infty} |U_{nij}|^2/\,|W_n| \;=\; 0\,, \qquad \forall\, i,j = 1,\ldots,m\,. \tag{9}$$

- Finally, for $j < i$, put $U_{nji} = \check{U}_{nij}$ to preserve symmetry in the covariance matrix estimate.

Based on the above-mentioned assumptions, define the estimator $\widehat{\Sigma}_n = (\widehat{\sigma}_{nij})$ by

$$\widehat{\sigma}_{nij} \;=\; \int_{U_{nij}} \widehat{\mathrm{Cov}}_{nij}(x)\, |W_n|\Gamma_{nij}(x)\, dx, \qquad n \geq 1 \tag{10}$$

with

$$\widehat{\mathrm{Cov}}_{nij}(x) \;=\; \int_{W_{ni}\cap(W_{nj}-x)} Y_i(y)Y_j(y+x)\, G_i(W_n,y)G_j(W_n,y+x)\, dy \cdot \Gamma^{-1}_{nij}(x) \;-\; \widehat{\mu}_{ni}\widehat{\mu}_{nj}. \tag{11}$$

We may assume, without loss of generality, that $|U_{nij}| > 0$ and $\Gamma_{nij}(x) > 0$ for all $x \in U_{nij}$, $i,j = 1,\ldots,m$ and any $n \in \mathbb{N}$. Notice that $\widehat{\sigma}_{nij} = \widehat{\sigma}_{nji}$ for any $i,j = 1,\ldots,m, n \geq 1$.

As an example, let $\mathrm{supp}(\mathrm{Cov}_{ij}) = \mathbb{R}^d$ and $W_n = nK_o$ for some $K_0 \in \mathcal{K}$ with $|K_0| > 0$ and $o \in \mathrm{int}(K_0)$. Define $\varrho_n = \sup\{r > 0 : B_r(o) \subseteq W_n\}$, and put $G_i(x) = \mathbb{1}(x \in B_{\varrho_n - r_0}(o))/|B_{\varrho_n - r_0}(o)|$ for all i, where $r_0 > 0$ satisfies $K_i \subseteq B_{r_0}(o)$ for $i = 1,\ldots,m$ and $n \in \mathbb{N}$ is large enough ensuring that $\varrho_n > r_0$. Then, conditions (9) and (8) are fulfilled with $\theta_{ij} \equiv 1$ if $U_{nij} = B_{\sqrt{\varrho_n}\,\varepsilon_n - r_0}(o)$ for some sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$ and $\lim_{n\to\infty} \sqrt{n}\,\varepsilon_n = \infty$.

Following along the lines of the subsequent proofs, one observes that $\widehat{\mathrm{Cov}}_{nij}(x)$, considered separately for fixed $x \in \mathbb{R}^d$, is an asymptotically unbiased and consistent estimator of $\mathrm{Cov}(x)$. For the special case, where $\mathbb{E}\, Y(o) = 0$ and $G(W_n, x) = \mathbb{1}(x \in W_n)/|W_n|$, further results on $\widehat{\mathrm{Cov}}_{nij}(x)$ can be found, for instance, in Ivanov & Leonenko (1989, Chapter 4).

Generally speaking, the averaging set U_{nij} in the definition (10) of the estimator $\widehat{\sigma}_{nij}$ mimics the integration over $\mathbb{R}^d$ in the definition (6) of the asymptotic covariance σ_{ij}, where both (8) and (9) are growth conditions on U_{nij}, bounding the rate with which U_{nij} fills the support $\mathrm{supp}(\mathrm{Cov}_{ij})$ of Cov_{ij}. If $\mathrm{supp}(\mathrm{Cov}_{ij}) = \mathbb{R}^d$, then $|U_{nij}| \to \infty$ as $n \to \infty$. This, together with (8) and (9), means from a practical point of view that the averaging set U_{nij} should be as large as possible to keep the bias of $\widehat{\sigma}_{nij}$ small, but not too large to get an estimator with small variance. In particular, the assumptions that (8) is fulfilled and that $|U_{nij}| \to \infty$ holds are used in the proof of Lemma 1 below, in order to show asymptotic unbiasedness of $\widehat{\sigma}_{nij}$, whereas (9) is used in the proof of Theorem 1 to show mean-square consistency of

$\widehat{\sigma}_{nij}$. As a rule, condition (9) suggests that the volume of U_{nij} should not be larger than the square root of the volume of the observation window W_n; see also the numerical examples considered in Section 6.

We also remark that the estimator matrix $\widehat{\Sigma}_n$ of the asymptotic covariance matrix Σ defined above is not necessarily positive semi-definite. To see this, we refer the reader to Section 6 where the numerical examples provide a clear evidence of that fact.

3.2 Asymptotic properties

Lemma 1. *The estimator $\widehat{\Sigma}_n = (\widehat{\sigma}_{nij})$ with $\widehat{\sigma}_{nij}$ as defined in* (10) *and* (11) *is asymptotically unbiased for the covariance matrix $\Sigma = (\sigma_{ij})$ given in* (6).

Proof. Insert $\pm\mu_i\mu_j$ into the defining equation (11) of $\widehat{\mathrm{Cov}}_{nij}(x)$ for any $i, j = 1, \ldots, m$ and apply Fubini's theorem to obtain

$$\mathbb{E}\,\widehat{\sigma}_{nij} \;=\; \int_{U_{nij}} \mathrm{Cov}_{ij}(x)\,|W_n|\Gamma_{nij}(x)\,dx - |W_n|\mathrm{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj}) \int_{U_{nij}} \Gamma_{nij}(x)\,dx \tag{12}$$

By (4) and (5) one can derive that $\lim_{n\to\infty} |W_n|\mathrm{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj}) = \sigma_{ij} < \infty$ applying the dominated convergence theorem. Together with

$$0 \;\le\; \limsup_{n\to\infty} \int_{U_{nij}} \Gamma_{nij}(x)\,dx \;\le\; c_1 \cdot \lim_{n\to\infty} \frac{|U_{nij}|}{|W_n|} \;=\; 0\,,$$

the second expression in (12) converges to zero as $n \to \infty$. On the other hand, the first summand can be split up as follows

$$\begin{aligned}
&\int_{U_{nij}} \mathrm{Cov}_{ij}(x)\,|W_n|\Gamma_{nij}(x)\,dx \\
&\quad= \;\theta_{ij} \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\,dx - \int_{U_{nij}} \mathrm{Cov}_{ij}(x)\big(\theta_{ij} - |W_n|\Gamma_{nij}(x)\big)\,dx - \;\theta_{ij} \int_{\mathbb{R}^d \setminus U_{nij}} \mathrm{Cov}_{ij}(x)\,dx\,.
\end{aligned}$$

By (7) we have $\lim_{n\to\infty} \int_{\mathbb{R}^d\setminus U_{nij}} |\mathrm{Cov}_{ij}(x)|\,dx \;=\; 0$, and from (8) it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} \Big| \int_{U_{nij}} \mathrm{Cov}_{ij}(x)(\theta_{ij} - |W_n|\Gamma_{nij}(x))\,dx \Big| \\
&\qquad\qquad \le \; \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\,dx \cdot \lim_{n\to\infty} \sup_{x\in U_{nij}} \big|\, \theta_{ij} - |W_n|\Gamma_{nij}(x) \,\big| = \; 0\,,
\end{aligned}$$

□

According to Lemma 1, asymptotic unbiasedness of $\widehat{\Sigma}_n$ holds under the same integrability assumption (7) needed so that Σ is well defined. Additional assumptions are necessary, however, to ensure mean-square consistency. Consistency is understood with respect to the matrix norm $||A|| = \big(\sum_{i,j=1}^m a_{ij}^2\big)^{1/2}$ for a matrix $A = (a_{ij}) \in \mathbb{R}^{m\times m}$.

Theorem 1. *Suppose that* $\mathbb{E}\, Y_i^4(o) < \infty$ *for* $i = 1, \ldots, m$. *Then it holds that* $\lim_{n\to\infty} \mathbb{E}\, ||\widehat{\Sigma}_n - \Sigma||^2 = 0$, *i.e.,* $\widehat{\Sigma}_n = (\widehat{\sigma}_{nij})$ *is mean-square consistent for* $\Sigma = (\sigma_{ij})$ *if the sampling window* U_{nij} *and the random fields* Y_i, $i, j = 1, \ldots, m$ *satisfy the following additional assumptions:*

$$\frac{1}{|U_{nij}|^2} \int_{U_{nij}} \int_{U_{nij}} \int_{\mathbb{R}^d} \left|\mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\big)\right| \, dy\, dx_1 dx_2 \;\leq\; \kappa_1 \tag{13}$$

and

$$\sup_{x_1,x_2\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left|\mathbb{E}\big((Y_i(o)-\mu_i)(Y_i(y)-\mu_i)Y_j(x_1)Y_j(x_2)\big)\right| dy \;\leq\; \kappa_2 \tag{14}$$

for some finite constants κ_1 *and* κ_2.

Proof. By Minkowski's inequality and since $\lim_{n\to\infty} \mathbb{E}\,\widehat{\sigma}_{nij} = \sigma_{ij}$ according to Lemma 1, it suffices to show that $\lim_{n\to\infty} \mathbb{E}\,(\widehat{\sigma}_{nij} - \mathbb{E}\,\widehat{\sigma}_{nij})^2 = 0$ for all $i, j = 1, \ldots, m$. To this end, define

$$S_{n1} \;=\; \int_{U_{nij}} \int_{W_{ni}\cap(W_{nj}-x)} \Big(Y_i(y)Y_j(y+x) - \mathbb{E}\,(Y_i(y)Y_j(y+x))\Big)\, |W_n| G_i(W_n, y) G_j(W_n, y+x)\, dy\, dx$$

and

$$S_{n2} \;=\; -\big(\widehat{\mu}_{ni}\widehat{\mu}_{nj} - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big) \int_{U_{nij}} |W_n| \Gamma_{nij}(x)\, dx\,.$$

Hence, we find $\widehat{\sigma}_{nij} - \mathbb{E}\,\widehat{\sigma}_{nij} = S_{n1} + S_{n2}$ and have to verify that $\lim_{n\to\infty} \mathbb{E}\, S_{n1}^2 = 0$ and $\lim_{n\to\infty} \mathbb{E}\, S_{n2}^2 = 0$. For the first expression, one obtains

$$\begin{aligned}
&\mathbb{E}\left(\int_{U_{nij}} \int_{W_{ni}\cap(W_{nj}-x)} \Big[Y_i(y)Y_j(y+x) - \mathbb{E}(Y_i(y)Y_j(y+x))\Big] |W_n| G_i(W_n, y) G_j(W_n, y+x)\, dy\, dx\right)^2 \\
&\quad\leq\; \int_{U_{nij}} \int_{U_{nij}} \int_{W_n} \int_{W_n} \left|\mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y_2-y_1)Y_j(x_2+y_2-y_1)\big)\right| \\
&\qquad\qquad \times |W_n|^2\, G_i(W_n, y_1) G_j(W_n, x_1+y_1) G_i(W_n, y_2) G_j(W_n, x_2+y_2)\, dy_1\, dy_2\, dx_1\, dx_2 \\
&\quad=\; \int_{U_{nij}} \int_{U_{nij}} \int_{\mathbb{R}^d} \left|\mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\big)\right| \\
&\qquad\qquad \times |W_n|^2 \int_{W_n} G_i(W_n, y) G_j(W_n, x_1+y) G_i(W_n, y+u) G_j(W_n, x_2+y+u)\, du\, dy\, dx_1\, dx_2 \\
&\quad\leq\; c_1^3 \cdot \kappa_1 \cdot \frac{|U_{nij}|^2}{|W_n|}\,,
\end{aligned}$$

where the second inequality follows from (5). Hence, the final expression converges to zero given (9). Furthermore, it holds that

$$\begin{aligned}\mathbb{E}\, S_{n2}^2 &= \mathbb{E}\big[\,\widehat{\mu}_{ni}\widehat{\mu}_{nj} - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big]^2 \cdot \left(\int_{U_{nij}} |W_n|\Gamma_{nij}(x)\,dx \right)^2 \\ &\le |W_n|\mathbb{E}\big[\,\widehat{\mu}_{ni}\widehat{\mu}_{nj} - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big]^2 \cdot c_1^2\, \frac{|U_{nij}|^2}{|W_n|}\,,\end{aligned}$$

which tends to zero as well if $\overline{\lim}_{n\to\infty}|W_n|\,\mathbb{E}\big(\widehat{\mu}_{ni}\widehat{\mu}_{nj} - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big)^2 < \infty$. The latter can be seen as follows. Inserting $\pm\mu_i\mu_j$ and $\pm\mu_i\widehat{\mu}_{nj}$ yields

$$\begin{aligned}|W_n|\,\mathbb{E}\big[\,\widehat{\mu}_{ni}\widehat{\mu}_{nj} - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big]^2 &= |W_n|\,\mathbb{E}\big[(\widehat{\mu}_{ni} - \mu_i)\,\widehat{\mu}_{nj} + \mu_i\,(\widehat{\mu}_{nj} - \mu_j) + \mu_i\mu_j - \mathbb{E}\,(\widehat{\mu}_{ni}\widehat{\mu}_{nj})\big]^2 \\ &\le 3|W_n|\Big[\mathbb{E}\big([\widehat{\mu}_{ni} - \mu_i]^2\,\widehat{\mu}_{nj}^2\big) + \mu_i^2\,\mathbb{E}\big(\widehat{\mu}_{nj} - \mu_j\big)^2 + \mathrm{Cov}^2\big(\widehat{\mu}_{ni}, \widehat{\mu}_{nj}\big)\Big],\end{aligned}$$

where $\lim_{n\to\infty}|W_n|\mathbb{E}\,(\widehat{\mu}_{nj} - \mu_j)^2 = \sigma_{jj} < \infty$ and $\lim_{n\to\infty}\sqrt{|W_n|}\mathrm{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj}) = 0$. From the definition of $\widehat{\mu}_{ni}, i = 1, ..., m$ and Fubini's theorem it follows that

$$\begin{aligned}|W_n|\,\mathbb{E}\big([\widehat{\mu}_{ni} - \mu_i]^2\,\widehat{\mu}_{nj}^2\big) &= |W_n| \int_{W_n^4} \mathbb{E}\big([Y_i(v_1) - \mu_i]\,[Y_i(v_2) - \mu_i]\,Y_j(v_3)Y_j(v_4)\big) \\ &\qquad \times G_i(W_n, v_1)G_i(W_n, v_2)G_j(W_n, v_3)G_j(W_n, v_4)\,dv_1\,dv_2\,dv_3\,dv_4 \\ &= |W_n| \int_{W_n}\int_{W_n - v}\int_{W_n - v}\int_{W_n - v} \mathbb{E}\big([Y_i(o) - \mu_i]\,[Y_i(y) - \mu_i]\,Y_j(x_1)Y_j(x_2)\big) \\ &\qquad \times G_i(W_n, v)G_i(W_n, y + v)G_j(W_n, x_1 + v)G_j(W_n, x_2 + v)\,dy\,dx_2\,dx_1\,dv \\ &\le c_1 \int_{W_n}\int_{W_n - v}\int_{W_n - v}\int_{\mathbb{R}^d} \big|\mathbb{E}\big([Y_i(o) - \mu_i]\,[Y_i(y) - \mu_i]\,Y_j(x_1)Y_j(x_2)\big)\big|\,dy \\ &\qquad \times G_i(W_n, v)G_j(W_n, x_1 + v)G_j(W_n, x_2 + v)\,dx_2\,dx_1\,dv \\ &\le c_1 \cdot \sup_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathbb{E}\big([Y_i(o) - \mu_i)]\,[Y_i(y) - \mu_i]\,Y_j(x_1)Y_j(x_2)\big)\big|\,dy\,.\end{aligned}$$

The last two lines are obtained by (3) and (5), that is, in particular, that $G(W_n, \cdot)$ integrates to one over W_n. By condition (14), we obtain the desired result. □

As mentioned before, similar estimation methods for Σ are used in Böhm *et al.* (2004, Section 3) for 0-1-valued vector fields or Schmidt & Spodarev (2005, Section 3.5) for arbitrary stationary vector fields with uniform weights. Both variants assume that conditions (7) and (13) hold to prove mean-square consistency. In the case of 0-1-valued random fields no further assumptions are needed, since Y is uniformly bounded on $\mathbb{R}^d$. For the more general setting considered in Schmidt & Spodarev (2005), mean-square consistency is shown using different arguments in the proof, which lead to stronger integrability conditions. We point out that formula (56) in Schmidt & Spodarev (2005) is a sufficient condition for (13). In addition, we have to assume that

$$\int_{\mathbb{R}^{3d}} \Big| \mathbb{E}\big([Y_i(o)-\mu_i]\,[Y_j(x_1)-\mu_i]\,[Y_i(x_2)-\mu_i]\,[Y_j(x_3)-\mu_i]\big) - \mathrm{Cov}_{ij}(x_1)\mathrm{Cov}_{ij}(x_2-x_3) \qquad (15)$$
$$-\mathrm{Cov}_{ii}(x_2)\mathrm{Cov}_{jj}(x_1-x_3) - \mathrm{Cov}_{ij}(x_3)\mathrm{Cov}_{ji}(x_1-x_2)\Big|\, dx_1\,dx_2\,dx_3 \;\le\; \kappa$$

for some finite constant κ and $i,j=1,\dots,m$.

3.3 Examples; Gaussian random fields

It can easily be verified that conditions (7) and (13)–(15) hold if the covariance function of (not necessarily Gaussian) vector random field Y has compact support, i.e., $\mathrm{Cov}_{ij}(x)=0$ if $|x|>r_0$ for some $r_0>0$.

Furthermore, a less trivial class of examples is given by stationary Gaussian random fields with exponential (cross)covariance structure (for which condition (7) is obviously satisfied). We show that conditions (13)–(14) hold in this case as well. For simplicity of calculations, assume $d=1$ (dependent Gaussian processes).

Let $Y_1=\{Y_1(t),\ t\in\mathbb{R}\}$ and $Y_2=\{Y_2(t),\ t\in\mathbb{R}\}$ be two dependent stationary centered Gaussian random processes with exponential covariance $C_i(t)=\mathbb{E}\left(Y_i(0)Y_i(t)\right)=e^{-|t|/a}$, $t\in\mathbb{R}$, $i=1,2$ and cross–covariance $C_{12}(t)=C_{21}(t)=\mathbb{E}\left(Y_1(0)Y_2(t)\right)=e^{-\sqrt{t^2+h^2}/a}$, $t\in\mathbb{R}$. Such processes can be constructed e.g. as $Y_1(t)=Z\big((t,0)\big)$ and $Y_2(t)=Z\big((t,h)\big)$ for some $h\in\mathbb{R}$ and each $t\in\mathbb{R}$, where $Z=\{Z(x),\ x\in\mathbb{R}^2\}$ is a stationary centered Gaussian random field with covariance function $C(x)=\mathbb{E}\left(Z(o)Z(x)\right)=e^{-|x|/a}$, $x\in\mathbb{R}^2$ for some constant $a>0$.

Note that condition (13) is satisfied if

$$\int_{\mathbb{R}} \big|\mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\big)\big|\, dy \;\le\; \kappa_1 \qquad (16)$$

for all $i,j\in\{1,2\}$ and $x_1,x_2\in\mathbb{R}$. Assume $i\neq j$, $h\neq 0$. The particular case $i=j$ will follow from the same reasoning by letting $h=0$. By K_{ij} we denote the covariance

$$K_{ij}=\mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2+y)\big)\,.$$

It can be shown that

$$K_{ij}=e^{-\big(|y|+|x_1-x_2-y|\big)/a}+e^{-\big(\sqrt{(y-x_1)^2+h^2}+\sqrt{(x_2+y)^2+h^2}\big)/a}$$

leading to the upper bound

$$|K_{ij}|\le e^{-|y|/a}+e^{-|y-x_1|/a}\,,\qquad x_1,x_2,y\in\mathbb{R}\,,$$

and hence

$$\int\limits_{\mathbb{R}} |K_{ij}|\,dy \le 2\int\limits_{\mathbb{R}} e^{-|y|/a}\,dy=\kappa_1<\infty\,.$$

Thus, the inequality (16) is proved. Analogously, condition (14) is satisfied if

$$\int_{\mathbb{R}} \left|\mathbb{E}\big(Y_i(o)Y_i(y)Y_j(x_1)Y_j(x_2)\big)\right| \, dy \;\leq\; \kappa_2 \tag{17}$$

for all $i, j \in \{1, 2\}$ and $x_1, x_2 \in \mathbb{R}$. As above, it can be shown that

$$\begin{aligned} \mathbb{E}\big(Y_i(o)Y_i(y)Y_j(x_1)Y_j(x_2)\big) &= e^{-\left(|y|+|x_1-x_2|\right)/a} + e^{-\left(\sqrt{x_2^2+h^2}+\sqrt{(y-x_1)^2+h^2}\right)/a} \\ &+ e^{-\left(\sqrt{x_1^2+h^2}+\sqrt{(y-x_2)^2+h^2}\right)/a} \leq e^{-|y|/a} + e^{-|y-x_1|/a} + e^{-|y-x_2|/a} \end{aligned}$$

for all $x_1, x_2, y \in \mathbb{R}$ leading to

$$\int_{\mathbb{R}} \left|\mathbb{E}\big(Y_i(o)Y_i(y)Y_j(x_1)Y_j(x_2)\big)\right| dy \leq 3 \int_{\mathbb{R}} e^{-|y|/a} \, dy = \kappa_2 < \infty \,,$$

i.e., the inequality (17) is also proved.

4 Random fields related to the Boolean model

The aim of this section is to give a simple sufficient condition for the integrability conditions (13) and (14) for a another special class of jointly stationary random fields, which are functionals of particular stationary random closed sets.

4.1 Boolean model

Let $X = \{X_l\}$ be a stationary Poisson point process on $\mathbb{R}^d$ with finite intensity $\lambda > 0$. To each *germ* X_l affix an independent copy M_l of a non-empty compact and convex random set M_0 called the *typical grain*. The sequence $M = \{M_l\}$ of grains has to be independent of the point process X. A Boolean model Ξ with convex grains is defined as the set-theoretic union of the translated RACS $M_l + X_l$, i.e.,

$$\Xi = \bigcup_{l=1}^{\infty} (M_l + X_l)\,. \tag{18}$$

The right-hand side of (18) is almost surely closed and different from $\mathbb{R}^d$ if

$$\mathbb{E}\,|M_0 \oplus \check{K}| < \infty \qquad \text{for any } K \in \mathcal{K}\,; \tag{19}$$

see Heinrich (1992). This condition is always satisfied e.g. in the case of grains $M_0 \subset B_R(o)$ a.s. for some $R > 0$. Moreover, a Boolean model Ξ with convex grains M_l can be represented as the union set of a Poisson particle process $\Psi = \{\Psi_l\}$ on $\mathcal{K}$, where particles Ψ_l are defined as $\Psi_l = M_l + X_l$, confer Schneider & Weil (2008). The intensity measure of Ψ is denoted by Λ, where $\Lambda(B) = \mathbb{E}\,\Psi(B)$ is the expected number of particles belonging to B for $B \in \sigma_{\mathcal{K}}$. Let $g_{\Psi(B)}(s) = \mathbb{E}\left(s^{\Psi(B)}\right)$, $s \in \mathbb{R}$ be the generating function of $\Psi(B)$. Then it holds that

$$g_{\Psi(B)}(s) = e^{(s-1)\,\Psi(B)}, \qquad s \in \mathbb{R}\,. \tag{20}$$

An application of Campbell's theorem for independently marked point processes in $\mathbb{R}^d$ (see, e.g. Schneider & Weil (2008)) yields

$$\begin{aligned}\Lambda(B) &= \mathbb{E}\sum_{\psi\in\Psi}\mathbb{I}(\psi\in B) \;=\; \mathbb{E}\sum_{y\in X, M_0\in M}\mathbb{I}((M_0+y)\in B)\\ &= \lambda\int_{\mathbb{R}^d}\mathbb{E}\,\mathbb{I}((M_0+y)\in B)\,dy \end{aligned}\tag{21}$$

for all $B \in \sigma_{\mathcal{K}}$. Considering $\mathcal{K}_K = \{K' \in \mathcal{K} : K' \cap K \neq \emptyset\}$, we find that $\Lambda(\mathcal{K}_K) = \lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|$ so that $g_{\Psi(\mathcal{K}_K)}(s) < \infty$ for any $s \in \mathbb{R}$ given that (19) is fulfilled.

4.2 Related random fields

Let $f : \mathcal{R} \to \mathbb{R}$ be a functional on the convex ring $\mathcal{R}$, which is the family of all finite unions of sets from $\mathcal{K}$. We say, f is a *valuation* on $\mathcal{R}$ if f is measurable and additive, i.e.,

$$f(K_1 \cup K_2) = f(K_1) + f(K_2) - f(K_1 \cap K_2)$$

for any $K_1, K_2 \in \mathcal{R}$ and $f(\emptyset) = 0$. Furthermore, we assume that f is *conditionally bounded* on $\mathcal{K}$, meaning that for any $K \in \mathcal{K}$ there exists a finite bound $c(K)$ such that for all $K' \in \mathcal{K}$ with $K' \subseteq K$ it holds that

$$|f(K')| \;\leq\; c(K)\,.$$

For any fixed convex body $K \in \mathcal{K}$ and for any Boolean model Ξ satisfying (19) the random set $\Xi \cap K$ belongs to $\mathcal{R}$ with probability one. Consequently, we may consider the random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ defined by

$$Y(x) = f\left((\Xi - x) \cap K\right), \qquad x \in \mathbb{R}^d\,. \tag{22}$$

In many cases, it can be thought of as a measurement of the random set Ξ within a moving scanning window $K + x$, $x \in \mathbb{R}^d$. Since Ξ is stationary, the random field Y is stationary as well. Notice that by (20) all moments of $Y(x), x \in \mathbb{R}^d$ are finite, since one can show that

$$\mathbb{E}\,|Y^p(x)| \;\leq\; c^p(K)e^{(2^p-1)\lambda\,\mathbb{E}\,|M_0\oplus\check{K}|} \qquad \text{for any } p > 0\,, \tag{23}$$

see for example Lemma 2.1 of Pantle *et al.* (2006). Finally, consider $m \in \mathbb{N}$ pairs $(f_i, K_i), i = 1, \ldots, m$ of conditionally bounded valuations $f_i : \mathcal{R} \to \mathbb{R}$ and convex test sets $K_i \in \mathcal{K}$. Then the vector field $Y = \{Y(x), x \in \mathbb{R}^d\}$ with $Y(x) = (Y_1(x), \ldots, Y_m(x))$ and $Y_i = \{f_i((\Xi - x) \cap K_i), x \in \mathbb{R}^d\}$ is of the form discussed in Section 3.

4.3 Integrability of mixed moments

In the following, we show that the vector field Y constructed above obeys the integrability conditions (13) and (14) under second moment conditions on the volume of the dilated primary grain $M_0 \oplus \check{K}$, $K \in \mathcal{K}$. To do this, we use the representation of Ξ as the union set of the generating Poisson particle process $\Psi = \{\Psi_l\}$ on $\mathcal{K}$ with intensity measure Λ as introduced above. In Lemma 4.1 of Pantle *et al.* (2006) it has already been shown that (7), i.e. $\int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, dx < \infty$, holds under the assumption $\mathbb{E}\,|M_0 \oplus \check{K}_i|^2 < \infty$ for $i = 1, \ldots, m$. The subsequent theorem yields that (13) and (14) are satisfied under the very same condition.

Theorem 2. *If* $\mathbb{E}\,|M_0 \oplus \check{K}_i|^2 < \infty$ *for* $i = 1, \ldots, m$*, then there exist constants* $\kappa_1, \kappa_2 < \infty$ *such that*

$$\sup_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathrm{Cov}\big(Y_i(o)Y_j(x_1), Y_i(y)Y_j(x_2 + y)\big)\big|\, dy \;\leq\; \kappa_1$$

and

$$\sup_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big|\mathbb{E}\big([Y_i(o) - \mu_i]\,[Y_i(y) - \mu_i]\, Y_j(x_1)\, Y_j(x_2)\big)\big|\, dy \;\leq\; \kappa_2$$

for all $i, j = 1, ..., m$*.*

The proof of this theorem is given in Appendix.

5 The empirical covariance

The estimator $\widehat{\sigma}_{nij}$ introduced in (10) makes use of the fact that σ_{ij} is basically the integral of $\mathrm{Cov}_{ij}(x)$ over $\mathbb{R}^d$. Thus, estimating the covariance and considering the integral over an unboundedly increasing window is a quite intuitive approach. Another possibility is to employ that $\sigma_{ij} = \lim_{n\to\infty} |W_n| \mathrm{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj})$. Provided that there exist samples $\sqrt{|W_n|}\widehat{\mu}_{ni,1}, \ldots, \sqrt{|W_n|}\widehat{\mu}_{ni,N(n)}$ and $\sqrt{|W_n|}\widehat{\mu}_{nj,1}, \ldots, \sqrt{|W_n|}\widehat{\mu}_{nj,N(n)}$ for some $N(n) \in \mathbb{N}$, a natural estimate is given by the empirical covariance of these samples. Initially, we assumed that there exists only one value $\widehat{\mu}_{ni}$ and $\widehat{\mu}_{nj}$, respectively. One way to use as much information from the observations in W_n as possible and to cut down the running time is to divide W_n into $N(n)$ smaller subwindows and perform the estimation of the means μ_{ni} of random fields Y_i on each subwindow separately. This will give us the required samples of size $N(n)$.

To be more precise, we need some further definitions and notation. Let $\{V_n\}$ with $V_n \subset W_n, n \geq 1$ be a sequence of monotonously increasing, bounded Borel sets fulfilling (1) and $\{N(n)\}$ an increasing sequence of integers with $\lim_{n\to\infty} N(n) = \infty$. For any $n \in \mathbb{N}$ choose some vectors $h_{n,1}, \ldots, h_{n,N(n)} \in \mathbb{R}^d$ and define $V_{n,k} = V_n + h_{n,k}$, $k = 1, \ldots, N(n)$ satisfying the following conditions:

- Suppose that

$$\bigcup_{k=1}^{N(n)} V_{n,k} \subseteq W_n \qquad \text{and} \qquad G_i(V_{n,k}, x) = G_i(V_n, x - h_{n,k}), \qquad x \in \mathbb{R}^d,\ i = 1, \ldots, m, n \in \mathbb{N}. \tag{24}$$

- Condition (5) holds for $\{V_{n,k}\}$ with constant $\tilde{c}_1$ and limits $\tilde{\theta}_{ij} \in (0,\infty)$, $i,j = 1\dots,m$. Without loss of generality we may assume that $\tilde{c}_1 = c_1$ and $\theta_{ij} = \tilde{\theta}_{ij}$.
- There exists some $r > 0$ such that

$$V_{n,k} \cap V_{n,\ell} \subset \partial V_{n,k} \oplus B_r(o) \qquad \text{for } k,\ell \in \{1,\dots,N(n)\} \text{ with } k \neq \ell. \tag{25}$$

Furthermore, denote by $\widehat{\mu}_{ni,k}$ the estimator of $\mu_i = \mathbb{E}\, Y_i(o)$ as given in (2), but based on observations within $V_{n,k}$ only: $\widehat{\mu}_{ni,k} = \int_{V_{n,k}} Y_i(x) G_i(V_{n,k},x)\,dx$. Now define a new estimator $\widetilde{\Sigma}_n = (\widetilde{\sigma}_{nij})$ by the formula

$$\widetilde{\sigma}_{nij} = \frac{|V_n|}{N(n)-1} \sum_{k=1}^{N(n)} \big(\widehat{\mu}_{ni,k} - \overline{\mu}_{ni}\big)\big(\widehat{\mu}_{nj,k} - \overline{\mu}_{nj}\big), \tag{26}$$

where $\overline{\mu}_{ni} = \dfrac{1}{N(n)} \sum_{k=1}^{N(n)} \widehat{\mu}_{ni,k}$ for $i = 1,\dots,m$.

The estimator $\widetilde{\Sigma}_n$ is asymptotically unbiased under the same assumptions as considered for $\widehat{\Sigma}_n$, but mean-square consistency requires integrability condition (15).

Lemma 2. *The estimator $\widetilde{\Sigma}_n$ defined in* (26) *is asymptotically unbiased for Σ as $n \to \infty$.*

Proof. Set $N = N(n)$ and employ the following representation

$$\widetilde{\sigma}_{nij} = \frac{|V_n|}{N} \sum_{k=1}^{N} \big(\widehat{\mu}_{ni,k} - \mu_i\big)\big(\widehat{\mu}_{nj,k} - \mu_j\big) - \frac{|V_n|}{N(N-1)} \sum_{\substack{k,\ell=1\\ k\neq\ell}}^{N} \big(\widehat{\mu}_{ni,k} - \mu_i\big)\big(\widehat{\mu}_{nj,\ell} - \mu_j\big). \tag{27}$$

This formula can be derived by elementary transformation of $\sum_{k=1}^{N} \big(\widehat{\mu}_{ni,k} - \overline{\mu}_{ni} \pm \mu_i\big)\big(\widehat{\mu}_{nj,k} - \overline{\mu}_{nj} \pm \mu_j\big)$. With regard to the expectation of the first summand in (27) we get

$$\begin{aligned}
\frac{|V_n|}{N} \sum_{k=1}^{N} \mathbb{E}\,\big(\widehat{\mu}_{ni,k} - \mu_i\big)\big(\widehat{\mu}_{nj,k} - \mu_j\big) &= \frac{|V_n|}{N} \sum_{k=1}^{N} \int_{V_{n,k}} \int_{V_{n,k}} \mathrm{Cov}_{ij}(y-x) G_i(V_{n,k},x) G_j(V_{n,k},y)\,dx\,dy \\
&= \frac{|V_n|}{N} \sum_{k=1}^{N} \int_{V_n} \int_{V_n} \mathrm{Cov}_{ij}(y-x) G_i(V_n,x) G_j(V_n,y)\,dx\,dy \\
&= \int_{\mathbb{R}^d} \mathrm{Cov}_{ij}(x)\,|V_n| \Gamma_{V_{nij}}(x)\,dx\,,
\end{aligned}$$

where we used condition (24) in the second line. The last expression converges to σ_{ij} as $n \to \infty$ by the dominated convergence theorem, given (7) and (5). It remains to show that the expectation of the second expression in (27) tends to zero as $n \to \infty$. In fact, one obtains for any $k \neq \ell$ that

$$\begin{aligned}
\mathbb{E}\,\big(\widehat{\mu}_{ni,k} - \mu_i\big)\big(\widehat{\mu}_{nj,\ell} - \mu_j\big) &= \int_{V_{n,k}} \int_{V_{n,\ell}} \mathrm{Cov}_{ij}(y-x) G_i(V_{n,k},x) G_j(V_{n,\ell},y)\,dx\,dy \\
&\leq c_1^2 \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)|\, \frac{|V_{n,k} \cap (V_{n,\ell} - x)|}{|V_n|^2}\,dx\,.
\end{aligned}$$

By (25), it holds that $|V_{n,k} \cap (V_{n,\ell} - x)| \leq |(\partial V_n \oplus B_{|x|+r}(o)) \cap V_n|$ for each $l \neq k, l, k = 1, \ldots, N$. Thus, assuming (1) and (7) we have

$$\mathbb{E}\Big(\frac{|V_n|}{N(N-1)} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^{N} \big(\widehat{\mu}_{ni,k} - \mu_i\big)\big(\widehat{\mu}_{nj,\ell} - \mu_j\big)\Big) \leq c_1^2 \int_{\mathbb{R}^d} |\mathrm{Cov}_{ij}(x)| \frac{|(\partial V_n \oplus B_{|x|+r}(o)) \cap V_n|}{|V_n|}\, dx \to 0$$

as n tends to infinity. □

Theorem 3. *If condition* (15) *holds, then* $\widetilde{\Sigma}_n$ *is a mean-square consistent estimator of* Σ.

Proof. By means of Lemma 2, it suffices to show that $\lim_{n\to\infty} \mathbb{E}\,(\widetilde{\sigma}_{nij} - \mathbb{E}\,\widetilde{\sigma}_{nij})^2 = 0$. To simplify notation, write $\omega_{ij,kn}(x,y) = G_i(V_{n,k},x)G_j(V_{n,k},y)$ for all $i,j = 1,...,m, x,y \in \mathbb{R}^d$. As in the proof of Lemma 2 consider the two summands of formula (27) separately, using the fact that the covariance of two random variables converges to zero whenever the variances do. For the variance of the first summand we observe that

$$\begin{aligned}
&\mathbb{E}\Big(\frac{|V_n|}{N}\sum_{k=1}^{N}\Big(\big(\widehat{\mu}_{ni,k} - \mu_i\big)\big(\widehat{\mu}_{nj,k} - \mu_j\big) - \int_{V_{n,k}}\int_{V_{n,k}} \mathrm{Cov}_{ij}(y-x)\,\omega_{ij,kn}(x,y)\,dx\,dy\Big)\Big)^2 \\
&= \frac{|V_n|^2}{N^2}\sum_{k,\ell=1}^{N}\int_{V_{n,k}^2}\int_{V_{n,\ell}^2}\Big(\mathbb{E}\left((Y_i(v_1)-\mu_i)(Y_j(v_2)-\mu_j)(Y_i(v_3)-\mu_i)(Y_j(v_4)-\mu_j)\right) \\
&\qquad - \mathrm{Cov}_{ij}(v_2-v_1)\mathrm{Cov}_{ij}(v_4-v_3)\Big)\,\omega_{ij,kn}(v_1,v_2)\,\omega_{ij,\ell n}(v_3,v_4)\,dv_1\,dv_2\,dv_3\,dv_4\,.
\end{aligned}$$

For any $v_1,v_2,v_3,v_4 \in \mathbb{R}^d$ set $c_{ij}^{(4)}(v_1,v_2,v_3,v_4) = \mathbb{E}\left((Y_i(v_1)-\mu_i)(Y_j(v_2)-\mu_j)(Y_i(v_3)-\mu_i)(Y_j(v_4)-\mu_j)\right)$ $- \mathrm{Cov}_{ij}(v_2-v_1)\mathrm{Cov}_{ij}(v_4-v_3) - \mathrm{Cov}_{ii}(v_3-v_1)\mathrm{Cov}_{jj}(v_4-v_2) - \mathrm{Cov}_{ij}(v_4-v_1)\mathrm{Cov}_{ij}(v_2-v_3)$. Then it follows that

$$\begin{aligned}
&\int_{V_{n,k}^2}\int_{V_{n,\ell}^2}\Big(\mathbb{E}\left((Y_i(v_1)-\mu_i)(Y_j(v_2)-\mu_j)(Y_i(v_3)-\mu_i)(Y_j(v_4)-\mu_j)\right) - \mathrm{Cov}_{ij}(v_2-v_1)\mathrm{Cov}_{ij}(v_4-v_3)\Big) \\
&\qquad \times \omega_{ij,kn}(v_1,v_2)\,\omega_{ij,\ell n}(v_3,v_4)\,dv_1\,dv_2\,dv_3\,dv_4 \\
&= \int_{V_{n,k}^2}\int_{V_{n,\ell}^2} c_{ij}^{(4)}(v_1,v_2,v_3,v_4)\,\omega_{ij,kn}(v_1,v_2)\,\omega_{ij,\ell n}(v_3,v_4)\,dv_1\,dv_2\,dv_3\,dv_4 \\
&\quad + \int_{V_{n,k}^2}\int_{V_{n,\ell}^2}\Big(\mathrm{Cov}_{ii}(v_3-v_1)\mathrm{Cov}_{jj}(v_4-v_2) + \mathrm{Cov}_{ij}(v_4-v_1)\mathrm{Cov}_{ij}(v_2-v_3)\Big) \\
&\qquad \times \omega_{ij,kn}(v_1,v_2)\,\omega_{ij,\ell n}(v_3,v_4)\,dv_1\,dv_2\,dv_3\,dv_4
\end{aligned}$$

Using the same arguments as in the proof of Lemma 2, one obtains

$$\begin{aligned}
&\frac{|V_n|^2}{N^2}\sum_{k,\ell=1}^{N}\int_{V_{n,k}^2}\int_{V_{n,\ell}^2}\Big(\mathbb{E}\left((Y_i(v_1)-\mu_i)(Y_j(v_2)-\mu_j)(Y_i(v_3)-\mu_i)(Y_j(v_4)-\mu_j)\right) \\
&\qquad - \mathrm{Cov}_{ij}(v_2-v_1)\mathrm{Cov}_{ij}(v_4-v_3)\Big)\,\omega_{ij,kn}(v_1,v_2)\,\omega_{ij,\ell n}(v_3,v_4)\,dv_1\,dv_2\,dv_3\,dv_4 \\
&\leq \frac{c_1^4}{|V_n|}\cdot\int_{\mathbb{R}^{3d}} |\,c_{ij}^{(4)}(o,x_1,x_2,x_3)|\,dx_1\,dx_2\,dx_3 +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c_1^4}{N^2}\sum_{k,\ell=1}^{N}\int_{\mathbb{R}^d}|\operatorname{Cov}_{ii}(x)|\,\frac{|V_{ni,k}\cap(V_{ni,l}-x)|}{|V_n|}\,dx\int_{\mathbb{R}^d}|\operatorname{Cov}_{jj}(x)|\,dx\\
&+\frac{c_1^4}{N^2}\sum_{k,\ell=1}^{N}\int_{\mathbb{R}^d}|\operatorname{Cov}_{ij}(x)|\,\frac{|V_{ni,k}\cap(V_{nj,l}-x)|}{|V_n|}\,dx\int_{\mathbb{R}^d}|\operatorname{Cov}_{ij}(x)|\,dx\\
\le\;&\frac{c_1^4}{|V_n|}\cdot\kappa+c_1^4\int_{\mathbb{R}^d}|\operatorname{Cov}_{ii}(x)|\cdot\left(\frac{1}{N}+\frac{|(\partial V_n\oplus B_{|x|+r}(o))\cap V_n|}{|V_n|}\right)dx\int_{\mathbb{R}^d}|\operatorname{Cov}_{jj}(x)|\,dx\\
&+c_1^4\int_{\mathbb{R}^d}|\operatorname{Cov}_{ij}(x)|\cdot\left(\frac{1}{N}+\frac{|(\partial V_n\oplus B_{|x|+r}(o))\cap V_n|}{|V_n|}\right)dx\int_{\mathbb{R}^d}|\operatorname{Cov}_{ij}(x)|\,dx\,.
\end{aligned}
$$

The last expression tends to zero, since $\lim_{n\to\infty}|V_n|=\lim_{n\to\infty}N(n)=\infty$. Besides that, it can now easily be derived that the variance of the second summand of (27) has similar upper bounds and, consequently, converges to zero as well. □

If $Y(x)=\mathbb{1}(x\in\Xi)$, where Ξ is a Boolean model with compact typical grain M_0, then it is known from Lemma 7 of Heinrich (2005) that a sufficient assumption for the conditions of Theorem 3 is given by $\mathbb{E}\,|M_0|^4<\infty$. Notice that $\widehat{\Sigma}_n$, on the contrary, is mean–square consistent in this particular case provided that M_0 is compact and convex and $\mathbb{E}\,|M_0|^2<\infty$, compare Theorem 2.

6 Numerical examples

We conclude with a numerical comparison of the estimators $\widehat{\Sigma}_n$ and $\widetilde{\Sigma}_n$ with respect to performance and computational effort.

6.1 Dependent Gaussian random processes

Revisiting the example of Section 3.3, the behaviour of the estimator $\widehat{\Sigma}_n$ with respect to the relationship between the observation window W_n and the averaging set U_n is investigated by simulations, see Table 1. Let $K(\alpha,x)$ denote the modified Bessel function of the second kind. Then, the asymptotic covariance matrix

$$
\Sigma=\begin{pmatrix}\int\limits_{\mathbb{R}}e^{-|y|/a}\,dy & \int\limits_{\mathbb{R}}e^{-\sqrt{y^2+h^2}/a}\,dy\\ \int\limits_{\mathbb{R}}e^{-\sqrt{y^2+h^2}/a}\,dy & \int\limits_{\mathbb{R}}e^{-|y|/a}\,dy\end{pmatrix}=\begin{pmatrix}2a & 2hK\left(1,\frac{h}{a}\right)\\ 2hK\left(1,\frac{h}{a}\right) & 2a\end{pmatrix}=\begin{pmatrix}100 & 87.374\\ 87.374 & 100\end{pmatrix}
$$

of the dependent Gaussian processes Y_1 and Y_2 with parameter values $a=50$ and $h=20$ and weight function $G_i(W_n,x)=\mathbb{1}(x\in W_n)\,/\,|W_n|$ (meaning $\theta_{ij}=1,\ i,j=1,2$) was estimated by $\widehat{\Sigma}_n$ as an average of 200 simulations. The value of the Bessel function $K(1,2/5)$ was assessed numerically using Maple. In Table 1, the values of the empirical standard deviation and the bias for $\widehat{\sigma}_{ij}$ are given as well. One can observe that the variance of the estimator increases with increasing subwindow U_n. Conversely, the bias of the estimator decreases with increasing U_n. The overall precision of the estimator increases with increasing W_n since both bias and variance become smaller.

6.2 Boolean Model and related random fields

Let Ξ be a stationary Boolean model with compact and convex uniformly bounded grains. Hence, condition (19) is satisfied. Then, the intrinsic volumes $V_j(\Xi \cap K), j = 0, ..., d$ of $\Xi \cap K$ are well defined for any convex body $K \in \mathcal{K}$, see e.g. Schneider (1993) or Schneider & Weil (2008) on intrinsic volumes. In the plane, i.e., $d = 2$, for instance, $V_2(\Xi \cap K)$ is the usual area, $2V_1(\Xi \cap K)$ is the boundary length and $V_0(\Xi \cap K)$ is the *Euler-Poincaré characteristic* of the set $\Xi \cap K$. Let $Y_i(x) = V_0(\Xi \cap B_{r_i}(x))$ for $m > d+1$ distinct radii $r_i, i = 1, ..., m$. In the following, we estimate the asymptotic integrated covariance matrix of the stationary vector–valued random field $Y = (Y_1, \ldots, Y_m)^\top$ from simulated realisations of Ξ. Since $Y_i(x)$ and $Y_j(y)$ are independent if $|x - y| > r_i + r_j + d_{M_0}$, $x, y \in \mathbb{R}^2$, where d_{M_0} is the maximal (deterministic) diameter of the typical grain M_0, we put $U_{nij} = B_{r_i+r_j+d_{M_0}}(o)$ in (10) for the estimator $\widehat{\Sigma}_n$.

First, let us motivate this simulation study by applications in stochastic geometry. Consider the so–called *intrinsic volume densities* of a Boolean model Ξ and the corresponding integrated covariance functions. For any sequence $\{K_n\}$ of convex bodies $K_n = nK_0$ with $K_0 \in \mathcal{K}$ such that $|K_0| > 0$ and $o \in \mathrm{int}(K_0)$, the limits

$$\overline{V}_j(\Xi) = \lim_{n\to\infty} \frac{\mathbb{E}\, V_j(\Xi \cap K_n)}{|K_n|}\,, \qquad j = 0, \ldots, d \tag{28}$$

exist and are called the intrinsic volume densities of Ξ. For some intrinsic volume densities, estimators of several types are considered in the literature. The following indirect estimation method has been proposed in Spodarev & Schmidt (2005):

$$\widehat{v}_n = \left(A^\top_{r_1,\ldots,r_m} A_{r_1,\ldots,r_m}\right)^{-1} A^\top_{r_1,\ldots,r_m} \widehat{\mu}_n\,, \tag{29}$$

where $\widehat{v}_n$ is a least-squares estimator for $v = (\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))$ minimizing $|A_{r_1,\ldots,r_m}\widehat{\mu}_n - v|$ on $\mathbb{R}^{d+1}$, $\widehat{\mu}_n$ is the mean value estimator of Y defined in (2) and $A_{r_1,\ldots,r_m}$ is a specific $m \times (d+1)$–dimensional matrix of rank $d+1$. We refer to the article Guderlei *et al.* (2006) for related implementation issues. A major advantage of this estimation method for our purpose is that the values $Y_i(x)$ can be determined for each point x inside the observation window W_n explicitly with acceptable runtime. To assess the quality of the estimates of Σ, the transformed estimators $\widehat{C}_n$ and $\widetilde{C}_n$ are compared, where

$$\widehat{C}_n \;=\; \left(A^\top_{r_1,\ldots,r_m} A_{r_1,\ldots,r_m}\right)^{-1} A^\top_{r_1,\ldots,r_m} \widehat{\Sigma}_n\, A_{r_1,\ldots,r_m} \left(A^\top_{r_1,\ldots,r_m} A_{r_1,\ldots,r_m}\right)^{-1}, \quad n \geq 1$$

and $\widetilde{C}_n$ is defined analogously. In other words, we compare the estimated values of the asymptotic covariance matrix C of $\sqrt{|W_n|}(\widehat{v}_n - v)$, whereas, from the computational point of view, the estimation of the asymptotic covariance matrix of random fields Y_i was performed first and then the linear transformation as in (29) was applied.

In the sequel, set $d = 2$ and skip the index n for simplicity. Several Boolean models Ξ with typical grain $M_0 = B_R(o)$, $R \sim U(20, 40)$ and intensity $\lambda = -\log(0.5)/\pi\, \mathbb{E}(R^2)$ were simulated in the observation window $W = [-1500, 1500]^2$ with in total 3000^2 data points (pixels). The intensity of the underlying point process is chosen so that $\overline{V}_2(\Xi) = 0.5$ for each setting. Notice that the values of $\overline{V}_1(\Xi)$ and $\overline{V}_0(\Xi)$ are also known for these models (see Stoyan *et al.* (1995, p. 76)). For the auxiliary vector $Y = (Y_1, \ldots, Y_m)^\top$, put $m = 16$ and $r_{i+1} = 4.2 + 1.3\, i$, $i = 0, \ldots, 15$; confer Guderlei *et al.* (2006) on the choice of these parameters. Any integral in the definitions of the estimators is discretized using observations on the grid $W \ominus B_r(o) \cap \Delta\mathbb{Z}^2$ for some grid mesh size $\Delta \in \mathbb{N}$ and $r = \max\{r_1, \ldots, r_m\}$. For the computation of $\widehat{\Sigma}$, we choose the discrete weighting functions $g_i(W, x) = \mathbb{1}(x \in (W \ominus B_r(o)) \cap \Delta\mathbb{Z}^2)/\mathrm{card}((W \ominus B_r(o)) \cap \Delta\mathbb{Z}^2)$. For the values of all estimation parameters, see Table 2. The evaluation of $\widetilde{\Sigma}$ is performed on N non–overlapping subwindows V_k. Again, uniform weights are assigned to each observation in $V_k \ominus B_r(o) \cap \Delta\mathbb{Z}^2$. A finer mesh size Δ is chosen for the second partition (cf. last line of Table 2) due to the reduced size of the subwindows. For each Boolean model, $k = 200$ simulations were performed. Typical simulation results are shown in the Tables 3–6 including the standard deviation (std) from the corresponding averaged values. On average, a Pentium IV (2.4 GHz) requires about 25 minutes for the evaluation of $\widehat{C}$ on $W = [-1500, 1500]^2$. As expected, since only very elementary operations are needed, the running time for $\widetilde{C}$ is shorter with 3 minutes on average on both partitions. Smaller values of grid mesh size Δ lead to slightly more accurate results, measured with respect to the estimates of v. The running time, however, is unreasonably higher.

Since analytic formulae for C are not available, a table of empirical co-/variances from 1000 independent samples of $\sqrt{|W|}(\widehat{v} - v)$ are displayed as reference values in Table 3. With respect to these reference values, $\widehat{C}$ provides the best results in most cases (see Table 4). The results for $\widetilde{C}$ usually get better the finer the subpartition is (cf. Tables 5 and 6). The fluctuations in the estimated values $\widetilde{C}$ are still higher than those of $\widehat{C}$. The deviation from the reference values increases for two components $\tilde{c}_{00}$ and $\tilde{c}_{02}$ on the second partition. This can be explained by the fact that here the estimation of v is performed on relatively small (sub)windows compared to the estimation procedure of reference values.

Experiments showed that the application of $\widetilde{C}$ is advisable only if the observation window W is sufficiently large so that it can be decomposed into "sufficiently many" disjoint, but "not too small" subwindows. In addition to the right choice of parameters m and $r_i, i = 1, \ldots, m$, the question of an adequate size and number of the subwindows makes $\widetilde{C}$ rather critical for application. Using $\widehat{C}$ at most 2% of the estimates were not positive semidefinite. On the contrary, the matrix $\widetilde{C}$ (resp. $\widetilde{\Sigma}$) is by definition always positive semidefinite and proved to be positive definite in all simulated examples. We also remark that all samples $\widehat{v}_1, ..., \widehat{v}_k$ were tested for multivariate normality using the test proposed in Henze & Zirkler (1990) with significance level $\alpha = 0.05$ and scaling parameter $\beta = 0.5, 1.0$ and 3.0, respectively. None of the tests led to rejection of the multivariate normality assumption.

Note that there exists a simple direct estimation method for the area density $p = \overline{V}_2(\Xi)$ considered separately, see Böhm *et al.* (2004). The random field used here is given by $Y(x) = \mathbb{1}(x \in \Xi),\ x \in \mathbb{R}^d$. For this method, an explicit formula exists for the asymptotic variance σ_{pp} of $\sqrt{|W|}(\int_W Y(x)G(W_n, x)dx - p)$, where $\sigma_{pp} \approx 678.097$ in the considered example. The corresponding estimates $\widehat{\sigma}_{pp}$ and $\widetilde{\sigma}_{pp}$ are attached to each table for comparison.

7 Summary

In this paper, we considered two asymptotically unbiased and (under some additional conditions) mean-square consistent estimators for the matrix of integrated cross covariances of a stationary vector–valued random field. The first one, $\widehat{\Sigma}$, used ideas of time series analysis in its construction. The second one, $\widetilde{\Sigma}$, is an empirical covariance–type estimator which was considered in the literature before. Both estimators allow for the construction of asymptotical test of the mean of vector–valued random fields. Our simulation study showed that $\widehat{\Sigma}$ (although not necessarily positive semidefinite) performs better than $\widetilde{\Sigma}$ concerning the accuracy and the variance of estimation. However, the performance of $\widetilde{\Sigma}$ gets better if the number of subwindows V_k (as well as their size) increases. Hence, the use of $\widetilde{\Sigma}$ is legitimized for rather large observation windows W. As for the estimator $\widehat{\Sigma}$, its bias decreases and its variance increases with increasing the subwindow U_n. On the other hand, the computation of the estimator $\widehat{\Sigma}$ is much more involved (and hence slower) than that of $\widetilde{\Sigma}$. For a large number of repeated tests, the use of $\widetilde{\Sigma}$ can be recommended for runtime reasons.

Acknowledgement

The authors are grateful to Daniel Meschenmoser and Wolfgang Karcher for their help in implementation of the estimation methods and numerical experiments.

References

Adler, R. J. & Taylor, J. E. (2007) *Random fields and geometry.* Springer.

Böhm, S., Heinrich, L. & Schmidt, V. (2004) Asymptotic properties of estimators for the volume fraction of jointly stationary random sets. *Statistica Neerlandica*, **58**, 388–406.

Bulinski, A. & Kryzhanovskaya, N. (2006) Convergence rate in CLT for vector–valued random fields with self–normalization. *Probab. Math. Stat.*, **26**(2), 261–281.

Guderlei, R., Klenk, S., Mayer, J., Schmidt, V. & Spodarev, E. (2007) Algorithms for the computation of Minkowski functionals of deterministic and random polyconvex sets. *Image and Vision Computing*, **25**, 464–474.

Heinrich, L. (1992) On existence and mixing properties of germ–grain models. *Statistics*, **23**, 271–286.

Heinrich, L. (2005) Large deviations of the empirical volume fraction for stationary Poisson grain models. *Ann. Appl. Probab.*, **15**(1A), 392–420.

Henze, N. & Zirkler, B. (1990) A class of invariant consistent tests for multivariate normality. *Comm. Statist. Theory Method*, **19**, 3595–3617.

Ivanov, A. V. & Leonenko, N. N. (1989) *Statistical analysis of random fields.* Dordrecht: Kluwer.

Pantle, U., Schmidt, V. & Spodarev, E. (2006) Central limit theorem for functionals of stationary germ–grain models. *Adv. Appl. Probab.*, **38**(1), 76–94.

Schmidt, V. & Spodarev, E. (2005) Joint estimators for the specific intrinsic volume of stationary random sets. *Stoch. Proc. Appl.*, **115**, 959–981.

Schneider, R. (1993) *Convex bodies: The Brunn–Minkowski theory.* Cambridge: Cambridge University Press.

Schneider, R. & Weil, W. (2008) *Stochastic and integral geometry.* Springer, Berlin.

Spodarev, E. & Schmidt, V. (2005) On the local connectivity number of stationary random closed sets. In C. Ronse, L. Najman & E. Decenciere (eds), *Proceedings of the 7th International Symposium on Mathematical Morphology*, pp. 343–354. Dordrecht: Springer.

Stoyan, D., Kendall, W. S. & Mecke, J. (1995) *Stochastic geometry and its applications*, 2nd edn. Chichester: J. Wiley & Sons.

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Appendix

Proof of Theorem 2. We use a similar construction as in the proof of Lemma 4.1 in Pantle *et al.* (2006). For simplicity, consider the case $i = j$ and omit the indices in the following. If $i \neq j$ the proof is analogous. With $\mathcal{K}_C = \{K' \in \mathcal{K} : K' \cap C \neq \emptyset\}$ for any $C \subset \mathbb{R}^d$ define $\mathcal{K}^* = \mathcal{K}^*(x_1, x_2, y) = (\mathcal{K}_K \cup \mathcal{K}_{K+x_1}) \cap (\mathcal{K}_{K+y} \cup \mathcal{K}_{K+x_2+y})$ for $x_1, x_2, y \in \mathbb{R}^d$. Now, consider the event $A = \{\Psi(\mathcal{K}^*) > 0\}$ and its complement A^c, where $\Psi(B)$ is the random number of particles of Ψ belonging to a set $B \subseteq \mathcal{K}$. Then, it holds that

$$\begin{aligned}
&\big|\mathrm{Cov}\big(Y(o)Y(x_1), Y(y)Y(x_2+y)\big)\big| \\
&\quad = \big|\mathbb{E}\big(\,Y(o)Y(x_1)\,(\mathbb{1}_A + \mathbb{1}_{A^c}) \cdot \big[Y(y)Y(x_2+y) - \mathbb{E}\,(Y(y)Y(x_2+y))\big]\big)\big| \\
&\quad = \big|\mathbb{E}\big(\,Y(o)Y(x_1)\,\mathbb{1}_A \cdot \big[Y(y)Y(y+x_2) - \mathbb{E}\,(Y(y)Y(y+x_2))\big]\big) \\
&\qquad + \mathbb{E}\big(\,Y(o)Y(x_1)\,\mathbb{1}_{A^c} \cdot \big[Y(y)Y(y+x_2) - \mathbb{E}\,(Y(y)Y(y+x_2))\big]\big)\big|. \qquad (30)
\end{aligned}$$

In the first step, we investigate the summand (30). Define $Y_B(x) = f\big((\bigcup_{l:\Psi_l \in B} \Psi_l - x) \cap K\big)$ for any $B \subseteq \mathcal{K}$ and $x \in \mathbb{R}^d$. By additivity of f we see that

$$\begin{aligned}
Y(x) &= Y_B(x) + Y_{\mathcal{K}\setminus B}(x) - f\big((\bigcup_{l:\Psi_l \in B} \Psi_l - x)\ \cap (\bigcup_{l:\Psi_l \in \mathcal{K}\setminus B} \Psi_l - x)\ \cap K\big) \\
&= Y_{B\cap \mathcal{K}_{K+x}}(x) + Y_{\mathcal{K}_{K+x}\setminus B}(x) - f\big((\bigcup_{l:\Psi_l \in B\cap \mathcal{K}_{K+x}} \Psi_l - x)\ \cap (\bigcup_{l:\Psi_l \in \mathcal{K}_{K+x}\setminus B} \Psi_l - x)\ \cap K\big)
\end{aligned}$$

Thus, we have $Y(x)\mathbb{I}(\Psi(\mathcal{K}^*)=0) = Y_{\mathcal{K}_{K+x}\setminus\mathcal{K}^*}(x)\mathbb{I}(\Psi(\mathcal{K}^*)=0)$ almost surely and obtain the following equations

$$Y(o)Y(x_1)\cdot \mathbb{I}_{A^c} = Y_{\mathcal{K}_K\setminus\mathcal{K}^*}(o)Y_{\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*}(x_1)\cdot \mathbb{I}_{A^c}$$

and

$$Y(y)Y(y+x_2)\cdot \mathbb{I}_{A^c} = Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2)\cdot \mathbb{I}_{A^c}.$$

The random variables $Y_{\mathcal{K}_K\setminus\mathcal{K}^*}(o)Y_{\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*}(x_1)\cdot \mathbb{I}_{A^c}$ and $Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2)$ are independent, since the particles Ψ_l involved are mutually independent. As a result we get

$$\begin{aligned}
&\mathbb{E}\big(Y(o)Y(x_1)\mathbb{I}_{A^c}\cdot\big[Y(y)Y(y+x_2)-\mathbb{E}\,(Y(y)Y(y+x_2))\big]\big)\\
&= \mathbb{E}(Y_{\mathcal{K}_K\setminus\mathcal{K}^*}(o)Y_{\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*}(x_1)\,\mathbb{I}_{A^c})\cdot\big[\mathbb{E}(Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2))-\mathbb{E}\,(Y(y)Y(y+x_2))\big]\big)\\
&= \mathbb{E}(Y(o)Y(x_1)\,\mathbb{I}_{A^c})\cdot\mathbb{E}\big([Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2)-Y(y)Y(y+x_2)]\,\mathbb{I}_A\big).
\end{aligned}$$

Insert the above equations into (30) to obtain by triangle inequality

$$\begin{aligned}
&\big|\mathrm{Cov}\big(Y(o)Y(x_1),Y(y)Y(x_2+y)\big)\big|\\
&\le \big|\mathbb{E}\big(Y(o)Y(x_1)\,\mathbb{I}_A\cdot\big[Y(y)Y(y+x_2)-\mathbb{E}\,(Y(y)Y(y+x_2))\big]\big)\big|\\
&\quad+\mathbb{E}\big|Y(o)Y(x_1)\big|\cdot\big|\mathbb{E}\big([Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2)-Y(y)Y(y+x_2)]\mathbb{I}_A\big)\big|.
\end{aligned}$$

For the first summand in the upper bound we can conclude that

$$\begin{aligned}
&\big|\mathbb{E}\big(Y(o)Y(x_1)\,\mathbb{I}_A\cdot\big[Y(y)Y(y+x_2)-\mathbb{E}\,(Y(y)Y(y+x_2))\big]\big)\big|\\
&\le c^2(K)\,\mathbb{E}\big(2^{\Psi(\mathcal{K}_K)+\Psi(\mathcal{K}_{K+x_1})}\cdot[c^2(K)\,2^{\Psi(\mathcal{K}_{K+y})+\Psi(\mathcal{K}_{K+y+x_2})}+\mathbb{E}\,|Y(y)Y(y+x_2)|\,]\cdot\mathbb{I}_A\big).
\end{aligned}$$

Since Ψ is a Poisson process, the random variables $\Psi(B)$ and $\Psi(B')$ are independent for any two disjount sets $B,B'\subseteq\mathcal{K}$, whence

$$\begin{aligned}
&\mathbb{E}\big(2^{\Psi(\mathcal{K}_K)+\Psi(\mathcal{K}_{K+x_1})+\Psi(\mathcal{K}_{K+y})+\Psi(\mathcal{K}_{K+y+x_2})}\cdot\mathbb{I}_A\big)\\
&= \mathbb{E}\,2^{\Psi(\mathcal{K}_K\setminus\mathcal{K}^*)+\Psi(\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*)}\cdot\mathbb{E}\,2^{\Psi(\mathcal{K}_{K+y}\setminus\mathcal{K}^*)+\Psi(\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*)}\cdot\mathbb{E}\big(2^{4\Psi(\mathcal{K}^*)}\,\mathbb{I}_A\big)
\end{aligned}$$

and

$$\mathbb{E}\left(2^{\Psi(\mathcal{K}_K)+\Psi(\mathcal{K}_{K+x_1})}\cdot\mathbb{I}_A\right) = \mathbb{E}\,2^{\Psi(\mathcal{K}_K\setminus\mathcal{K}^*)+\Psi(\mathcal{K}_{K+x_1}\setminus\mathcal{K}^*)}\cdot\mathbb{E}\left(2^{2\,\Psi(\mathcal{K}^*)}\,\mathbb{I}_A\right).$$

For the second summand of the considered upper bound, we analogously obtain

$$\begin{aligned}
&\big|\mathbb{E}\big(\big[Y(y)Y(y+x_2)-Y_{\mathcal{K}_{K+y}\setminus\mathcal{K}^*}(y)Y_{\mathcal{K}_{K+y+x_2}\setminus\mathcal{K}^*}(y+x_2)\big]\,\mathbb{I}_A\big)\big|\\
&\le c^2(K)\,\mathbb{E}\,2^{\Psi(\mathcal{K}_{K+y})+\Psi(\mathcal{K}_{K+y+x_2})}\,\mathbb{E}\big([2^{2\Psi(\mathcal{K}^*)}+1]\,\mathbb{I}_A\big).
\end{aligned}$$

Due to the fact that $c(K)<\infty$, $\mathbb{E}\,|Y(x)Y(y)|<\infty$ and $\mathbb{E}\,2^{\Psi(\mathcal{K}_{K+x}\setminus\mathcal{K}^*)+\Psi(\mathcal{K}_{K+y}\setminus\mathcal{K}^*)}\le g_{\Psi(\mathcal{K}_K)}(4)<\infty$ for any $x,y\in\mathbb{R}^d$, we may concentrate on $\mathbb{E}\,(s^{\Psi(\mathcal{K}^*)}\,\mathbb{I}_A)$, say, for arbitrary $s\in\mathbb{R}_+$. This quantity, however, is bounded by

$$\begin{aligned}
\mathbb{E}\,(s^{\Psi(\mathcal{K}^*)}\mathbb{I}_A) &= \mathbb{E}\,(s^{\Psi(\mathcal{K}^*)})-\mathbb{E}\,(\mathbb{I}_{A^c}) = e^{(s-1)\Lambda(\mathcal{K}^*)}-e^{-\Lambda(\mathcal{K}^*)}\\
&\le s\,e^{(s-1)\Lambda(\mathcal{K}^*)}\Lambda(\mathcal{K}^*) \le s\,e^{(s-1)\lambda\,\mathbb{E}\,|M_0\oplus\check{K}|}\Lambda(\mathcal{K}^*)
\end{aligned}$$

using that $1 - e^{-x} \leq x$ for any $x \in \mathbb{R}_+$ and $\Lambda(\mathcal{K}^*) \leq \Lambda(\mathcal{K}_K) + \Lambda(\mathcal{K}_{K+x_1}) = 2\lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|$ by (21). It remains to show that $\Lambda(\mathcal{K}^*) = \Lambda(\mathcal{K}^*(x_1, x_2, y))$ is integrable with respect to $y \in \mathbb{R}^d$ and that the integral admits an upper bound uniformly in $x_1, x_2 \in \mathbb{R}^d$. Employing equation (21) and Fubini's theorem we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Lambda(\mathcal{K}^*(x_1, x_2, y))\, dy \\
&\quad = \lambda\,\mathbb{E} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}\big[z \in \check{M}_0 \oplus K \cup (\check{M}_0 \oplus (K + x_1))\big]\,\mathbb{1}\big[z - y \in (\check{M}_0 \oplus K) \cup (\check{M}_0 \oplus (K + x_2))\big]\, dy\, dz \\
&\quad \leq \lambda\,\mathbb{E} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(\mathbb{1}[z \in \check{M}_0 \oplus K] + \mathbb{1}[z - x_1 \in \check{M} \oplus K]\big) \\
&\qquad\qquad \times\ \big(\mathbb{1}[z - y \in \check{M}_0 \oplus K] + \mathbb{1}[z - y \in \check{M}_0 \oplus (K + x_2)]\big)\, dy\, dz \\
&\quad = 4\lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|^2 < \infty.
\end{aligned}$$

for arbitrary $x_1, x_2 \in \mathbb{R}^d$. Combining all estimates, we conclude that there exists a constant $\kappa_1 = \kappa_1(M_0, \lambda, K)$, which depends on M_0 and K through $|M_0 \oplus \check{K}|$ and $c(K)$, such that for all $x_1, x_2 \in \mathbb{R}^d$ it holds $\int_{\mathbb{R}^d} |\mathrm{Cov}(Y(o)Y(x_1), Y(y)Y(x_2 + y))| \leq \kappa_1$.

The second assertion of Theorem 2 can be shown following the same idea. Let $\mathcal{K}^{**} = \mathcal{K}^{**}(x_1, x_2, y) = \mathcal{K}_{K+y} \cap (\mathcal{K}_K \cup \mathcal{K}_{K+x_1} \cup \mathcal{K}_{K+x_2})$ for $x_1, x_2, y \in \mathbb{R}^d$ and consider the event $\bar{A} = \{\Psi(\mathcal{K}^{**}) > 0\}$. Then, by the same arguments as before, it follows that

$$\begin{aligned}
\big|\mathbb{E}\big([Y(o) - \mu]\,[Y(y) - \mu]\,Y(x_1)\,Y(x_2)\big)\big| &\leq \big|\mathbb{E}\big([Y(o) - \mu]\,[Y(y) - \mu]\,Y(x_1)\,Y(x_2)\,\mathbb{1}_{\bar{A}}\big)\big| \\
&\quad + \mathbb{E}\big|[Y(o) - \mu]\,Y(x_1)\,Y(x_2)\big|\big|\mathbb{E}\big([Y_{\mathcal{K}_{K+y}}(y) - Y(y)]\mathbb{1}_{\bar{A}}\big)\big| \\
&\leq \kappa(M_0, \lambda, K) \cdot \Lambda(\mathcal{K}^{**}(x_1, x_2, y))
\end{aligned}$$

for some finite bound $\kappa(M_0, \lambda, K)$ depending on K and M_0 through $c(K)$ and $\mathbb{E}\,|M_0 \oplus \check{K}|$ only. Furthermore, we have for any $x_1, x_2 \in \mathbb{R}^d$ that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Lambda(\mathcal{K}^{**}(x_1, x_2, y))\, dy \\
&\quad = \lambda\,\mathbb{E} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}\big[y - z \in M_0 \oplus \check{K}\big]\,\mathbb{1}\big[z \in (\check{M}_0 \oplus K) \cup (\check{M}_0 \oplus (K + x_1)) \cup (\check{M}_0 \oplus (K + x_2))\big]\, dy\, dz \\
&\quad \leq 3\lambda\,\mathbb{E}\,|M_0 \oplus \check{K}|^2 < \infty.
\end{aligned}$$

□

Table 1: Average values for $\widehat{\sigma}_{nij}$ (out of 200 runs) on different observation windows W_n with various choices for U_n.

$\|W_n\|$	$\|U_n\|$		$\widehat{\sigma}_{00}$	$\widehat{\sigma}_{01}$	$\widehat{\sigma}_{11}$
10000	100	mean	**62.72**	**52.13**	**62.61**
		bias	-37.28	-47.87	-37.39
		std	8.67	8.23	9.26
10000	200	mean	**83.63**	**71.72**	**83.21**
		bias	-16.37	-28.28	-16.80
		std	15.84	14.89	16.11
10000	400	mean	**95.77**	**82.89**	**93.73**
		bias	-4.23	-17.11	-6.27
		std	27.35	26.05	26.90
10000	800	mean	**92.67**	**80.47**	**90.31**
		bias	-7.33	-19.53	-9.69
		std	37.60	34.08	36.06
40000	100	mean	**62.69**	**51.88**	**62.45**
		bias	-37.31	-48.13	-37.55
		std	4.83	4.56	4.86
40000	200	mean	**85.83**	**73.91**	**85.92**
		bias	-14.17	-26.09	-14.08
		std	7.92	7.45	8.07
40000	400	mean	**96.53**	**84.68**	**97.43**
		bias	-3.47	-15.32	-2.57
		std	13.05	12.22	12.75
40000	800	mean	**98.68**	**87.01**	**100.11**
		bias	-1.32	-12.99	0.11
		std	19.39	17.11	18.00

Table 2: Estimation parameters for random fields related to Boolean models. Here W is the simulation window, Δ is the grid mesh size for integral approximations; $n_{\max}$ is the maximal number of available data points in W (subwindows V_k, respectively), which is the ratio of the total number of pixels in the window to Δ; N is the number of non–overlapping subwindows for the estimator $\widetilde{\Sigma}$; $\sharp V_k$ is the number of pixels in each V_k.

Estimator	W	Δ	$n_{\max}$	N	$\sharp V_k$
$\widehat{\Sigma}$	$[-1500, 1500]^2$	10	300^2	——	——
$\widetilde{\Sigma}$	$[-1500, 1500]^2$	10	100^2	9	1000×1000
$\widetilde{\Sigma}$	$[-1500, 1500]^2$	5	100^2	36	500×500

Table 3: Empirical co-/variances of 1000 independent samples. Reference values for the unknown asymptotic covariance matrix.

	$\overline{c}_{0j}$	$\overline{c}_{1j}$	$\overline{c}_{2j}$
$\overline{c}_{i0}$ std	**1.65e-4** 0.29e-4	**-7.48e-4** 4.51e-4	**-0.12** 0.04
$\overline{c}_{i1}$ std		**0.11** 0.02	**5.81** 1.18
$\overline{c}_{i2}$ std			**676.68** 119.59

Table 4: Average value for $\widehat{C}$ (out of 200 runs) on observation window $W = [-1500, 1500]^2$. Average estimate of the variance of the area density $\widehat{\sigma}_{pp} \approx 678.93$ with relative standard deviation $\delta \approx 4.70\,\%$.

	$\widehat{c}_{0j}$	$\widehat{c}_{1j}$	$\widehat{c}_{2j}$
$\widehat{c}_{i0}$ std	**1.75e-4** 0.30e-4	**-1.27e-3** 0.52e-3	**-0.17** 0.03
$\widehat{c}_{i1}$ std		**0.19** 0.02	**7.09** 0.73
$\widehat{c}_{i2}$ std			**688.13** 34.90

Table 5: Average value for $\widetilde{C}$ (out of 200 runs) on observation windows $V_k = [-500, 500)^2 + h_k$, $h_k = (k_1\,1000, k_2\,1000)^\top$, $k = (k_1, k_2), k_1, k_2 = -1, 0, 1$. Average estimate of the variance of the area density $\widetilde{\sigma}_{pp} \approx 648.06$ with relative standard deviation $\delta \approx 51.38\,\%$.

	$\widetilde{c}_{0j}$	$\widetilde{c}_{1j}$	$\widetilde{c}_{2j}$
$\widetilde{c}_{i0}$	**1.75e-4**	**-1.34e-3**	**-0.18**
std	0.32e-4	0.63e-3	0.03
$\widetilde{c}_{i1}$		**0.19**	**6.95**
std		0.02	0.82
$\widetilde{c}_{i2}$			**691.33**
std			39.68

Table 6: Average value for $\widetilde{C}$ (out of 200 runs) on observation windows $V_k = [-250, 250)^2 + h_k$, $h_k = (k_1\,500, k_2\,500)^\top$, $k = (k_1, k_2), k_1, k_2 = -2, -1, 0, 1, 2$. Average estimate of the variance of the area density $\widetilde{\sigma}_{pp} \approx 651.15$ with relative standard deviation $\delta \approx 23.95\,\%$.

	$\widetilde{c}_{0j}$	$\widetilde{c}_{1j}$	$\widetilde{c}_{2j}$
$\widetilde{c}_{i0}$	**1.42e-4**	**-9.20e-4**	**-0.18**
std	0.31e-4	7.25e-4	0.03
$\widetilde{c}_{i1}$		**0.17**	**6.68**
std		0.02	0.89
$\widetilde{c}_{i2}$			**675.10**
std			46.03