Anisotropic dilated Poisson k-flat processes

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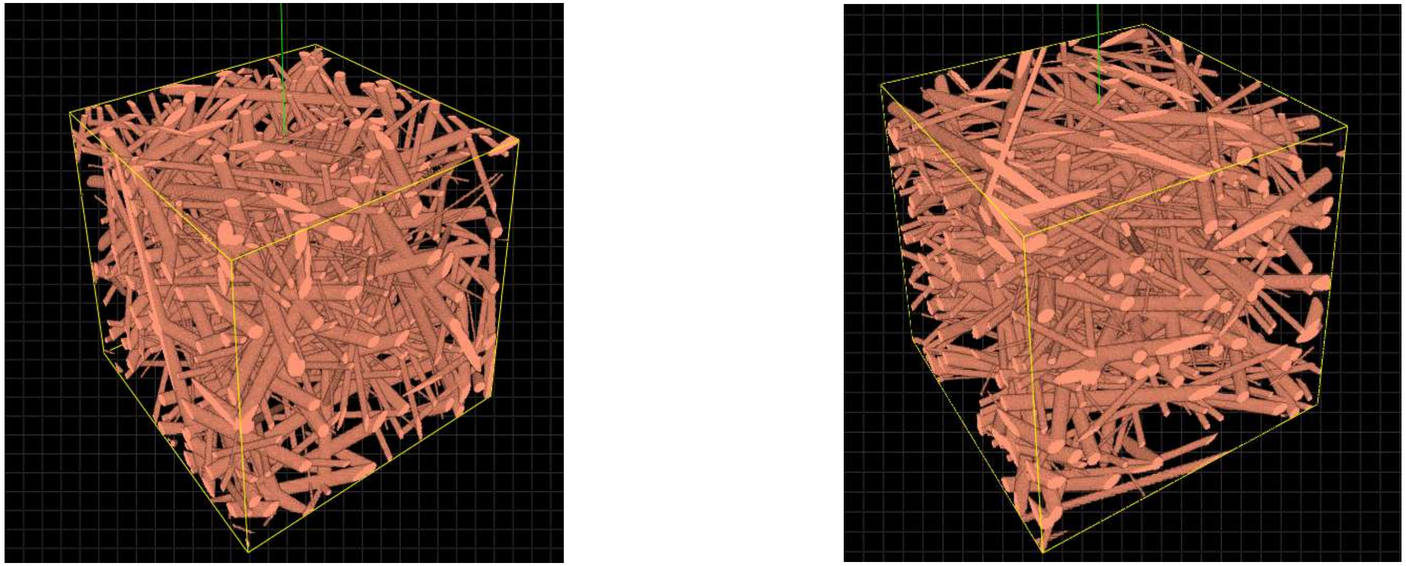
Modelling the structure of materials

- Optimizing the acoustic absorbtion properties of pressed non–woven



Before pressing: isotropy

After pressing: anisotropy

Images are made using MAVI 1.0. Source: K. Schladitz et al. (2005) “Design of acoustic trim based on geometric modeling and flow simulation for non-woven”. Preprint.

Modelling approach

- Poisson processes of thick flats (cylinders, lamellae, membranes)
  - Matheron (1975)
  - Davy (1978)
  - Schneider (1987)
  - Weil (1987)
  - Ohser, Mücklich (2000)
  - Michel, Paroux (2002)
- Explicit formulae mostly for the isotropic processes
- Anisotropic case?

Overview

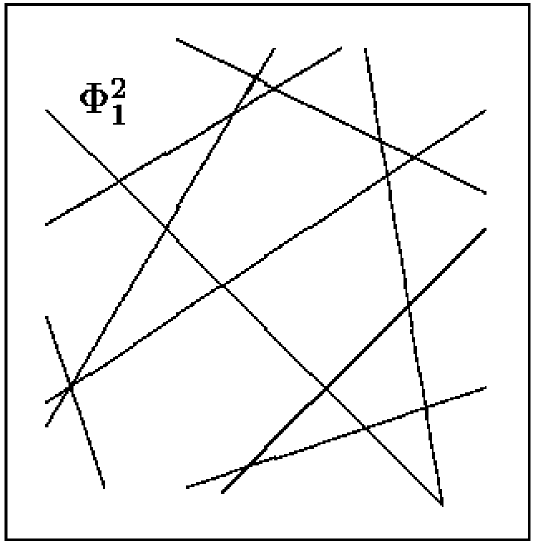
- Processes of k–dimensional flats
- Stationary anisotropic Poisson processes of cylinders
- Capacity functional and related characteristics
    - Covariance function
    - Contact distribution
- Specific intrinsic volumes
    - Volume fraction
    - Specific surface area
- Outlook

Preliminaries

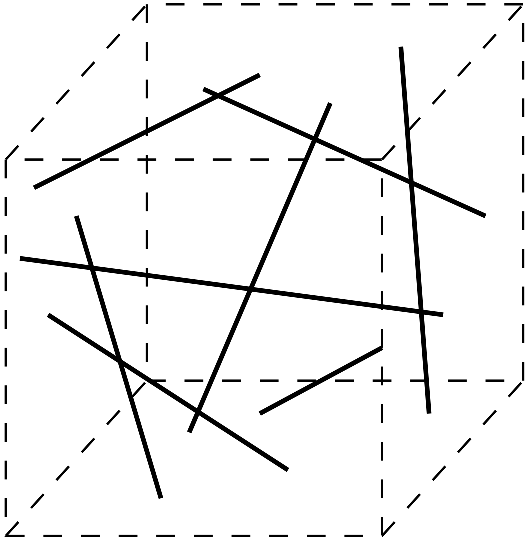
$\mathcal{K}$	family of all compact convex sets (bodies) in $\mathbb{R}^d$
$\mathcal{R}$	$= \{\bigcup_{i=1}^{n} K_i : K_i \in \mathcal{K},\ i = 1, \dots, n,\ \forall n\}$ convex ring
$F(k,d)$	the set of affine k–flats in $\mathbb{R}^d$
$G(k,d)$	the set of linear k–subspaces in $\mathbb{R}^d$ (Grassmannian)
$\mathfrak{F},\ \mathfrak{G}$	Borel σ–algebras of $F(k,d)$, $G(k,d)$
$B_r(a)$	ball with center in a and radius r
$\mathbf{S}^{d-1}$	unit sphere in $\mathbb{R}^d$
$\omega_j(k_j)$	surface area (volume) of $B_1(o)$ in $\mathbb{R}^j$, $j = 0, \dots, d$
$K_1 \oplus K_2$	$= \{x_1 + x_2 : x_1 \in K_1,\ x_2 \in K_2\}$ Minkowski addition

Processes of k–flats

A process Φ_k^d of k–dimensional flats in $\mathbb{R}^d$ is a random element $(\Omega, \Gamma, P) \to (\mathcal{M}, \mathfrak{M})$, where $\mathcal{M}$ is the set of all at most countable "locally finite" systems of k–flats.

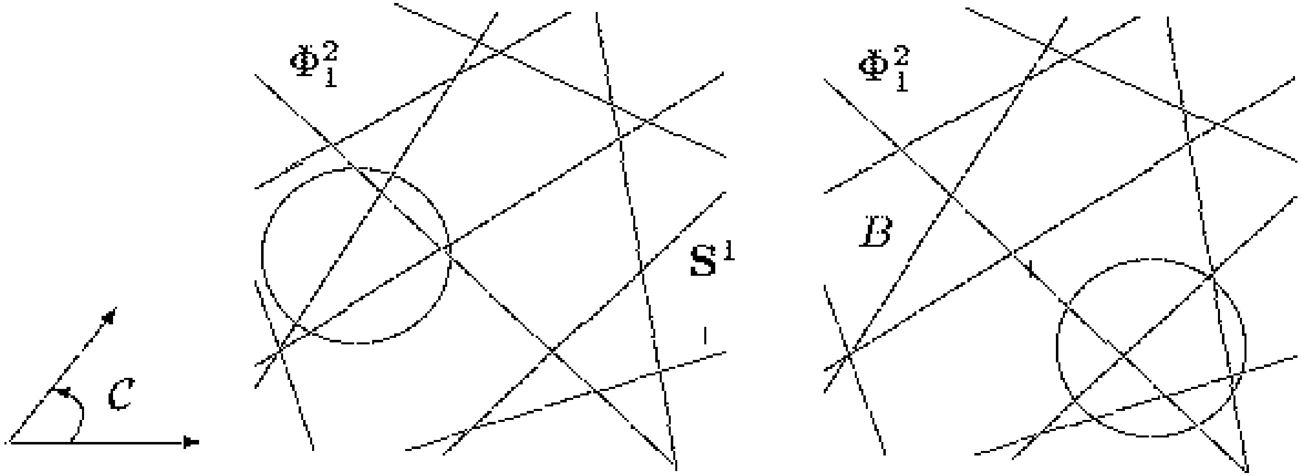


Φ_1^2



Φ_1^3

Processes of k–flats



- Stationarity: $P\{\Phi_k^d \in \cdot\} = P\{x + \Phi_k^d \in \cdot\}$ for all $x \in \mathbb{R}^d$
- Intensity: $\lambda = \frac{E\nu_k(\Phi_k^d \cap B)}{\nu_d(B)}$ for all bounded $B \subset \mathbb{R}^d$, where $\nu_k(\cdot)$ is the Lebesgue measure in $\mathbb{R}^k$
- Directional distribution: $\theta(\mathcal{C}) = \frac{E|\{\xi \in \Phi_k^d \colon \xi \cap \mathbf{S}^{d-1} \neq \emptyset,\, r(\xi) \in \mathcal{C}\}|}{\lambda k_{d-k}}$, $\mathcal{C} \in \mathfrak{G}$, where $r(\xi) \in G(k,d)$, $r(\xi) \parallel \xi$ is the direction of $\xi \in F(k,d)$

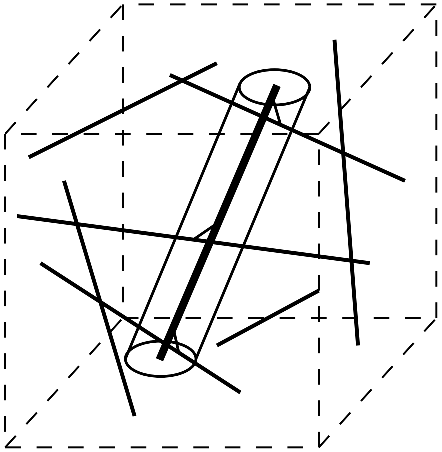
Stationary Poisson processes of k–flats

- Intensity measure: $\Lambda(\mathcal{B}) = E\Phi_k^d(\mathcal{B})$ for any Borel set $\mathcal{B} \in \mathfrak{F}$. It holds:

  $$\Lambda(\mathcal{B}) = \lambda \int\limits_{G(k,d)} \int\limits_{\xi^\perp} I_{\mathcal{B}}(y+\xi)\, \nu_{d-k}^{\xi^\perp}(dy)\, \theta(d\xi)\,,$$

  where $\nu_{d-k}^{\xi^\perp}(\cdot)$ is the Lebesgue measure on $\xi^\perp$.
- Isotropy: $\theta(d\xi) = d\xi$, the unique probability measure on $G(k,d)$ that is invariant with respect to rotations (Haar measure).
- Poisson processes: The stationary process Φ_k^d is Poisson if $\Phi_k^d(\mathcal{B}) \sim$ Poisson $(\Lambda(\mathcal{B}))$ for all $\mathcal{B} \in \mathfrak{F}$ with $\Lambda(\mathcal{B}) < \infty$.

Random closed sets of dilated k–flats



Introduce the stationary RACS $\Xi = \bigcup_{i=1}^{\infty} (\xi_i \oplus \Xi_i)$, where

- $\Phi_k^d = \{\xi_1, \xi_2, \xi_3, \ldots\}$ is a stationary Poisson process with intensity λ and directional distribution θ
- $\Xi_1, \Xi_2, \Xi_3, \ldots$ is a sequence of iid random closed sets, $\Xi_i \overset{d}{=} \Xi_0 \in \mathcal{K}$, $o \in \Xi_0$, int$(\Xi_0) \neq \emptyset$ a.s.

Capacity functional

- Cross section of a cylinder: $\Xi_0(\xi) = (\xi \oplus \Xi_0) \cap \xi^\perp$ for any $\xi \in F(k,d)$
- Capacity functional: $T_\Xi(C) = P(\Xi \cap C \neq \emptyset)$ for any compact set C. It holds

$$T_\Xi(C) = 1 - \text{exp}\left\{ -\lambda \int\limits_{G(k,d)} E\,\nu_{d-k}\left(\check{\Xi}_0(\xi) \oplus \text{Pr}_{\xi^\perp}(C)\right)\,\theta(d\xi)\right\},$$

  where $\check{K} = -K$, $\text{Pr}_{\xi^\perp}$ is the orthogonal projection onto $\xi^\perp$.
- The distribution of Ξ depends only on $\Xi_0(\xi)$, $\xi \in G(k,d)$!

Covariance function

$$C_\Xi(h) = P(o, h \in \Xi) - P^2(o \in \Xi) = 1 - T_\Xi(\{o, h\}) - (1 - T_\Xi(\{o\}))^2$$

$$C_\Xi(h) = e^{-2\lambda \int\limits_{G(k,d)} E\,\nu_{d-k}(\Xi_0(\xi))\,\theta(d\xi)} \times \left(e^{\lambda \int\limits_{G(k,d)} E\,\nu_{d-k}\left(\Xi_0(\xi)\cap(\Xi_0(\xi)-\mathrm{Pr}_{\xi^\perp} h)\right)\theta(d\xi)} - 1 \right), \quad h \in \mathbb{R}^d.$$

- Example (Isotropic dilated Poisson line process in $\mathbb{R}^2$):
  $d = 2$, $k = 1$, $\Xi_0 = B_a(o)$, $\theta(d\varphi) = d\varphi/\pi$, $\varphi \in [0, \pi)$.

$$C_\Xi(h) = \begin{cases} e^{-4\lambda a}\left(e^{2\lambda(a-|h|/\pi)} - 1\right), & |h| \leqslant 2a, \\ e^{-4\lambda a}\left(e^{2\lambda a-\lambda/\pi\left(4a\arccos(2a/|h|)+2|h|(1-\sqrt{1-4a^2/|h|^2})\right)} - 1\right), & |h| > 2a. \end{cases}$$

Contact distribution function

For any compact set $C \subset \mathbb{R}^d$, $o \in C$, $r > 0$, introduce

$$H_C(r) = P(\Xi \cap rC \neq \emptyset | o \notin \Xi) = \frac{T_\Xi(rC) - T_\Xi(\{o\})}{1 - T_\Xi(\{o\})}$$

In our case:

$$H_C(r) = 1 - e^{-\lambda \int\limits_{G(k,d)} E\left[\nu_{d-k}\left(\check{\Xi}_0(\xi) \oplus \mathrm{Pr}_{\xi^\perp}(rC)\right) - \nu_{d-k}\left(\Xi_0(\xi)\right)\right] \theta(d\xi)}$$

Contact distribution function

- Linear contact distribution function: $C = h \in \mathbf{S}^{d-1}$.

$$H_C(r) = 1 - e^{-\lambda c_0 r},$$

where

- $c_0 = \frac{(d-k+1)\,\omega_{d-k+1}}{2\pi(d-k)\,\omega_{d-k}} \int\limits_{G(k,d)} E\, S_{d-k-1}\big(\Xi_0(\xi)\big)\,[h,\xi]\,\theta(d\xi),$
- $S_{d-k-1}\big(\Xi_0(\xi)\big)$ is the surface area of $\Xi_0(\xi)$ in $\xi^\perp$ and
- $[h,\xi]$ is the volume of the parallelepiped spanned by the vector h and an orthonormal basis in ξ.

Contact distribution function

- Example (Anisotropic dilated Poisson line process in $\mathbb{R}^3$):

  $d = 3$, $k = 1$, $\Xi_0 = B_a(o)$, $\theta(\cdot) = \omega_3(\cdot)/\omega_3$

  $$H_C(r) = 1 - e^{-2aT_{22}\theta(h^\perp)r},$$

  where

  - $T_{22}\theta(h^\perp) = \lambda \int\limits_{G(1,3)} |\sin \angle(h,\xi)|\, \theta(d\xi)$ is the rose of intersections of a stationary Poisson process of hyperplanes Φ_2^3 in $\mathbb{R}^3$ with the hyperplane $h^\perp$.
  - Φ_2^3 has intensity λ and directional distribution $\theta^\perp$ given by $\theta^\perp(d\eta) = \theta(d\eta^\perp)$, $\eta \in G(2,3)$.

Contact distribution function

- Spherical contact distribution function: $C = B_1(o)$.

$$H_C(r) = 1 - e^{-\lambda \sum_{j=0}^{d-k-1} k_{d-k-j} r^{d-k-j} \int_{G(k,d)} E\, V_j^{(d-k)}\left(\Xi_0(\xi)\right) \theta(d\xi)},$$

where $V_j^{(d-k)}(\Xi_0(\xi))$, $j = 0, \ldots, d-k-1$ are the intrinsic volumes of the $(d-k)$–dimensional RACS $\Xi_0(\xi)$.

- Example (Anisotropic dilated Poisson line process in $\mathbb{R}^3$):
$d = 3$, $k = 1$, $\Xi_0 = B_a(o)$,

$$H_C(r) = 1 - e^{-\lambda \pi r (r+2a)}$$

Specific intrinsic volumes

- Specific intrinsic volumes: $\overline{V}_j(\Xi) = \lim_{n\to\infty} \frac{E\,V_j(\Xi \cap W_n)}{\nu_d(W_n)}$

  for $j = 0, \ldots, d$, where $\{W_n\}$ is a sequence of monotonously increasing sampling windows $W_n = nW$ with $W \in \mathcal{K}$ and $\nu_d(W) > 0$

- Volume fraction: $p_\Xi = \overline{V}_d(\Xi) = P(o \in \Xi) = T_\Xi(\{o\})$.

  In our case:

$$p_\Xi = 1 - \text{exp}\left\{ -\lambda \int\limits_{G(k,d)} E\,\nu_{d-k}\left(\Xi_0(\xi)\right)\,\theta(d\xi) \right\}.$$

Specific intrinsic volumes

- Specific surface area: $S_\Xi = 2\overline{V}_{d-1}(\Xi)$. In our case:

$$S_\Xi = -\lambda(1-p_\Xi)\frac{d\,\omega_d}{\omega_{d-1}} \int\limits_{G(k,d)} \int\limits_{G(1,d)} E\left(\gamma'_{\Xi_0(\xi),\,Pr_{\xi^\perp}\zeta}(o)\right)[\xi,\zeta]\,d\zeta\,\theta(d\xi),$$

where

- $\gamma_{\Xi_0(\xi)}(h) = \nu_{d-k}\left(\Xi_0(\xi) \cap (\Xi_0(\xi) - h)\right)$, $h \in \xi^\perp$ is the set covariogram of $\Xi_0(\xi)$ in direction h and
- $\gamma'_{\Xi_0(\xi),\,Pr_{\xi^\perp}\zeta}(o)$ is the derivative of the above set covariogram at zero in direction $Pr_{\xi^\perp}\zeta$.

Specific intrinsic volumes

- Anisotropic dilated Poisson process of hyperplanes in $\mathbb{R}^d$:

$\forall\, d$, $k = d-1$, arbitrary $\Xi_0 \in \mathcal{K} \Longrightarrow \gamma'_{\Xi_0(\xi),\, Pr_{\xi^\perp}\zeta}(o) = -1$ and

$$S_\Xi = 2\lambda \exp\left\{ -\lambda \int\limits_{G(1,d)} E\, b_\xi(\Xi_0)\, \theta^\perp(d\xi) \right\} ,$$

where $b_\xi(\Xi_0)$ is the breadth of Ξ_0 in direction ξ.

Outlook

- Other specific intrinsic volumes $\overline{V}_j(\Xi)$, $j \leqslant d-2$
- More general dilation sets Ξ_i
  - polyconvex sets $\Xi_i \in \mathcal{R}$ a.s.
  - lower dimensional sets

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