The number of integer-valued vectors in the interior of a polytope

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1 Introduction

Let $P = |X|$ be a d-dimensional (convex) polytope in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$, let

$$L(X, n) = \sharp\{(n \cdot |X|) \cap \mathbb{Z}^d\}$$

be the number of integer-valued vectors in $n \cdot |X|$ and let

$$L(X - \partial X, n) = \sharp\{int(n \cdot |X|) \cap \mathbb{Z}^d\}$$

be the number of integer-valued vectors in the interior of $n \cdot |X|$, where $n \in \mathbb{N}$.

It was shown that there exists a polynomial f_X of degree d, such that

$$L(X, n) = f_X(n).$$

We will now show that

$$L(X - \partial X, n) = (-1)^d \cdot f_X(-n).$$

Definition 1.1. *A set of* $p+1$ *points in* $\mathbb{R}^d$, $\{a_0, a_1, ..., a_p\}$, *is said to be* (affine) independent*, if for real coefficients* $\lambda_0, ..., \lambda_p$

$$\left(\sum_{i=0}^{p} \lambda_i a_i = 0 \text{ and } \sum_{i=0}^{p} \lambda_i = 0\right) \Rightarrow \left(\lambda_i = 0\ \forall i \in \{0, ..., p\}\right).$$

Remark 1.2. *A single point is always independent.*
If $p + 1 \geq 2$*, then* $p + 1$ *points are independent if and only if they do not lie in an affine subspace of dimension* $\leq p - 1$.

Definition 1.3. *Let* $\{a_0, a_1, ..., a_p\}$ *be an indenpendent set of* $p+1$ *points in* $\mathbb{R}^d$*. The* (open) p-simplex σ *with vertices* $a_0, a_1, ..., a_p$ *is given by*

$$\sigma = (a_0, a_1, ..., a_p) = \{\sum_{i=0}^{p} \lambda_i a_i \mid \sum_{i=0}^{p} \lambda_i = 1 \text{ and } \lambda_i > 0\ \forall i \in \{0, ..., p\}\ \}.$$

Definition 1.4. *A simplex* τ *is a* face *of the simplex* $\sigma = (a_0, a_1, ..., a_p)$ *if the set of vertices of* τ*,* $vert\, \tau$*, is a subset of* $vert\, \sigma = \{a_0, a_1, ..., a_p\}$*. In this case we write* $\tau \preceq \sigma$.

Definition 1.5. *A* finite simplicial complex X *is a finite collection of simplexes, such that*

1. $(\sigma \in X \text{ and } \tau \preceq \sigma) \Rightarrow \tau \in X$ *(X is "closed") and*
2. $(\sigma_1, \sigma_2 \in X \text{ and } \sigma_1 \neq \sigma_2) \Rightarrow \sigma_1 \cap \sigma_2 = \emptyset$ *(distinct simplexes of X are disjoint).*

Definition 1.6. *For a collection of simplexes* X *we write* $|X|$ *for the underlying space of* X*, i.e.*

$$|X| = \bigcup_{\sigma \in X} \sigma.$$

If X *is a simplicial complex,* $|X|$ *is called* polyhedron*.*

Proposition 1.7. *Every polytope* P *is a polyhedron. Moreover, for every polytope* P *exists a simplicial complex* X *such that* $|X| = P$ *and* $vert\, X = vert\, P$.

Proof. Constructively, using induction on the dimension of the polytope P. □

Definition 1.8. *The topological closure of the simplex* $\sigma = (a_0, a_1, ..., a_p)$*,* $\overline{\sigma}$*, is called* closed p-simplex*.*

$$\overline{\sigma} = \{\sum_{i=0}^{p} \lambda_i a_i \mid \sum_{i=0}^{p} \lambda_i = 1 \text{ and } \lambda_i \geq 0\ \forall i \in \{0, ..., p\}\}.$$

In an abuse of notation we shall sometimes write $\overline{\sigma}$ *for the simplicial complex whose polyhedron is* $\overline{\sigma}$*, i.e.*

$$\overline{\sigma} = \{\tau \mid \tau \preceq \sigma\}.$$

2 The main theorem for a simplex

Theorem 2.1. *Let σ be a simplex in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$. Then for $n \in \mathbb{N}$:*

$$L(\sigma, n) = (-1)^{dim\,\sigma} L(\overline{\sigma}, -n).$$

Proof. Without loss of generality $dim\,\sigma = d$.
(If $dim\,\sigma < d$, then the affine hull spanned by σ intersected with $\mathbb{Z}^d$ is an affine sublattice of $\mathbb{Z}^d$ whose underlying sublattice is $\mathbb{Z}$-generated by $dim\,\sigma$ linearly independent vectors. We can look at this lattice in the corresponding subspace of dimension $dim\,\sigma$.)
Let $e_1, ..., e_d$ a basis of $\mathbb{Z}^d$. See $\mathbb{R}^d$ as a subspace of $\mathbb{R}^{d+1}$ and let $e_0, e_1, ..., e_d$ a basis of $\mathbb{Z}^{d+1}$.
Let $\sigma = (u_0, ..., u_d)$, $v_i = e_0 + u_i$ $(0 \leq i \leq d)$ and $\sigma' = (v_0, ..., v_d)$.
Let $M = \mathbb{Z}v_0 + ... + \mathbb{Z}v_d$ be the sublattice of $\mathbb{Z}^{d+1}$ that is $\mathbb{Z}$-generated by $v_0, ..., v_d$.
Note that $v_0, ..., v_d$ is an $\mathbb{R}$-basis for $\mathbb{R}^{n+1}$.

$$\Gamma := \{x \in \mathbb{Z}^{d+1} \mid x = \sum_{i=0}^{d} \mu_i v_i \text{ with } 0 \leq \mu_i < 1\,\forall i\}$$

is a complete set of represesentatives for M in $\mathbb{Z}^{d+1}$. Especially the index $[\mathbb{Z}^{d+1} : M]$ is equal to $|\Gamma|$, the number of points in Γ.

$$\Gamma' = \{x \in \mathbb{Z}^{d+1} \mid x = \sum_{i=0}^{d} \mu_i' v_i \text{ with } 0 < \mu_i' \leq 1\,\forall i\}$$

is also a complete set of representatives for M.

$L(\overline{\sigma}, n)$ is equal to the number of points $y \in \mathbb{Z}^{d+1}$ that lie in $n\,\overline{\sigma'} = \overline{(n\,v_0, ..., n\,v_d)}$
Each point $y \in \mathbb{Z}^{d+1} \cap \overline{n\,\sigma'}$ is congruent mod M to exactly one point x of Γ, i.e. there exists integers $m_0, ..., m_d$, s.t.

$$y = x + \sum_{i=0}^{d} m_i\,v_i \tag{1}$$

Here $m_i \geq 0\ \forall i$, since provided $x = \sum_{i=0}^{d} \mu_i\,v_i$ with $\mu_i \in [0, 1)\ \forall i$ we have $x + \sum_{i=0}^{d} m_i\,v_i = \sum_{i=0}^{d} \frac{(\mu_i + m_i)}{n}\,n\,v_i \in n\,\overline{\sigma'}$ and therefore $\mu_i + m_i \geq 0\ \forall i$.

Comparing the e_0-coordinates of both sides of (1) gives

$$n = x_0 + \sum_{i=0}^{d} m_i \tag{2}$$

where x_0 is the e_0-coordinate of x.
So each point $y \in \mathbb{Z}^{d+1} \cap n\overline{\sigma'}$ gives rise to exactly one solution $(m_0, ..., m_d)' \in \mathbb{Z}^{d+1}_{\geq 0}$ of (2).
Viceversa, if $m_0, ..., m_d$ are non-negative integers that solve (2), then they give rise to a point $y \in \mathbb{Z}^{d+1} \cap n\overline{\sigma'}$.
So the number of those points $y \in \mathbb{Z}^{d+1} \cap n\overline{\sigma'}$ that are congruent to a fix $x \in \Gamma$ is equal to the number of solutions in $\mathbb{Z}^{d+1}_{\geq 0}$ of (2). This is the number of possibilities of adding $d + 1$ non-negative integers to x_0 to get n. This number is equal to the coefficient of u^n in

$$u^{x_0}(1 + u + u^2 + ...)^{d+1} = u^{x_0}(\sum_{k=0}^{\infty} \binom{k+d}{d} u^k).$$

So it is equal to $\binom{n+d-x_0}{d}$.
Hence

$$L(\overline{\sigma}, n) = \sum_{x\in\Gamma} \binom{n+d-x_0}{d}. \tag{3}$$

This is a polynomial in n of degree d.

Similarly, $L(\sigma, n)$ is equal to the number of points $y \in \mathbb{Z}^{d+1}$ that lie in $n\,\sigma' = (n\,v_0, ..., n\,v_d)$. Using now Γ' as set of representatives for M in $\mathbb{Z}^{d+1}$ we see that every $y \in \mathbb{Z}^{d+1} \cap n\,\sigma'$ has a unique representation

$$y = x' + \sum_{i=0}^{d} m_i\, v_i \tag{4}$$

with $x' \in \Gamma'$ and non-negative integers $m_0, ..., m_d$.
Comparing the e_0-coordinates of (4) gives now

$$n = x_0' + \sum_{i=0}^{d} m_i \tag{5}$$

where x_0' is the e_0-coordinate of x'.
Viceversa, non-negative integers $m_0, ..., m_d$ that solve (5) give rise to a point $y \in \mathbb{Z}^{d+1} \cap n\,\sigma'$. Hence this time

$$L(\sigma, n) = \sum_{x'\in\Gamma'} \binom{n+d-x_0'}{d}. \tag{6}$$

The mapping $\phi : \Gamma \to \Gamma'$ defined by $\phi(x) = v_0 + ... + v_d - x$ is bijective. The e_0-coordinate of $\phi(x)$ is $d+1-x_0$, where x_0 is again the e_o-coordinate of x. Therefore

$$L(\sigma, n) = \sum_{x\in\Gamma} \binom{n+d-(d+1-x_0)}{d} = \sum_{x\in\Gamma} \binom{n-1+x_0}{d}.$$

Finally

$$L(\sigma, -n) = \sum_{x\in\Gamma} \binom{-n-1+x_0}{d} = \sum_{x\in\Gamma} \frac{(-n-1+x_0)\cdot ... \cdot(-n-d+x_0)}{d!}$$

$$= \sum_{x\in\Gamma} (-1)^d \frac{(n+d-x_0)\cdot ... \cdot (n+1-x_0)}{d!} = (-1)^d \sum_{x\in\Gamma} \binom{n+d-x_0}{d} = (-1)^d L(\overline{\sigma}, n).$$

□

3 The main theorem

To generalize theorem 2.1 for poloytopes we need another tool:

Definition 3.1. *Let X be a simplicial dissection of a polytope P, i.e. X is a simplicial complex with $|X| = P$. We define the* boundary subcomplex *∂X of X as the collection of simplexes whose points are on the topological boundary of $P = |X|$ in the affine hull of P.*

Remark 3.2. *Let P be a d-dimensional polytope in $\mathbb{R}^d$ whith integer-valued vertices. Then there exists a simplicial complex X with integer-valued vertices such that $|X| = P$ (Proposition 1.7). For this complex we have:*

$$int\, P = |X - \partial X|\,.$$

Lemma 3.3. *Let P be a d-dimensional polytope and X be a simplicial complex with $|X| = P$. For every $\tau \in X$ we have*

$$\sum_{\sigma \succeq \tau} (-1)^{dim\,\sigma - dim\,\tau} = \begin{cases} (-1)^{d - dim\,\tau}, & if\ \tau \notin \partial X \\ 0, & if\ \tau \in \partial X \end{cases}$$

Proof. Since we need to know a good amount of Algebraic Topology in order to prove this, we skip the proof here. □

From now on let X be a simplicial complex whose underlying space $|X| = \bigcup_{\sigma \in X} \sigma$ is a d-dimensional polytope.
Let V be a real vector space and $\phi : X \to V$ be a function.
For any subset Y of X we define

$$S(Y, \phi) = \sum_{\sigma \in Y} (-1)^{1 + dim\,\sigma} \phi(\sigma). \tag{7}$$

And we define the function $\phi^* : X \to V$ by

$$\phi^*(\sigma) = S(\overline{\sigma}, \phi) = \sum_{\tau \preceq \sigma} (-1)^{1 + dim\,\tau} \phi(\tau). \tag{8}$$

Proposition 3.4.

$$S(X, \phi^*) = (-1)^{d+1} \cdot S(X - \partial X, \phi)$$

Proof.

$$S(X, \phi^*) = \sum_{\sigma \in X} (-1)^{1 + dim\,\sigma} \phi^*(\sigma) = \sum_{\sigma \in X} (-1)^{1 + dim\,\sigma} \sum_{\tau \preceq \sigma} (-1)^{1 + dim\,\tau} \phi(\tau)$$

$$= \sum_{\tau \in X} \phi(\tau) \sum_{\sigma \succeq \tau} (-1)^{dim\,\sigma - dim\,\tau} = \sum_{\tau \notin \partial X} (-1)^{d - dim\,\tau} \phi(\tau) = (-1)^{d+1} S(X - \partial X, \phi).$$

The third equality holds since for fix $\tau \in X$ the coefficient for $\phi(\tau)$ is $(-1)^{dim\,\tau} \sum_{\sigma \succeq \tau} (-1)^{dim\,\sigma}$. For the fourth equality we use Lemma 3.3. □

Now we are able to prove the main result.

Theorem 3.5. *Let P be a polytope with integer-valued vertices and let X be a simplicial complex that triangulates P, s.t. $|X| = P$ and $vert\, X = vert\, P$. Then*

$$L(X - \partial X, n) = (-1)^d L(X, -n).$$

Proof. Define $\phi : X \to \mathbb{R}[n]$ by $\phi(\tau) = (-1)^{1+dim\,\tau} L(\tau, n)$.
Then by definition for any subset Y of X:

$$S(Y, \phi) = \sum_{\tau \in Y} (-1)^{1+dim\,\tau} \phi(\tau) = \sum_{\tau \in Y} L(\tau, n) = L(Y, n)$$

and

$$\phi^*(\sigma) = S(\overline{\sigma}, \phi) = L(\overline{\sigma}, n) = (-1)^{dim\,\sigma} L(\sigma, -n).$$

Therefore

$$S(X, \phi^*) = \sum_{\sigma \in X} (-1)^{1+dim\,\sigma} \phi^*(\sigma) = -\sum_{\sigma \in X} L(\sigma, -n) = -L(X, -n).$$

Using Proposition 3.4 we conclude

$$L(X, -n) = -S(X, \phi^*) = (-1)^d S(X - \partial X, \phi) = (-1)^d L(X - \partial X, n).$$

□