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Practice 2: Real Functions

Exercise 4

We introduce the Function

$$f(x) := \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases}.$$

Complete the function for $f(0)$, so that

(*i*) $f \in V$ and

(*ii*) $f \in -V$.

Does $f(x)$ belong to V and/or $-V$ if $f(0) = 0$?

Exercise 5

A function $f : I \subset \mathbb{R} \to \mathbb{R} \cup -\infty$ is said to be upper semicontinuous at the point $a \in I$, if for all $\varepsilon > 0$ there is $\delta > 0$ with

$$f(x) < f(a) + \varepsilon \text{ for all } x \in I, |x - a| < \delta.$$

Analoguously a function $f : I \subset \mathbb{R} \to \mathbb{R} \cup +\infty$ is said to be lower semicontinuous at the point $a \in I$ if for all $\varepsilon > 0$ there is $\delta > 0$ with

$$f(x) > f(a) - \varepsilon \text{ for all } x \in I, |x - a| < \delta.$$

Show that f is upper semicontinuous if and only if $f \in -V$ and that f is lower semicontinuous if and only if $f \in V$.

Exercise 6

Let $\pi : a = x_0 < x_1 < \cdots < x_n = b$ be a decomposition of the compact interval $I = [a, b], a < b$ and $f : I \to \mathbb{R}$ a bounded function, i. e. there is $Z > 0$ so that $|f(x)| \leq Z$ for all $x \in I$. We define

$$Z_k := \sup_{x \in I_k} f(x), z_k := \inf_{x \in I_k} f(x)$$

for $1 \leq k \leq n$. Then f is Riemann-integrable over I if for all $\varepsilon > 0$ there is a decomposition $\pi = \pi_\varepsilon$ with

$$\sum_{k=1}^{n} Z_k |I_k| - \sum_{k=1}^{n} z_k |I_k| < \varepsilon.$$

Show that every Riemann-integrable function $f : I \to \mathbb{R}$ is also Lebesgue-integrable.