Concentration of Measure and Value-at-Risk estimation for non-linear portfolios

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Raymond Brummelhuis
(joint work with Brad Baxter)

School of Econ., Math. and Stat.
Birkbeck College, Univ. London

Basic Problem: Compute/estimate the Value-at-Risk of a portfolio

$$\mathfrak{P} = \mathfrak{P}(X_1, \cdots, X_n)$$

depending on large number of risk-factors $X = (X_1, \cdots, X_n)$

- Large portfolio or multi-asset option (e.g. k-th to default swap, CDO)
- Restrict to Gaussian X: typically log-returns of underlying
- Also: Gaussian copula models: Li model of multiple default

Value-at-Risk:

$$\mathrm{VaR}_\alpha(\mathfrak{P}) := q_{+,\alpha}\left(\mathfrak{P}_0 - \mathfrak{P}\right)$$

$$\begin{aligned} q_{+,\alpha}(Y) &:= \sup\{V \in \mathbb{R} : \mathbb{P}\left(Y > V\right) \geq \alpha\} \\ &= F_Y^{-1}(\alpha) \ \text{ if invertible} \end{aligned}$$

VaR_α = upper α-th quantile of portfolio loss-function

usual financial interpretation

Trivial example:

Linear Gaussian portfolios:

$$\mathfrak{P}_0 - \mathfrak{P} = \mu + \langle v, X \rangle,$$
$$v = (v_1, \cdots, v_n) \in \mathbb{R}^n,$$
$$X = (X_1, \cdots, X_n) \sim N(0, \mathrm{Id}_{n \times n})$$

$$\mathrm{VaR}_\alpha = \mu + |v| q_\alpha^N,$$
$$(|v| = \text{Euclidean norm })$$

$$\int_{q_\alpha^N}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} = \alpha.$$

More complicated non-linear cases:

- Linear- or Δ-approximation:

$$\begin{aligned}\mathfrak{P}_0 - \mathfrak{P} &= F(X_1, \cdots, X_n) \\ &\simeq F(0) + \langle \nabla F(0), X \rangle\end{aligned}$$

- $\Delta - \Gamma$-approximation:

$$\begin{aligned}&\simeq F(0) + \langle \nabla F(0), X \rangle + \frac{1}{2}\langle \nabla^2 F(0) X, X \rangle \\ &:= F(0) + \langle \Delta_0, X \rangle + \frac{1}{2}\langle \Gamma_0 X, X \rangle\end{aligned}$$

Already no exact answer $\Rightarrow$ MC / Characteristic Function Method / Asymptotics $\alpha \to 0$

- Fully non-linear portfolios $\Rightarrow$ basically obliged to do MC

(*Exception:* single factor portfolios $F = F(X)$ with increasing F:

$$\mathrm{VaR}_\alpha\left(F(X)\right) = F(q_\alpha^N);$$

decreasing F: similar

Examples: single puts and calls)

Proposal:

Derive rigorous analytic upper/lower bounds for non-linear Value-at-Risk of multi-asset portfolios or instruments.

Critique: Everything in Finance computable by (straightforward or clever) MC, so why bother?

Defense:

- computational speed
- theoretical insight: parameter dependence (portfolio loads, stress testing)
  effects of parameter uncertainty on statistical parameters
- checks on MC computations
- mathematically interesting problem

Main mathematical tool:

Concentration of Measure Theorem ('3 Russians thm')

$f : \mathbb{R}^n \to \mathbb{R}$ Lipschitz (bounded derivative). Then:

$$\mathbb{P}\left(f(X) - \mathbb{E}f > x\right) \leq 2\overline{\Phi}\left(\frac{x}{||\nabla f||_\infty}\right)$$

$$\overline{\Phi}(x) = \int_x^\infty e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}$$

Corollary: (CoM quantile estimate)

$$q_{+,\alpha}\left(f(X)\right) \leq \mathbb{E}f + ||\nabla f||_\infty \, q^N_{\alpha/2}$$

Observe (comparison with Δ-approx.):

- $\mathbb{E}f$ instead of $f(0)$
- $||\nabla f||_\infty$ ('maximum Delta') instead of $\nabla f(0)$
- Price to pay: $q^N_{\alpha/2}$ instead of q^N_α

Comments:

- *Gaussian* concentration of measure
- Classical conc. of meas. (Lévy): n-dimensional sphere - connection with isoperimetric $\geq$'s
- RHS $\leq e^{-|x|^2/2||\nabla f||_\infty}$
- 3-Russian's original proof: stochastic calculus (Ito formula, stochastic time-change)

- Later proofs (Pisier): LHS $\leq e^{-c|x|^2/||\nabla f||_\infty}$ for some absolute $c > 0$ with usually $c < 1/2$; $c = 1/2$ best possible
- Used in local theory of Banach spaces - Dvoretzky's theorem & generalizations (Milman, Pisier, ...): c doesn't matter
- For our purposes: sharp form important

Test-case: affine $f(X)$

$$f(X) = \mu + \langle v, X \rangle \quad \Rightarrow_{\mathrm{CoM}}$$

$$q_{+,\alpha}(f) \leq \mu + |v| q^N_{\alpha/2}$$

$\alpha \setminus$ quantile	$q^N_{\alpha/2}$	q^N_α	$q^N_{\alpha/2}/q^N_\alpha$
0.05	1.960	1.645	1.192
0.01	2.576	2.326	1.107
0.001	3.291	3.09	1.065

$$\frac{q^N_{\alpha/2}}{q^N_\alpha} \to 1, \ \alpha \to 0$$

$(q^N_\alpha \sim \sqrt{\log \alpha^{-2}}\ )$

(Compare with result of Pisier's estimate with $c = 2/\pi^2$:

$\alpha \setminus$ quantile	$q^{\mathrm{CoM}}_{+,\alpha}$	$q^N_{+,\alpha}$	$q^{\mathrm{CoM}}_{+,\alpha}/q^N_{+,\alpha}$
0.05	3.845	1.645	2.337
0.01	4.767	2.326	2.05
0.001	5.839	3.09	1.89

)

Applications to Portfolios

$$\mathfrak{P}_0 - \mathfrak{P} = F(X_1, \cdots, X_n) \ :$$

- ∇F possibly unbounded ($S = S_0 e^{\mu t + \sigma\sqrt{t}X}$)

$\Rightarrow$ Transform portfolio loss-function, e.g.

$$f = \log F$$

(if $F \geq 0$, otherwise $\max(F, 0)$)

- shorted call $C = C(S,t)$: $\nabla_X \log C$ unbounded as $S \to 0$!

$\Rightarrow$ Further modifications necessary

- $\mathbb{E}f$ problematic if $f \neq F$

E.g. not possible to evaluate explicitly & Jensen wrong way round $\Rightarrow$?

- $||\nabla f||_\infty$ too big

$\Rightarrow$ Can replace by something smaller?
$c \cdot \mathbb{E}\nabla f(X)$?

Two Worked Examples:
(Classical Black & Scholes setting)

- portfolio of shorted puts
- 1. Portfolio of ordinary stock
  (Nonlinear? - Yes!)

Puts

$$P_t = P(S,t;K,T) = Ke^{-r\tau}\Phi(-d_-) - S\Phi(-d_+),$$

$$d_\pm = \frac{\log(S/Ke^{-r\tau}) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \quad \tau = T - t$$

Portfolio:

$$\mathfrak{P}(S_t,t) := -\sum_{j=1}^{N} q_j P_j(S_{j,t},t)$$

$$\frac{dS_{j,t}}{S_{j,t}} = \mu_j dt + (HW_t)_j, \ W_t \ n\text{-dim. Brownian},$$

$$\Sigma = H^t H$$

Fix $t > 0$ small ($t = 1/365$), put $X := t^{-1/2} W_t \sim N(0, \mathrm{Id}_{n\times n})$

$$f(X) := \mathfrak{P}(S_t, t) - \mathfrak{P}(S_0, 0)$$
$$S_t = S_0 \exp\Big((\mu - \sigma^2/2)t + \sigma\sqrt{t} H X\Big) \quad \text{vector}$$

Theorem.

$$\mathrm{VaR}_\alpha(\mathfrak{P}) \le \mathfrak{P}_0 - \widetilde{\mathfrak{P}}_0 + \mathfrak{s}\, q^N_{\alpha/2},$$

where

$$\mathfrak{s}^2 := t \max_{u_j \in \{0,\, q_j K_j e^{-r\tau_j}\}} \langle \Sigma u, u\rangle.$$

$$\widetilde{\mathfrak{P}}_0 := \sum_j q_j \widetilde{P}_{j,0},$$

$\widetilde{P}_{j,0} = P_{j,0}$ if $\mu = r$; otherwise correction

Performance?

- Not too bad for in -the-money puts, much worse for at- and out-of-the-money
- Numerical results *suggest*:
$$||\nabla f||_\infty \;\rightarrow\; \mathbb{E}\nabla f(X)$$
still give upper bound & better estimate.
- Theoretical justification?

2. Stock Portfolio

$$\mathfrak{P} = \sum_{j=1}^{N} q_j S_j = \sum_{j=1}^{N} q_j S_{j,0}\, e^{\mu_j + \sqrt{t}(HX)_j}$$

Linear Approx. $S_j \simeq S_{j,0}\,(1 + \mu_j + (HX)_j)$

Estimates for fully non-linear VaR?

Apply CoM to: $f(X) = \log\left(\mathfrak{P}_0/\mathfrak{P}\right)$

$$\nabla f(x) = H^t \mathfrak{w}(x),$$
$$\mathfrak{w}_j(x) = \frac{q_j S_j}{\mathfrak{P}}|_{X=x} \text{ portfolio weights}$$

Theorem.

$$\mathrm{VaR}_\alpha(\mathfrak{P}) \leq$$
$$\mathfrak{P}_0 \left(1 - \exp\left(-\sum \mathfrak{w}_j(0)(\mu_j - \sigma_j^2)t - (\max_j \sigma_j) q^N_{\alpha/2}\right)\right).$$

Critique: absence of correlation effects

Modified CoM-estimate: For any $v \in \mathbb{R}^n$:

$$\mathbb{P}\left(f(X) - \mathbb{E}f - \langle v, X\rangle > x\right) \leq 2\overline{\Phi}\left(\frac{x}{||\nabla f - v||_\infty}\right)$$

Idea: choose v s.th. $||\nabla f - v||_\infty$ minimized.

Useful?