

Modelling Simultaneous Defaults and Valuation of Defaultable Claims

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Agenda

- 1.) Introduction and literature review
- 2.) Illustration of the modelling
- 3.) Stochastic modelling of default mechanism
- 4.) Calculation of the default correlation and comparison to benchmarks
- 5.) Valuation formulas for defaultable claims
- 6.) Concluding remarks



Existence of default dependencies

- nearly simultaneous defaults possible and observable
 - Southeast Asian bank crisis
 - potential default of LTCM → potential default of related banks
 - defaults of 10 US hotel chains and casino in 1999
- empirical studies (Lucas (1995), Das et al. (2002) or de Servigny/Renault (2002)) confirm this examples:
 - default correlation is positive
 - default correlation can reach relatively high values

⇒ default correlation and simultaneous default should be taken into consideration in a model



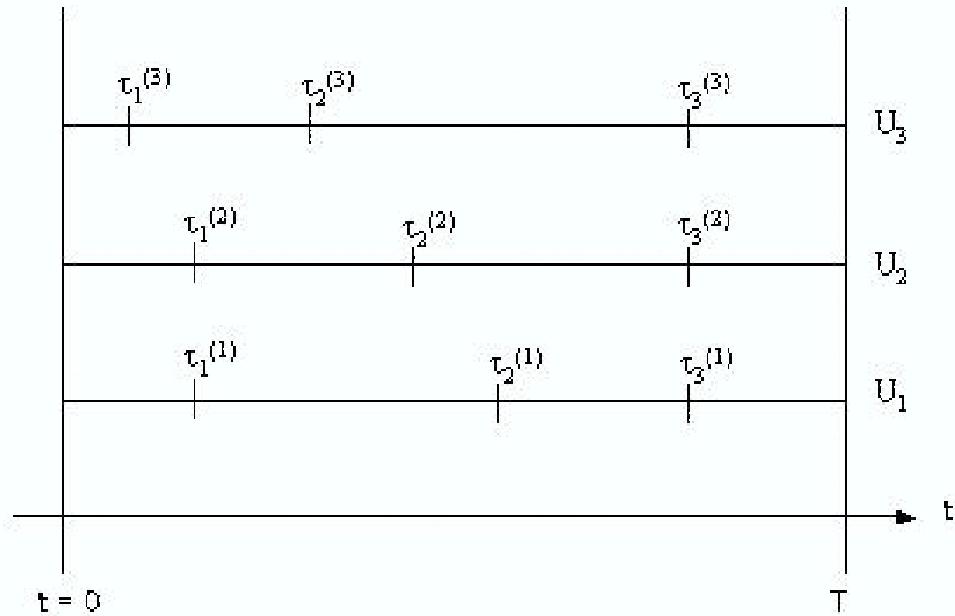
Related literature - A review

- classical reduced form models:
 - Lando (1998), Duffie/Singleton (1999a) → no correlation between firms considered
 - simple extension: correlation between default intensities
 - induced default correlation too low comparing to empirical results
- results for structural models similar to reduced form models
- extensions of the reduced form approach
 - Duffie/Singleton (1999b) → simulation procedures for correlated defaults
 - Jarrow/Yu (2001) → default of firm i increases default probability of firm j , approach difficult to handle in general and for many firms
- Davis/Lo (1999,2001) → infectious defaults, leads to default of firm j immediately after default of i



Basic idea (1)

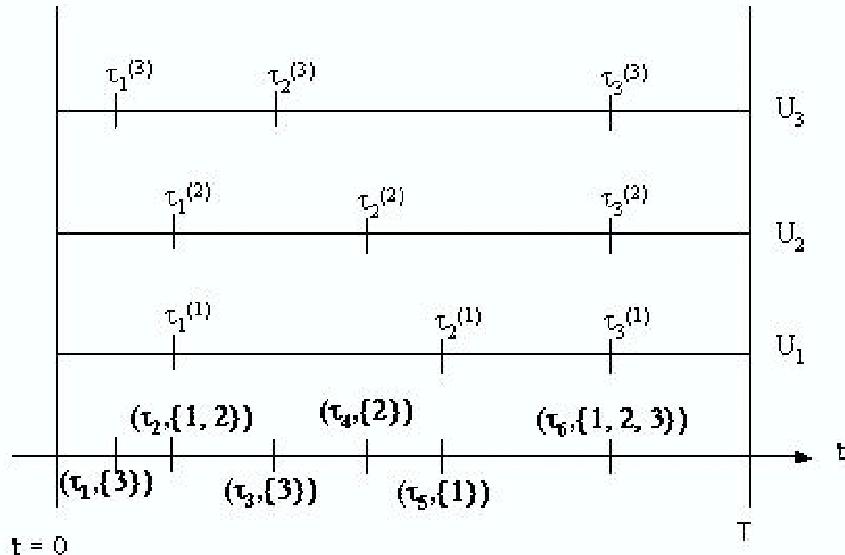
consider economy with n firms with default counting processes $N^{(i)}(t) = \sum_{i=1}^m \mathbf{1}_{\{\tau^{(i)} \leq t\}}$ with intensity $\lambda^{(i)}$



Basic idea (2)

instead of modelling n individuell default processes

→ use one marked Poisson Process $\Phi = (\tau_m, A_m)_{m \in \mathbb{N}}$ with intensity $\bar{\lambda}$



$$\tau_m = \inf \left\{ t > 0 : \int_{\tau_{m-1}}^t \bar{\lambda}_s ds > E_m \right\}$$

with $E_m \sim \exp(1)$ iid

A_m represents set of defaulted firm in τ_m

⇒ arbitrary dependencies can be modelled

⇒ $\bar{\lambda}$ can be estimated on larger data set

Specification of the reduced form model

- stochastic components
 - n firms with default counting processes $N^{(i)}(t) = \sum_{i=1}^m \mathbf{1}_{\{\tau^{(i)} \leq t\}}$
 - \mathbb{R}^d -valued background process $X = (X_1, \dots, X_d)$
 - mark space $\mathbb{X} = \mathcal{P}(\{1, \dots, n\}) \setminus \emptyset \rightarrow$ possible default sets
 - intensity for default of at least one firm $\bar{\lambda}_t = \bar{\lambda}(X, t)$
 - stochastic interest rate $r_t = r(X, t)$
- available information generated by
 - state variables: $\mathcal{F}_t = \sigma(X_s : s \leq t)$
 - default status: $\mathcal{H}_t^i = \sigma(N_s^{(i)} : s \leq t)$ resp. $\mathcal{H}_t = \bigvee_{i=1}^n \mathcal{H}_t^i$
 - both components: $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$



Problems

- $N(t) = (N^{(1)}(t), \dots, N^{(n)}(t))$ can have jumps in several components
- intensity $\lambda^{(i)}$ belonging to $N^{(i)}(t)$ unknown

⇒ changing the definition of the counting process and introduce new intensity

⇒ classic analysis of vector valued Poisson Processes can be applied



Transformation of counting process

consider a counting process not for every firm but for every possible set, i.e.

$$\tilde{N}(t) = (\tilde{N}^{\{1\}}, \dots, \tilde{N}^{\{1, \dots, n\}})$$

\Rightarrow no common jumps in \tilde{N} possible

to each component of \tilde{N} exists an intensity λ_t^A such that

- usual technical condition are fulfilled
- $\tilde{N}^A(t) - \int_0^t \lambda_s^A ds$ is a martingal with respect to \mathcal{G}_t

$$\tau^A = \inf \left\{ t \geq 0 : \int_0^t \lambda_s^A ds > E_\nu \right\} \quad (\nu = 1, \dots, 2^n - 1)$$

\rightarrow result of the transformation: $2^n - 1$ Cox Processes



Characterization of default intensity for a set

How can λ^A be calculated using $\bar{\lambda}_t$? → result from Brémaud (1981):

$$\frac{\lambda_t^A}{\bar{\lambda}_t} = \mathbb{P}(A \text{ defaults} \mid \mathcal{G}_t) = p^A(t)$$

$$\Rightarrow \quad \lambda_t^A = p^A(t) \cdot \bar{\lambda}_t$$

modelling default probability $p^A(t)$ using structural elements

- $(V_t^{(i)})_{t \geq 0}$ is \mathcal{F}_t -adapted process → firm value
- $L_t^{(i)}$ outstanding liabilities of firm i payable in t
- $\hat{p}^A = \mathbb{P}(V_T^{(i)} \leq L_T^{(i)} \ \forall i \in A, V_T^{(j)} > L_T^{(j)} \ \forall j \in A^c)$ → closed form solution possible

$$\Rightarrow \quad p^A(t) = \frac{\hat{p}^A}{\sum_{A \in \mathbb{X}} \hat{p}^A}$$



Computation of default correlation

Computation of correlation between firm i and j : $\rho_{ij}(t) = \frac{\text{Cov}\left(N^{(i)}(t), N^{(j)}(t)\right)}{\sqrt{\text{Var}\left(N^{(i)}(t)\right)} \sqrt{\text{Var}\left(N^{(j)}(t)\right)}}$

using representation over default sets: $N^{(i)}(t) = \sum_{A \in \mathbb{X}} \tilde{N}^A(t) \cdot \mathbf{1}_{\{i \in A\}}$

$$\begin{aligned} \text{Cov}(N^{(i)}(t), N^{(j)}(t)) &= \sum_{i \in A} \sum_{j \in B} \text{Cov}(\tilde{N}^A(t), \tilde{N}^B(t)) \\ &= \sum_{i,j \in A} \text{Var}(\tilde{N}^A(t)) + \sum_{\substack{i \in A \\ j \in B \\ A \neq B}} \text{Cov}(\tilde{N}^A(t), \tilde{N}^B(t)) \end{aligned}$$

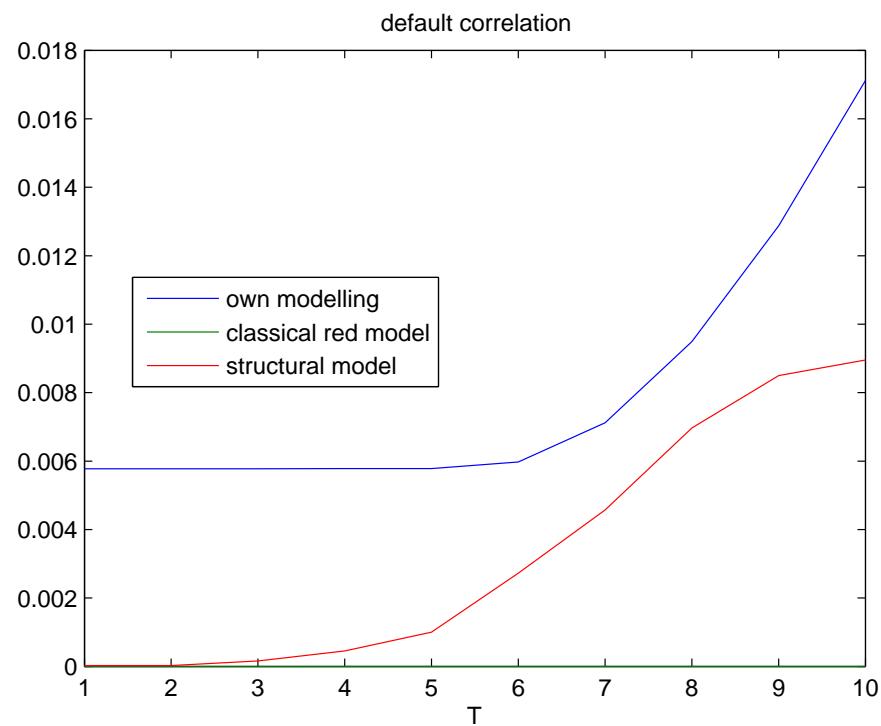
with $\text{Var}(\tilde{N}^A(t)) = \mathbb{E} [\Lambda_t^A]$

$$\text{Cov}(\tilde{N}^A(t), \tilde{N}^B(t)) = \mathbb{E} [\Lambda_t^A \Lambda_t^B] - \mathbb{E} [\Lambda_t^A] \mathbb{E} [\Lambda_t^B] \quad (A \neq B)$$

$$\Lambda_t^A = \int_0^t p_A(s) \bar{\lambda}_s ds$$



Comparison of default correlation in different approaches



$V_0^{(1)}$	1.1	$\lambda_0^{(1)}$	0.2
$V_0^{(2)}$	0.9	$\lambda_0^{(2)}$	0.2
μ_1	0.04	μ_1	0.1
μ_2	0.05	μ_2	0.1
σ_1	0.2	σ_1	0.2
σ_2	0.2	σ_2	0.2
$\bar{\lambda}_0$	0.1	ρ_{12}	0.3
κ	2	ρ_{13}	0.2
θ	0.8	ρ_{23}	0.15
η	0.4	L_i	1

Valuation formula - three building blocks (1)

1.) payment Y at fixed point in time

$$\mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) Y \cdot \mathbf{1}_{\{\tau^{(i)} > T\}} \middle| \mathcal{G}_t \right] = \mathbb{E} \left[\exp \left(- \int_t^T \left(r_s + \sum_{A \in \mathbb{X}, i \in A} \lambda_s^A \right) ds \right) Y \middle| \mathcal{F}_t \right]$$

with $\tau^{(i)} = \bigwedge_{A \in \mathbb{X}, i \in A} \tau^A$ and corresponding intensity $\sum_{A \in \mathbb{X}} \lambda_s^A \cdot \mathbf{1}_{\{i \in A\}}$

→ Valuation of expectations via Monte Carlo Simulation

Valuation formula - three building blocks (2)

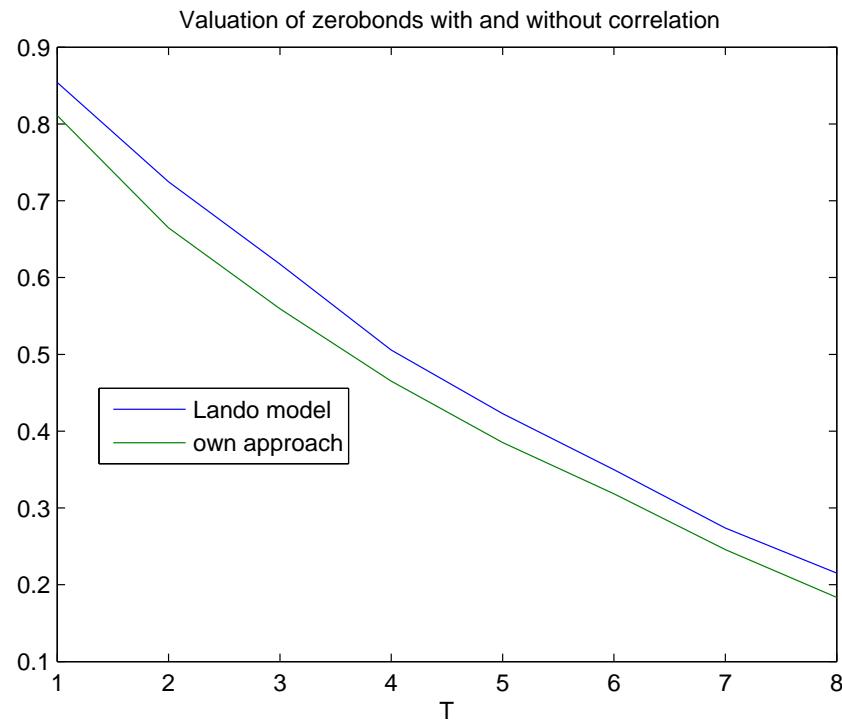
2.) continuous payments C_s up to default or maturity

$$\mathbb{E} \left[\int_t^T \exp\left(-\int_t^s r_u du\right) C_s \cdot \mathbf{1}_{\{\tau^{(i)} > s\}} ds \middle| \mathcal{G}_t \right] = \mathbb{E} \left[\int_t^T \exp\left(-\int_t^s \left(r_u + \sum_{A \in \mathbb{X}, i \in A} \lambda_u^A\right) du\right) C_s \middle| \mathcal{F}_t \right]$$

3.) payment at default $Z_{\tau^{(i)}}$

$$\begin{aligned} \mathbb{E} \left[\exp\left(-\int_t^{\tau^{(i)}} r_u du\right) Z_{\tau^{(i)}} \cdot \mathbf{1}_{\{\tau^{(i)} \leq T\}} \middle| \mathcal{G}_t \right] = \\ \mathbb{E} \left[\int_t^T \sum_{A \in \mathbb{X}, i \in A} \lambda_s^A \exp\left(-\int_t^s \left(r_u + \sum_{A \in \mathbb{X}, i \in A} \lambda_u^A\right) du\right) Z_s ds \middle| \mathcal{F}_t \right]. \end{aligned}$$

Simulation results for prices of defaultable zero bonds



$V_0^{(1)}$	1.1	$\lambda_0^{(1)}$	0.2
$V_0^{(2)}$	0.9	$\lambda_0^{(2)}$	0.2
μ_1	0.04	μ_1	0.1
μ_2	0.05	μ_2	0.1
σ_1	0.2	σ_1	0.2
σ_2	0.2	σ_2	0.2
$\bar{\lambda}_0$	0.1	ρ_{12}	0.3
κ	2	ρ_{13}	0.2
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η	0.4	L_i	1

Concluding remarks

- new modelling approach for dependencies in reduced form models → correlation more adequate modelled
- hybrid approach with advantages of the reduced form framework
- relation to Lando's (1998) model
 - + presented model extension concerning default dependencies
 - + Lando model special case for: $p^A(t) \equiv 0 \forall A \in \mathbb{X}$ with $|A| \geq 2$
 - + interpretable intensity and wider data basis for estimation
- extension to multiple defaults of one firm can easily be made

