Large Portfolio Approximation

in an Elliptical Distributions Framework

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Financial Modelling Workshop Ulm, September 2005

Outline

Introduction (1)

We want to price and analyze **Credit Structures** such as CDOs.

Essentially needed: **distribution of losses** in underlying portfolio

Introduction (2)

Credit structures with **large underlying portfolio** ($>$ 100 names)

- **Problem:**
  portfolio loss distribution
  $\longrightarrow$ mathematically difficult to obtain;
  pricing and sensitivity analysis via MC-Simulation
  $\longrightarrow$ very time consuming

- **Aim/Solution:**
  factor setup
  $\longrightarrow$ reduction of dimensionality;
  approximation of portfolio losses
  $\longrightarrow$ distribution to be obtained **(semi)-analytically**

Like Merton's model [2] (one-period model $[0, T]$):

default is triggered by fall of firm value S_j below "appropriate" threshold at time T.

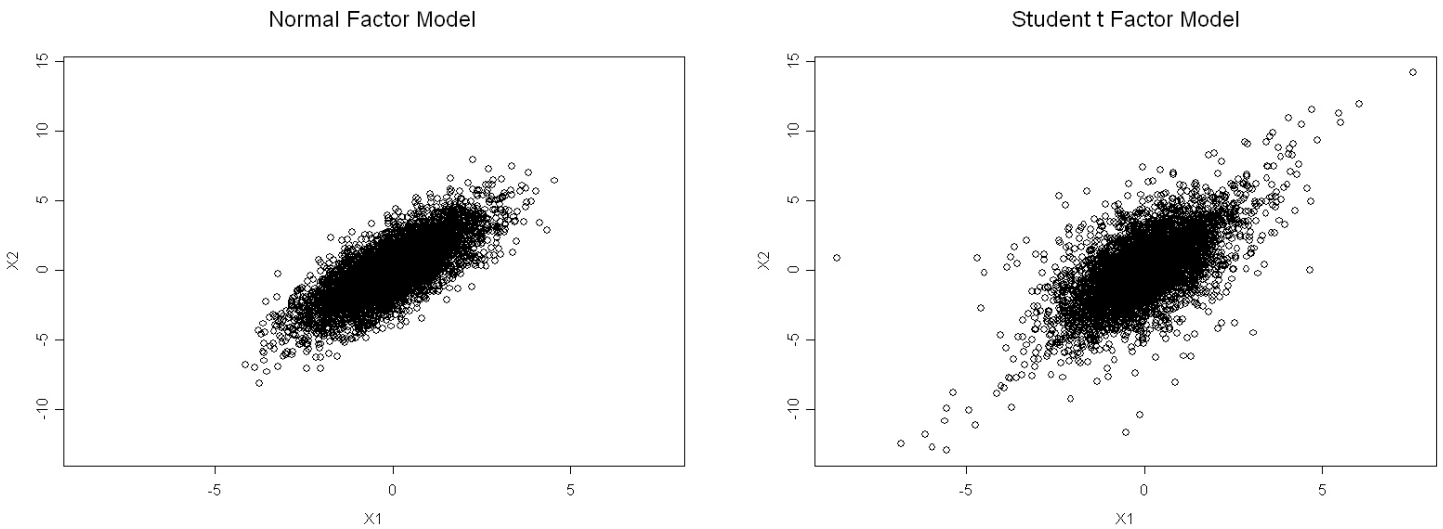

Figure: 5000 realizations of (S_1, S_2) with different dependence structures

More general: we want to use **elliptical distributions**.

Company j, $j = 1, \ldots, n$, with asset value S_j at time T:

	Merton	our setup
ratings at $t = 0$:	(non-)default	$k_j \in \{1, \ldots, r\}$; $k = 1$ for default;
given probabilities:	PDs	p_{kl}: going from k at $t = 0 \to l$ at $t = T$; $p_{11} = 1$;
thresholds:	DT_j	$c_{kl}^{(j)} \in \overline{\mathbb{R}}$: corresponding to p_{kl};
rating at $t = T$:	$\mathbf{1}_{\{S_j < DT_j\}}$	$L_j := \sum_{l=1}^{r} \mathbf{1}_{\{c_{k_j,l-1}^{(j)} < S_j \leq c_{k_j,l}^{(j)}\}} \cdot l$;
losses at $t = T$:	$(Not. - S_j)^+$	$\pi(j, k_j, l_j, \Psi)$, if from k_j to $L_j = l_j$, macro-influence Ψ.

$\to$ characteristics $(S_j, k_j, L_j, \pi(j, k_j, L_j, \Psi))$.

(See e.g. also [3] for setup with several rating classes)

Outline

Definition (Portfolio credit loss C_n at $t = T$)

$$C_n := \sum_{j=1}^{n} \pi(j, k_j, L_j, \Psi) = \sum_{j=1}^{n} \sum_{l_j=1}^{r} \pi_{j,l_j} \cdot Z_{j,l_j}$$

where $Z_{j,l} := \mathbb{1}_{\{L_j = l_j\}}$ *and* $\pi_{j,l} := \pi(j, k_j, l, \Psi)$.

Definition (asset value vector $S^{(n)}$)

$$S^{(n)} := (S_1, \ldots, S_n)^T = \beta^{(n)} F + \varepsilon^{(n)},$$

$$\rightarrow S_j = \beta_j^T F + \varepsilon_j$$

where

$\beta^{(n)} = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^{n \times m}$, $\beta_j \in \mathbb{R}^m$: *factor-loadings* $\quad (m << n)$

$\varepsilon^{(n)} = (\varepsilon_1, \ldots, \varepsilon_n)^T$, ε_j: *individual behavior / idiosyncratic risk.*

$\varepsilon^{(n)}$*independent of* (F, Ψ).

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$\varepsilon^{(n)}$*independent of* (F, Ψ).

$$\boxed{S^{(n)} = \beta^{(n)} F + \varepsilon^{(n)}}$$

Special choice of

$$F \sim EC_m(\mathbf{0}, \Omega_F, \phi_F) \qquad \text{and} \qquad \varepsilon^{(n)} \sim EC_n(\mathbf{0}, \Sigma_n, \phi) :$$

$$\begin{array}{rl}
F \stackrel{d}{=} R_1 \cdot W & \varepsilon^{(n)} \stackrel{d}{=} R \cdot Y^{(n)} \\
R_1 \geq 0, R_1 \sim G_1 & R \geq 0, R \sim G_2 \\
W \sim N_m(\mathbf{0}, \Omega_F) & Y^{(n)} \sim N_n(\mathbf{0}, \Sigma_n) \\
\Omega_F > 0 & \Sigma_n = \mathrm{diag}(\omega_1, \ldots, \omega_n) > 0 \\
\mathbb{E}(R_1^2) = 1 & \mathbb{E}(R^2) = 1
\end{array}$$

$$R_1, R, W, Y^{(n)} \text{ ind.}$$

$$\Longrightarrow \quad S_j = \beta_j^T F + \varepsilon_j \sim F_j = H_{\beta_j^T \Omega_F \beta_j, \omega_j} \qquad \text{with}$$

$$H_{\sigma_1^2, \sigma_2^2}(x) := \int_{-\infty}^{x} \int_0^{\infty} \int_0^{\infty} n(y; 0, r^2\sigma_1^2 + s^2\sigma_2^2) dG_{R_1}(r) dG_{R_2}(s) dy.$$

Assumption 1:

a.) $\exists$ Borel fct. $f_j : \mathbb{R}^{2+m+d} \to \mathbb{R}$ s.t.

$$\pi(j, k_j, L_j, \Psi) = f_j(Y_j, R, F, \Psi);$$

b.) $\exists$ strictly increasing $(b_n)_{n \geq 0}$ with $b_n \to \infty$, for $n \to \infty$, s.t.

$$\sum_{n=1}^{\infty} \left(\frac{\log n}{b_n} \right)^2 \mathbb{E}\left(\left[\pi_{n,L_n} - \mathbb{E}(\pi_{n,L_n} | R, F, \Psi) \right]^2 \right) < \infty.$$

Assumption 2:

a.) $\pi(j, k_j, l_j, \Psi)$ measurable w.r.t. $\sigma(R, F, \Psi)$, for all $1 \leq l_j \leq r$.

b.) Case 1: Ψ ind. of $\sigma(R, F) \vee \sigma(S_j) \subseteq \sigma(R, F, Y_j)$,

<u>or</u> case 2: $\sigma(R, F, \Psi) = \sigma(R, F)$

Outline

Under above factor structure and assumptions 1 and 2, $C_n \in L_1$:

Definition (Conditional portfolio credit losses $(B_n)_{n\geq 0}$)

$$B_n := \mathbb{E}\left(C_n | R, F, \Psi\right) = \sum_{j=1}^{n} \sum_{l_j=1}^{r} \pi(j, k_j, l_j, \Psi) \cdot \hat{\Phi}_{j,l_j}\left(F, R\right) \text{ with}$$

$$\hat{\Phi}_{j,l}\left(f, r\right) := \Phi\left(\frac{c^{(j)}_{k_j,l} - \beta_j^T f}{\sqrt{\omega_j} r}\right) - \Phi\left(\frac{c^{(j)}_{k_j,l-1} - \beta_j^T f}{\sqrt{\omega_j} r}\right), f \in \mathbb{R}^m, r \in \mathbb{R}.$$

Theorem

$$\frac{C_n - B_n}{b_n} \longrightarrow 0, \quad \text{as } n \to \infty, \text{ almost surely.}$$

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A simple example (1)

2 rating classes: 1 for default, 2 for non-default; let

- $m = 1$, $\beta_j \equiv \rho$, $\mathbb{V}ar(F) = 1$, $\omega_j \equiv 1 - \rho^2$.
- $\pi(j, 1, l_j, \Psi) = \begin{cases} N_j, & \text{for } l_j = 1; \\ 0, & \text{for } l_j = 2. \end{cases}$; N_j: size of loan j.

$\Rightarrow B_n = \Phi\left(\frac{\rho F - c_{2,2}}{\sqrt{1-\rho^2}R}\right) \cdot \sum_{j=1}^{n} N_j$

$$\Rightarrow \boxed{\frac{C_n}{b_n} - \frac{1}{b_n}\sum_{j=1}^{n} N_j \cdot \Phi\left(\frac{\rho F - c_{2,2}}{\sqrt{1-\rho^2}R}\right) \longrightarrow 0,} \quad \text{as } n \to \infty, \text{ a.s..}$$

A simple example (2)

If $\frac{1}{b_n}\sum_{j=1}^{n} N_j \overset{n\to\infty}{\longrightarrow} 1$:
For portfolio loss distribution:

in the limit only distribution of $\Phi\left(\frac{\rho F - c_{2,2}}{\sqrt{1-\rho^2}R}\right)$ important!

This (scaled) approximated portfolio loss distribution is directly given via a two-dimensional integral $\rightarrow$ analytical evaluation:

$$x \mapsto \int_0^\infty \int_0^\infty \Phi\left(\frac{\sqrt{1-\rho^2}r\Phi^{-1}(x) + c_{2,2}}{\rho r_1}\right) dG_1(r_1)dG(r).$$

Summary

- Factor model plus Gaussian mixture distributions yield approximation for portfolio credit losses.
- Approximation gives rise to (semi-)analytical valuation and analysis of credit structures.

- Outlook
  - Consequences for the analysis of credit structures?
  - Does this model imply a correlation smile similar to the market implied correlation smile?

For Further Reading I

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