Shot Noise Processes for Modeling Energy Forwards

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Characteristics of Energy Markets

1. Mean reversion
2. Spikes (energy is not easily storable, inelastic demand)
3. High volatility
4. Seasonalities

We basically discuss (2) and the consequences.

Shot-Noise Processes

Definition

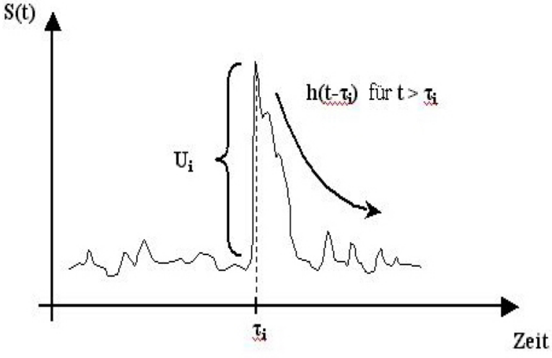
A simple shot-noise process is of the following form:

$$S_t = S_0 + \mu t + \sigma W_t + \sum_{i=1}^{N_t} U_i\, h(t - \tau_i),$$

where

- μ, σ *are constans*
- (W) *is a Brownian motion*
- (N) *is a Poisson process with intenitsy* λ *and jumps* $\tau_1, \tau_2, \dots$
- $U_1, U_2, \dots$ *are i.i.d.*
- $h : \mathbb{R}^+ \mapsto [0, 1]$ *is a deterministic function,* $h(0) = 1$
- $(W), (N), U_1, U_2, \dots$ *are mutually independent*

The Shot-Noise Effect



- Of course, this can be generalized ...
- The process is Markovian, iff $h(x) = e^{-ax}$
- The jump part is a piecewise deterministic process
- Characteristic function available in suitable form

Definition

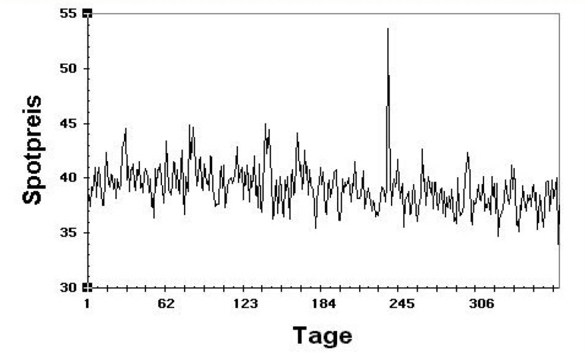
We will work with a multiplicative shot-noise process:

$$S_t = S_0 \exp\Big((\mu - \frac{\sigma^2}{2})t + \sigma W_t \Big) \prod_{i=1}^{N_t} \big(1 + U_i h(t - \tau_i)\big),$$

with $U_i \geq -1$.

Simulations

A simulated spot price process looks like



A Model for Forwards

Motivated by the foreign exchange analogy from Hinz et. al. we consider

$$F(t, T) = S(t) \cdot Z(t, T),$$

where the factor Z is of the form

$$Z(t, T) = \exp\Big(- \int_t^T f(t, u)\, du \Big).$$

Furthermore,

$$\begin{aligned} dS(t) &= S(t-)\big[m(t)\, dt + dM_t \big], \\ df(t, T) &= \alpha(t, T)\, dt + \sigma(t, T)\, dB_t. \end{aligned}$$

Basically, (B) can also be replaced by a Lévy process.

Under independence of B and M we have

Theorem

The considered measure is a martingale measure, iff

$$(i) \qquad \alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)\,du$$

$$(ii) \qquad f(t,t) = -m(t)$$

Proof. Derive dynamics of $Z(t,T) = G\big(f(t,\cdot)\big)(T)$ via Itô - formula for Hilbert spaces.

Obtain $dF(t,T) = d\big(S_t \cdot Z(t,T)\big)$ via Product rule.

$\Rightarrow$ Drift:

$$f(t,t) + m(t) + \sigma^*(t,T)d < M, B >_t - \alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2 \overset{!}{=} 0.$$

Generalizations

- More general shot-noise: i.i.d. processes $(U_i(t))_{t\geq 0}$
- Alternative structure of the forward rate:

$$F(t,T) = M(t,T) + \tilde{S}(t)Z(t,T),$$

  where spot price is $S(t) = M(t,t) + \tilde{S}(t)$
- Regime Switching is then a special case

$$U_i(t) = 1_{\{t\in[0,\tau]\}}u_i(t)$$

Pricing Caps and Floors

Assumptions:

- Constant interest rate r
- Pricing of a call on the forward rate

$$
\begin{aligned}
C(t,T) &= e^{-r(T-t)}\,\mathbb{E}^Q\Big[\big(F(T,T')-K\big)^+\Big|\,\mathcal{F}_t\Big] \\
&\propto \underbrace{\mathbb{E}^Q\Big[F(T,T')1_{\{F(T,T')>K\}}\Big|\,\mathcal{F}_t\Big]}_{=:A(t,T)} - K\,\underbrace{Q\Big[F(T,T')>K\Big|\,\mathcal{F}_t\Big]}_{=:B(t,T)}
\end{aligned}
$$

Theorem

We obtain for $B(t,T)$

$$e^{-\lambda(T'-T)} \sum_{n\geq 0} \frac{[\lambda(T'-T)]^n}{n!} \int_{[T,T']^m} \int_{\mathbb{R}^m} \Phi\Big(d_2(n,\mathbf{u},\mathbf{t})\Big) \frac{f_U(\mathbf{u})\, d\mathbf{u}\, d\mathbf{t}}{(T'-T)^n}$$

$$d_2 = \cdots - \ln\left(\frac{K}{F(t,T)\tilde{P}(t,T)\prod_{i=1}^n (1+\mathbf{u}_i h(T'-\mathbf{t}_i))}\right)$$

and a similar expression for A.

Of course, there is a put-call relationship and we obtain prices for floors and collars.

Swing Options

A swing option is characterized as follows:

- Tenor structure $T_1, \ldots, T_n$, strike price K
- Owner has the right to receive $\theta_i \in [\theta_{min}, \theta_{max}]$ MWh at T_i
- He has the obligation to fulfill $\Theta = \sum_{i=1}^n \theta_i \in [\Theta_{min}, \Theta_{max}]$
- Typically $\Theta_{min} > n\theta_{min}$ (also for max)

The price of the swing option is

$$Sw(t, T, T') = \sup_{\theta} \mathbb{E}^Q \Big[\sum_{i=1}^{n} e^{-r(T_i - t)} \theta_i [F(T_i, T_{i+1}) - K] \Big| \mathcal{F}_t \Big]$$

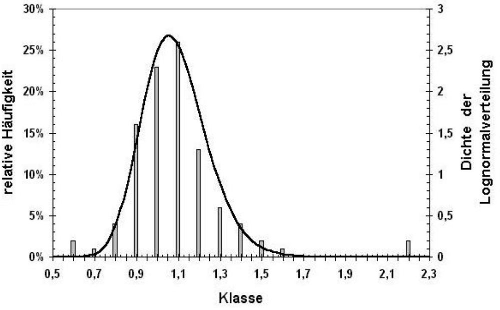
Need the optimal strategy w.r.t. local and global constraints.

Estimation of Shot-Noise Processes

- Observation in discrete time
- Assume Markovianity, i.e. $h(x) = e^{-ax}$
- Need to recognize jumps $\Rightarrow$ choose a suitable threshhold
- Can also be justified by typically *large* jumps
- Estimate distribution of U and λ, a

The Distribution of the jumps

A fit to the log-normal distribution:



Leads to a mixture of lognormal plus extreme jumps.

- Under the assumption of a mixture of log-normal jumps mean and variance can easily be estimated
- Jump intensity $\hat{\lambda} = \frac{N_t}{t}$
- Decay parameter is estimated using a least squares approach
- Alternatively, estimate $h(\cdot)$ nonparametrically (wavelets)

Open Questions

- What is a suitable density between P and Q
- Need also to determine market price of risk to join estimation and pricing
- Hedging
- Suitable generalizations of the model

Motivation

Consider the situation after Enron's default.

- Investor seeks impact on other companies
- She does not know, if company X has the same trouble or not
- Assumption: true hazard rate is

$$\lambda(\omega) = \begin{cases} \lambda_1 & \text{with prob. } p \\ \lambda_2 & \qquad\qquad 1-p \end{cases}$$

- Investor only observes $\{\tau > t\}$
- Behaving rational he estimates λ via

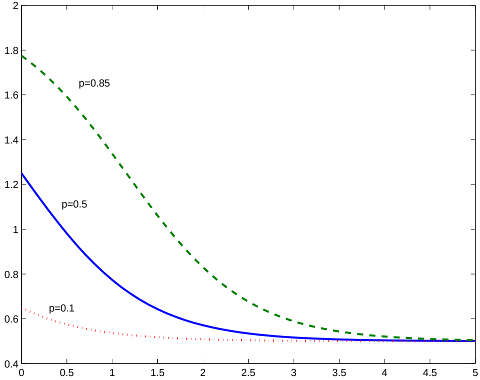
$$\hat{\lambda}_t := \mathbb{E}(\lambda | \tau > t)$$

First,

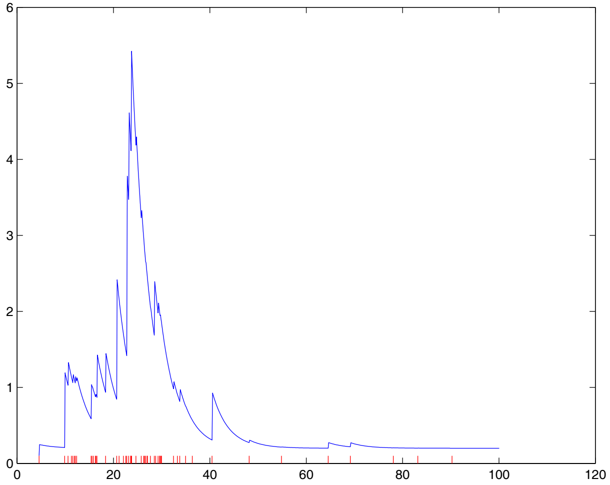
$$\mathbb{P}\big(\lambda = \lambda_1 | \tau > t\big) = \frac{\mathbb{P}(\lambda = \lambda_1, \tau > t)}{\mathbb{P}(\tau > t)} \\ = \frac{p e^{-\lambda_1 t}}{p e^{-\lambda_1 t} + (1-p) e^{-\lambda_2 t}}.$$

and hence

$$\hat{\lambda}_t = \frac{\lambda_1 p e^{-\lambda_1 t} + \lambda_2 (1-p) e^{-\lambda_2 t}}{p e^{-\lambda_1 t} + (1-p) e^{-\lambda_2 t}}.$$


p=0.85
p=0.5
p=0.1
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

- The shot-noise process, used in the default intensity, is self-exciting
- This process produces default clusters
- Useful for modeling CDOs


0
1
2
3
4
5
6
0
20
40
60
80
100
120

Literature

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