Multivariate Variance Gamma Modelling with Applications in Equity and Credit Risk Derivatives Pricing

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Abstract

We propose a multivariate model for financial assets which incorporates jumps, skewness, kurtosis (and if wanted stochastic volatility), and discuss its applications in the context of equity and credit risk. In the former case we describe the stochastic behavior of a series of stocks or indexes, in the latter we apply the model in a multi-firm, value-based default model.

Starting from an independent Brownian world, we will introduce jumps and other deviations from normality, as well as non-Gaussian dependence, by the simple but very strong technique of stochastic time-changing. We work out the details in the case of a Gamma time-change, thus obtaining a multivariate Variance Gamma (VG) setting.

A main feature of the model however is the fact that its risk neutral dependence can be calibrated from univariate derivative prices. Furthermore, the model is computationally friendly, since numerical results require a modest amount of time and the number of parameters grows linearly with the number of assets.

Examples of calibration exercises on the equity and credit market show the goodness of fit attained.

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OUTLINE

- From an independent Gaussian world to a dependent VG world.
- The multivariate VG model and its dependency structure.
- Applications in Equity modelling :
 - multivariate VG stock price model;
 - -joint calibration on multiple underlyers;
 - $-\operatorname{extention}$ to stochastic volatility.
- Applications in Credit Risk modelling :
 - multivariate VG firm value model;
 - CDS pricing by solving PIDE;
 - -joint calibration on multiple underlyers;
- Conclusion

BLACK-SCHOLES WORLD - UNIVARIATE

• The geometric Brownian motion univariate asset price model:

 $A_t = A_0 \exp(\theta t + \sigma W_t), t \ge 0,$

where $\{W_t, t \ge 0\}$ is standard Brownian Motion and σ is the usual volatility.

- This model is underlying the celebrated Black-Scholes model for stocks and indexes.
- This model is used to describe the firm's asset value process in credit default models by Merton, Black and Cox, CreditGrades, etc.



BLACK-SCHOLES WORLD - UNIVARIATE

- Shortcomings
 - Normal Distribution: log-returns are normally distributed, whereas empirical data typically shows skewness and excess kurtosis. This was the main motivation for the consideration of Lévy Processes.
 - Continuous Sample Paths:
 - * Brownian motion has continuous sample paths, whereas in reality prices are driven by jumps.
 - * The Brownian motion needs a substantial amount of time to reach a low barrier, whereas in reality jumps can cause an almost immediate move over the barrier.
 - Extreme Events: The model is not able to give realistic probabilities of extreme events: the Normal distribution has too light tails.

BLACK-SCHOLES WORLD - UNIVARIATE Empirical Density vs Normal Density



Figure 1: Normal and Gaussian Kernel density estimator - daily log-returns SP500 index

BLACK-SCHOLES WORLD - MULTIVARIATE

• In order to describe n dependent price processes

 $(A_t^{(1)},\ldots,A_t^{(n)})$

one can consider a vector of n dependent Brownian motions

 $(W_t^{(1)},\ldots,W_t^{(n)}).$

- \bullet The dependence is uniquely defined by the correlation matrix of the W 's.
- Each individual price process is model by geometric Brownian motion $A_t^{(i)} = A_0^{(i)} \exp(\theta_i t + \sigma_i W_t^{(i)}), t \ge 0,$

where σ_i is the *i*th prices process' volatility.

BLACK-SCHOLES WORLD - MULTIVARIATE

- Shortcomings
 - -Gaussian Dependence: multivariate Brownian motion leads to the Gaussian copula, which has, in contrast to empirical data, no tail dependency.
 - -Estimation of Correlation:
 - * This is typically done on the basis of historical data. Under more sophisticated models there is no garantuee any more that the historical dependence structure coincides with the risk-neutral one.
 - * Moreover, as with all historical data, a sudden change into the regime has only a marginal effect on the historical estimate, whereas the effect on the risk-neutral one can be much more pronounced.
 - Quadratic Growth of Parameters: The number of parameters grows quadratically in n through the correlation matrix, however the available market data, say derivative instrument prices, are usually linearly in n.

BUILDING IN DEPENDENCE BY TIME-CHANGING

- By using the technique of stochastic time-changing we build a NEW MULTIVARIATE MODEL with
 - non-Normal underlying distribution: skewness and kurtosis;
 - with jumps;
 - with more realistic extreme events probabilities;
 - -a non-Gaussian dependence structure;
 - with the number of parameters linearly in n;
 - -which can be calibrated using only the liquid univariate standard derivatives in the market !

BUILDING IN DEPENDENCE BY TIME-CHANGING

- We starting from the independent Gaussian case.
- To introduce dependence, we time-change ALL the asset prices by a COMMON stochastic time-change (new business time).
- Interpretation of time-change: a model of the information arrival of a common economic environment (Mandelbrot, C, G, M, Y).
 - no information is lossed \rightarrow time change is non-decreasing.
 - amount of new information is not affected by the amount already released \rightarrow independent increments.
 - amount of information released in a certain interval is only depending on the length of the interval \rightarrow stationary increments.
- Hence we opt for a non-decreasing Lévy processes (subordinator) for the time-change.
- We work out the details for the Gamma Lévy processes as new business time.

THE GAMMA TIME-CHANGING

• The Gamma(a, b) distribution has a density

$$f(x;a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx), \quad x \ge 0.$$

and is infinitely divisible.

- A Gamma process $G = \{G_t, t \ge 0\}$ is a stochastic process with
 - $-G_0 = 0$ (no initial information)
 - -stationary and independent increments
 - $-G_t G_s \approx \text{Gamma}(a(t-s), b),$
- Normalization : $E[G_t] = t$, which implies $a = b := 1/\nu$.



THE MULTIVARIATE VG MODEL

• Every price process is now modeled as Gamma-time-changed geometric Brownian motion.

$$A_t^{(i)} = A_0^{(i)} \exp(\theta_i G_t + \sigma_i W_{G_t}^{(i)}) := A_0^{(i)} \exp(X_t^{(i)}), \quad t \ge 0.$$

- Brownian motion with drift time-changed by a Gamma processes leads to the Variance Gamma (VG) model (Madan, Seneta, Carr, ...).
- The *i*th price process is the exponential of a VG-process $X^{(i)} = \{X_t, t \ge 0\}$ with parameters $(\sigma_i, \nu, \theta_i)$.
- The variance Var $\left[X_1^{(i)}\right] = \sigma_i^2 + \nu \theta_i^2$ decompose into
 - -idiosyncratic component σ_i^2 ;
 - -exogenous component $\nu \theta_i^2$;
 - θ_i governs the exposure to the global market uncertainty (ν).
- The univariate VG model has already proven its modeling capabilities.

THE MULTIVARIATE VG MODEL

- The processes are due to the common time-change dependent.
- Risk-neutral modelling can be done as in the univariate case:

$$A_t^{(i)} = A_0^{(i)} \exp((r - q_i)t + X_t^{(i)} + \omega_i t), \quad t \ge 0,$$

where r is the continuously compound interest rate, q_i the continuous dividend yield of the *i*th stock and

$$\omega_i = \nu^{-1} \log \left(1 - \frac{1}{2} \sigma_i^2 \nu - \theta_i \nu \right)$$



THE DEPENDENCY STRUCTURE

- Linear correlation of $X_1^{(1)}$ and $X_1^{(2)}$: $\rho = \frac{\theta_1 \theta_2 \nu}{\sqrt{\sigma_1^2 + \theta_1^2 \nu} \sqrt{\sigma_2^2 + \theta_2^2 \nu}}$.
- Because, conditioned on the time-change the log-returns are independent, the underlying copula can easily be compute numerically and scatter plots can be provided.
- Spearman's rho can be numerically calculated.
- Numerical calculations show lower-tail dependence.

$\overline{\sigma_1}$	θ_1	σ_2	θ_2	ν	ρ	$ ho_S$	λ_L
0.2	-0.10	0.25	-0.15	1	0.23	0.04	0.16
0.2	-0.10	0.25	-0.15	2.5	0.43	0.10	0.21
0.2	-0.25	0.25	-0.20	1	0.49	0.14	0.20
0.2	-0.25	0.25	-0.20	2.5	0.70	0.25	0.38
0.2	-0.35	0.25	-0.30	1	0.67	0.45	0.32
0.2	-0.35	0.25	-0.30	2.5	0.83	0.49	0.55

THE DEPENDENCY STRUCTURE

• Contour plots and scatter plots



APPLICATIONS: EQUITY

• Multivariate VG stock price model:

 $S_t^{(i)} = S_0 \exp((r - q_i)t + \theta_i G_t + \sigma_i W_{G_t}^{(i)} + \omega_i t), \quad i = 1, \dots, n, \quad t \ge 0.$

- It has been known that univariate VG can nicely capture the smile at one maturity.
- Calibration is done on univariate vanilla data !!!!
 - Do vanilla pricing with Carr-Madan formula using FFT.
 - Calibrate the models by imposing a common ν -parameter and by using essentially n times your univariate calibation algorithm.
- Example: Calibration on vanillas on SP500, Eurostoxx50 and Nikkei-225 on April, 5, 2005 maturing in approx. 1 year.
- \bullet Correlation coming of of the calibration make sense: e.g. ρ for SP500-Eurostoxx50 equal 0.71.
- If wanted one can impose correlation an take this into the calibration algorithm.

APPLICATIONS: MULTIVARIATE VG CALIBRATION - EQUITY



APPLICATIONS: MULTIVARIATE VG + SV - EQUITY

- If you like you can build in stochastic volatility by time-changing again.
- Speed at which time passes by should be mean reverting and positive.
- CGMY proposed i.a. to use CIR process to do this.
- In a multivariate setting, just take for each individual stock an independent CIR process:

$$S_t^{(i)} = S_0^{(i)} \frac{\exp((r - q_i)t)}{E\left[\exp\left(X_{Y_t^{(i)}}^{(i)}\right)\right]} \exp\left(X_{Y_t^{(i)}}^{(i)}\right),$$

where

 $-X^{(i)} = \{X_t^{(i)} = \theta_i G_t + \sigma_i W_{G_t}^{(i)}, t \ge 0\} \text{ is a VG process (JUMPS)} - Y_t^{(i)} \text{ are mutually independent integrated CIR-processes (SV).}$

• univariate VG-SV can nicely capture globally the smile at all maturities.

APPLICATIONS: MULTIVARIATE VG + SV - EQUITY





APPLICATIONS: MULTIVARIATE VG + SV - EQUITY

APPLICATIONS: FIRM VALUE UNIVARIATE VG - CREDIT

- Firm-value models (Merton, Black-Cox, CreditGrades, ...) for Credit Risk are typically using geometric Brownian motions to describe the dynamics of the value of the firm.
- In order to get skewness, kurtosis and jumps (leading to possiblity of instantaneus default) we propose to model the firm value V as the exponential of a Lévy process (e.g. VG process) X

$$V_t = V_0 \exp(X_t), \quad V_0 > 0.$$

- Default occurs at the first crossing of some predetermined barrier H.
- The risk-neutral *probability of no-default* between 0 and t is given by

$$P(t) = P_Q(V_s > H, \text{ for all } 0 \le s \le t);$$

= $E_Q \left[1 \left(\min_{0 \le s \le t} V_s > H \right) \right].$

APPLICATIONS: FIRM VALUE UNIVARIATE VG - CDS

• Recall that the par spread c of a CDS with maturity T equals.

$$c = \frac{(1-R)\left(1-\exp(-rT)P(T)-r\int_0^T \exp(-rs)P(s)ds\right)}{\int_0^T \exp(-rs)P(s)ds}$$
$$= \frac{(1-R)\left(1-BDOB(T,H)-r\int_0^T BDOB(s,H)ds\right)}{\int_0^T BDOB(s,H)ds}.$$

where

- * R is the firm specific recovery rate; r is default-free discount rate.
- * BDOB(T, H) is the price of a digital down-and-out barrier option with maturity T and barrier level H.
- Hence pricing CDS's comes down to pricing Barriers.

APPLICATIONS: FIRM VALUE UNIVARIATE VG - CDS

• For VG, Barrier pricing can be done fast (< 5 sec.) by numerically solving PIDE's (Madan, Hirsa, Carr, Cariboni-Schoutens) of the form

$$\int_{-\infty}^{+\infty} \left[F(V_{t-}e^{x}, t) - F(V_{t-}, t) - \frac{\partial F}{\partial V}(V_{t-}, t)V_{t-}(e^{x} - 1) \right] k(\mathrm{d}x) + \frac{\partial F}{\partial t}(V_{t}, t) + rV_{t}\frac{\partial F}{\partial V}(V_{t}, t) - rF(V_{t}, t) = 0,$$

where k(dx) is the Lévy measure of the underlying Lévy process.

- The Lévy measure for the VG explicitely now in closed formula.
- Fast CDS pricing allows fast calibration on CDS data.
- Example: calibration on 1, 3, 5, 7 and 10 year CDS on October 26, 2004.

APPLICATIONS: UNIVARIATE MODEL COMPARISON

• Below the Gaussian case, the classical CreditGrades model (with fixed barrier standard deviation; typically one sets $\lambda = 0.3$) and the Credit-Grades model with a free λ parameter is compared with the VG model.



APPLICATIONS: MULTIVARIATE VG DEFAULT MODEL

- In a multivariate setting just model each firm-value by a geometric Brownian Motion time-changed by a common Gamma process.
- Hence each individual firm value process is following an exponential VG process.
- \bullet The *i*th firm defaults the first time its asset value process

$$A_t^{(i)} = A_0^{(i)} \exp\left(X_t^{(i)}\right), t \ge 0.$$

hits a low barrier H_i .

 \bullet The the risk-neutral $probability\ of\ no-default$ between 0 and t is given by:

$$P_i(t) = P_Q \left(A_s^{(i)} > H_i, \text{ for all } 0 \le s \le t \right);$$

= $E_Q \left[1 \left(\min_{0 \le s \le t} A_s^{(i)} > H_i \right) \right].$

APPLICATIONS: MULTIVARIATE VG DEFAULT MODEL

- Calibration can easily be done by imposing a common ν -parameter and by using essentially n times your univariate calibation algorithm.
- Example: Calibration on CDSs of 5 underlyers.

Company	Moody		1y	3 <i>y</i>	5y	7y	10y
Autozone	Baa2	Market	25	65	102	117	127
Autozone	Baa2	Model	21	69	101	117	126
Ford Credit Co.	A3	Market	75	154	203	225	238
Ford Credit Co.	A3	Model	66	165	209	224	224
Kraft	A3	Market	4	19	31	40	51
Kraft	A3	Model	4	18	32	41	50
Walt Disney	Baa2	Market	6	21	36	45	56
Walt Disney	Baa2	Model	5	21	36	46	55
Whirlpool	Baa1	Market	16	36	66	73	86
Whirlpool	Baa1	Model	12	40	63	76	85

APPLICATIONS: MULTIVARIATE VG DEFAULT MODEL



CONCLUSION

- We have proposed a new Multivariate Variance Gamma model by timechanging independent multivariate geometric Brownian motions by a common Gamma process.
- The takes into account: skewness, kurtosis, jumps, non-gaussian dependence.
- Univariate techniques (pricing, MC, introducing SV, ...) can be readily transformed to the multivariate case.
 - If you don't have univariate VG running, you should.
 - $-\operatorname{If}$ you do, the extension to multivariate VG is straightforward.
- We have discussed applications in Equity and Credit Risk.
- Calibration can be done on UNIVARIATE data.
- Calibration fits are very satisfactory.
- The technique can also be used for other (Lévy) models, like NIG, CGMY, GH, Meixner, ...

- THE END -