

# Asset-based Estimates for Default Probabilities for Commercial Banks

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# Outline

## Structural Models

### KMV-Model

### CreditGrades Model

Model Inputs and Outputs

Model Description

Classical CreditGrades vs. “exact” CreditGrades

## Empirical Results

# Challenges of modelling default probabilities for banks

## Two main challenges

- ▶ Banks have opaque liabilities.
  - ▶ Unclear maturity profile: distinction between long-term and short-term liabilities difficult.
  - ▶ Banks are highly leveraged but do not behave like other highly leveraged industrial companies (e.g. w.r.t. asset volatilities).
- ▶ Number of banks  $\ll$  number of industrials.
  - ▶ Less default data for banks available.
  - ▶ Model calibration difficult.

# Structural Models

## Black-Scholes-Merton Default Model (1973/1974)

Firm's value process  $V$  is given by  $V_t = E(V_t) + D(V_t)$  (equity+debt) and follows

$$dV_t = V_t(\mu_V dt + \sigma_V dW_t), \quad (1)$$

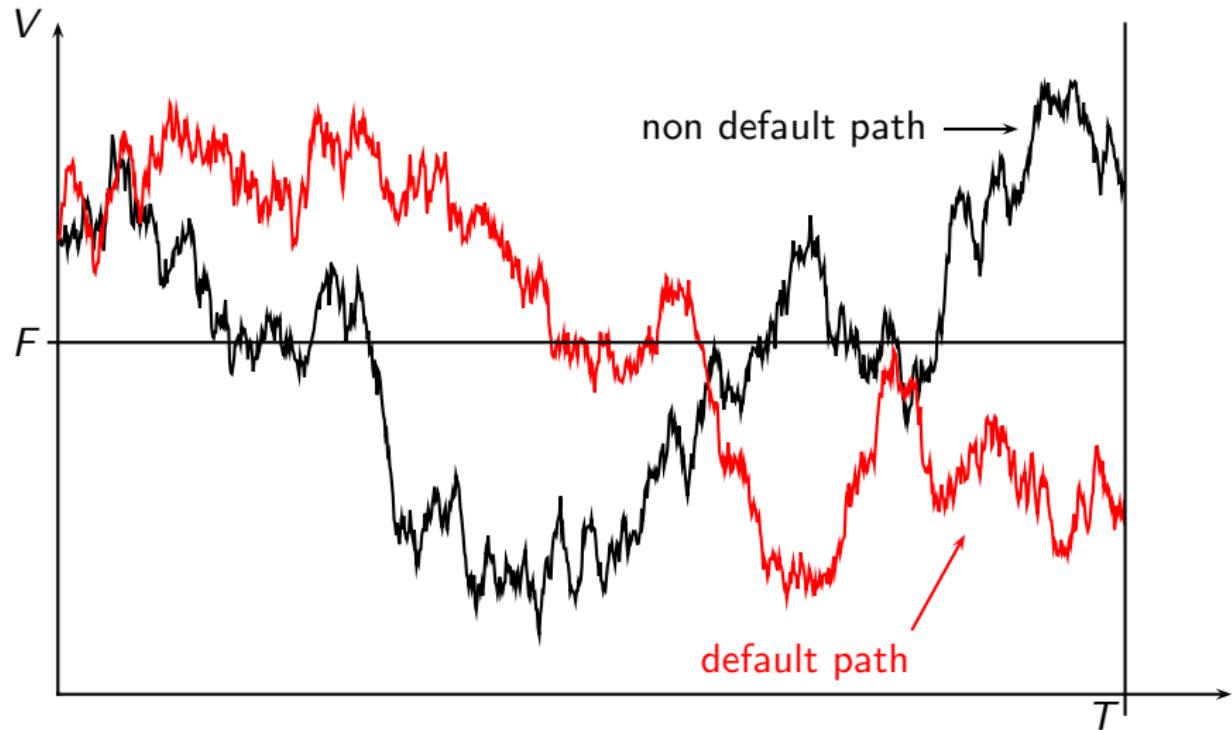
where  $\mu_V, \sigma_V > 0$  constant.

Default can only occur at maturity of the debt  $T$ . The default probability is given by

$$\mathbb{P}(V_T < F),$$

where  $F$  is the notional value of the firm's debt.

## Black-Scholes-Merton



The probability of default at time  $T$  given the asset value  $V_t$  at time  $t$  is given by

$$\begin{aligned}\mathbb{P}(V_T < F | V_t) &= \mathbb{P}(\log(V_T) < \log(F) | V_t) \\ &= \mathbb{P}\left(\frac{W_T - W_t}{\sqrt{T-t}} < -\frac{\log\left(\frac{V_t}{F}\right) + \left(\mu_V - \frac{\sigma_V^2}{2}\right)(T-t)}{\sigma_V \sqrt{T-t}}\right) \\ &= \Phi\left(-\frac{\log\left(\frac{V_t}{F}\right) + \left(\mu_V - \frac{\sigma_V^2}{2}\right)(T-t)}{\sigma_V \sqrt{T-t}}\right)\end{aligned}$$

# Structural Models

## First-Passage-Time Models (Black and Cox, 1976)

Default can now occur at any time and not only at the maturity of the debt  $T$ .

### Definition

The *passage time*  $T_b$  to a level  $b \in \mathbb{R}$  is defined by

$$T_b^Y(\omega) = \inf\{t \geq 0 : Y_t(\omega) = b\},$$

where  $Y$  is a stochastic process.

## Theorem

Consider the process  $Y$  that equals  $Y_t = at + bW_t$  with constant  $a$  and  $b$ ,  $b > 0$  and  $W$  a standard Brownian motion. Let us denote by  $m_t^Y$  the running minimum of  $Y$ , i.e.  $m_t^Y = \min_{s \in [0, t]} Y_s$ . Then the following formula is valid for every  $y \leq 0$ :

$$\mathbb{P}(m_t^Y \geq y) = \Phi\left(\frac{-y + at}{b\sqrt{t}}\right) - e^{2ayb^{-2}} \Phi\left(\frac{y + at}{b\sqrt{t}}\right),$$

where  $\Phi$  denotes the standard normal cumulative distribution function.

The relationship between this running minimum and passage times is given by

$$\mathbb{P}(m_t^Y \geq y) = \mathbb{P}(T_y^Y \geq t).$$

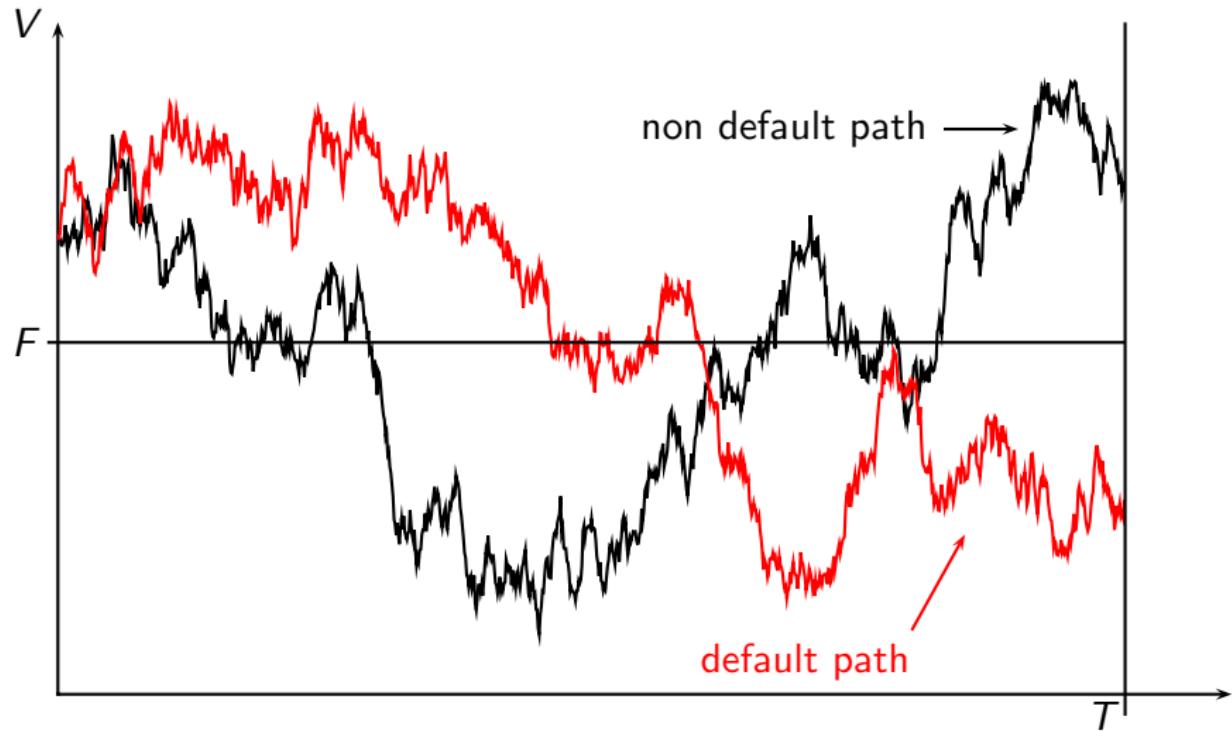
# Survival Probability

The probability that the asset value process does not reach the boundary until time  $T$  given that no default has occurred until time  $t$  is given by

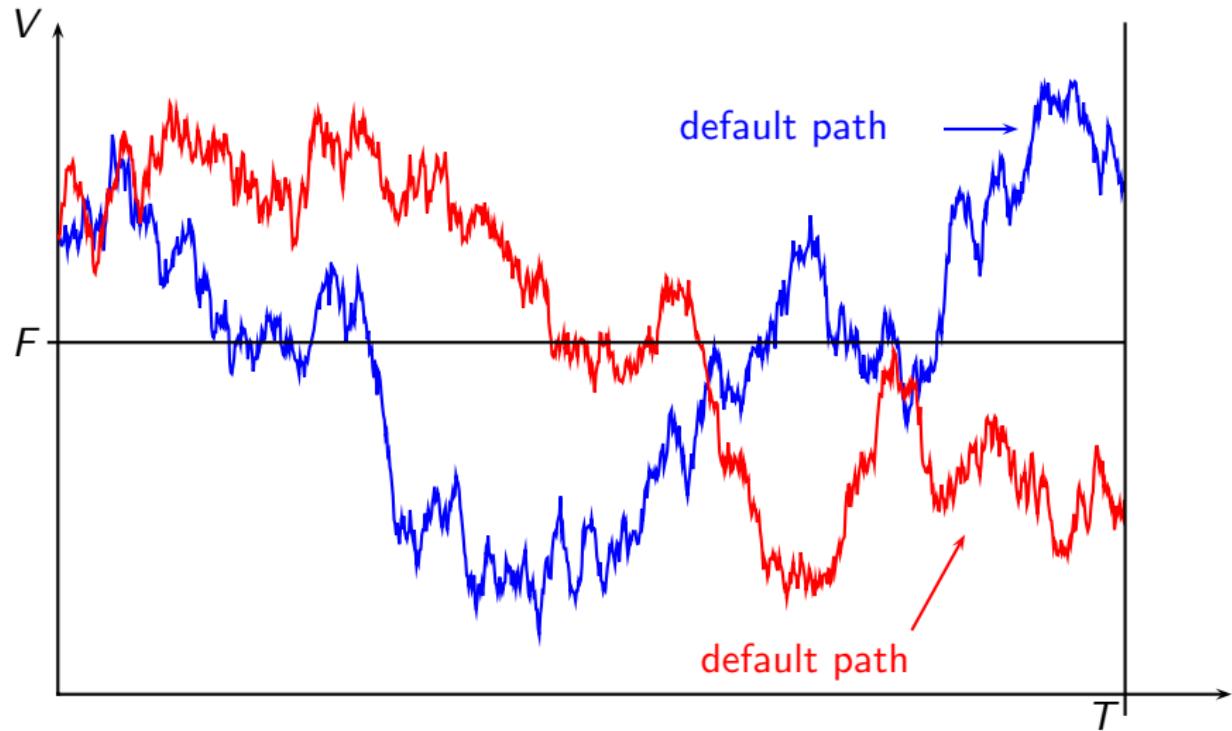
$$\begin{aligned} & \mathbb{P}(\log(V_s) \geq \log(F), \forall s \in [t, T] \mid V_t) \\ &= \mathbb{P}(m(s-t) + (W_s - W_t) \geq -DD_t, \forall s \in [t, T] \mid V_t) \\ &= \Phi\left(\frac{DD_t + m(T-t)}{\sqrt{T-t}}\right) - \exp(-2mDD_t)\Phi\left(\frac{-DD_t + m(T-t)}{\sqrt{T-t}}\right), \end{aligned}$$

where  $m := \frac{1}{\sigma_V}(\mu_V - \frac{\sigma_V^2}{2})$ ,  $DD_t := \frac{\log(V_t) - \log(F)}{\sigma_V}$ .

## Black-Scholes-Merton



## Black-Cox



# KMV-Model

## Basic idea:

Firm's equity is interpreted as a perpetual option.  
Default point is the absorbing barrier of firm's asset value.  
Default occurs as soon as the asset value hits the default point.

## Estimation requires three steps:

- ▶ Estimation of asset value and asset volatility
- ▶ Calculation of distance to default.
- ▶ Calculation of default probability.

KMV-Model is not completely published; it uses a proprietary data base.

# Estimation of Asset Value and Asset Volatility

Capital structure: equity, short-term debt, long-term debt (modelled as perpetuity).

The value of equity  $V_E$  and equity volatility  $\sigma_E > 0$  are given by

$$\begin{aligned} V_E &= f(V, \sigma_V, K, c, r) \quad \text{and} \\ \sigma_E &= g(V, \sigma_V, K, c, r), \end{aligned} \tag{2}$$

where

- ▶  $K$  leverage ratio of capital structure,
- ▶  $c$  average coupon paid on the long-term debt,
- ▶  $r$  risk-free interest rate,
- ▶ functions  $f, g$  determined by option pricing theory.

Equations (2) can be solved simultaneously for  $V$  and  $\sigma_V$ .

# Calculation of Distance-to-default

The distance to default ( $DD$ ) is given by

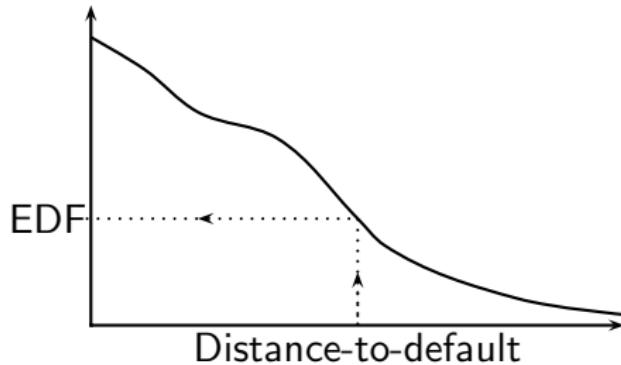
$$DD = \frac{\text{market value of assets} - \text{default point}}{(\text{market value of assets})(\text{asset volatility})},$$

where the default point ( $DP$ ) is given by

$$DP = \text{short-term debt} + \frac{1}{2} \text{ long-term debt.}$$

# Calculation of Default Probability

No analytic, strictly model-based default probability is computed.  
A large historical default database is used to assign a default probability ( $EDF=Expected\ Default\ Frequency$ ) to different levels of the distance to default.



# CreditGrades: Model Inputs and Outputs

Market  
Observables:

- ▶ Equity prices
- ▶ Equity volatility
- ▶ Risk-free Rate

+

Balance Sheet  
Information:

- ▶ Debt per share
- ▶ Recovery rate

CreditGrades  
Model:

- ▶ Credit Spreads
- ▶ Default Probabilities

CreditGrades is in contrast to KMV an **open** model.

# Model Description

In the CreditGrades model default is defined as the first time that a stochastic process  $V$  crosses the default barrier  $LD$ .

- ▶ The **stochastic process  $V$**  is the asset value process on a per share basis for the firm, and

$$dV_t = \sigma V_t dW_t,$$

with  $\sigma > 0$ .

- ▶ The **default barrier** is the amount of the firm's assets that remain in the case of default, which is  $LD$  where  $L$  is the recovery rate and  $D$  the firm's debt-per-share.

# Modelling the Recovery Rate

- ▶ Historical data show randomness of  $L$ .
- ▶ The recovery rate  $L$  is modelled as a lognormal random variable

$$L = \bar{L} e^{\lambda Z - \lambda^2/2},$$

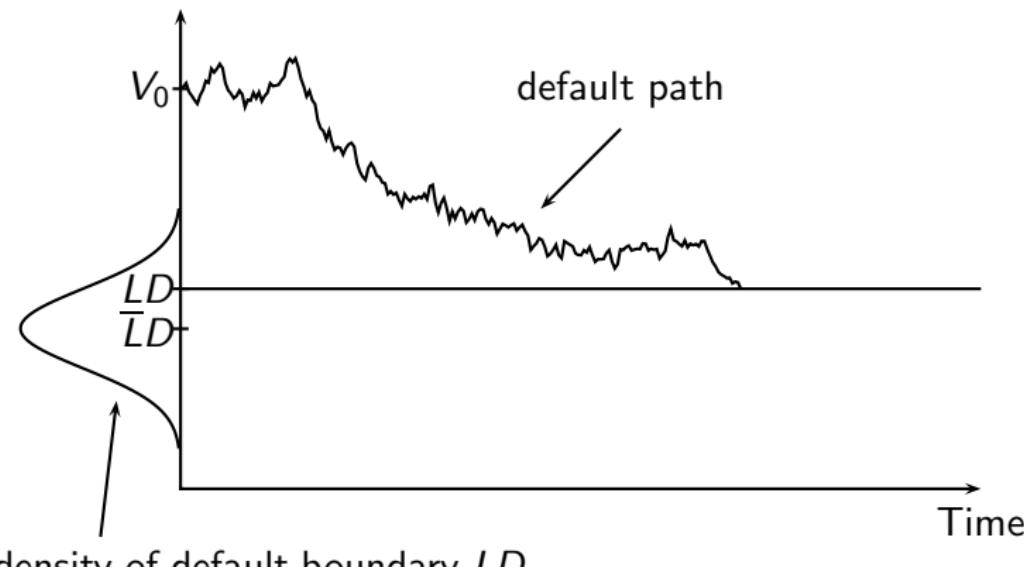
with  $\lambda, \bar{L} \in \mathbb{R}_+$ ,  $Z \sim N(0, 1)$ .

$Z$  and hence  $L$  is assumed to be independent of  $W$ .

- ▶ The default barrier  $LD$  is then given by

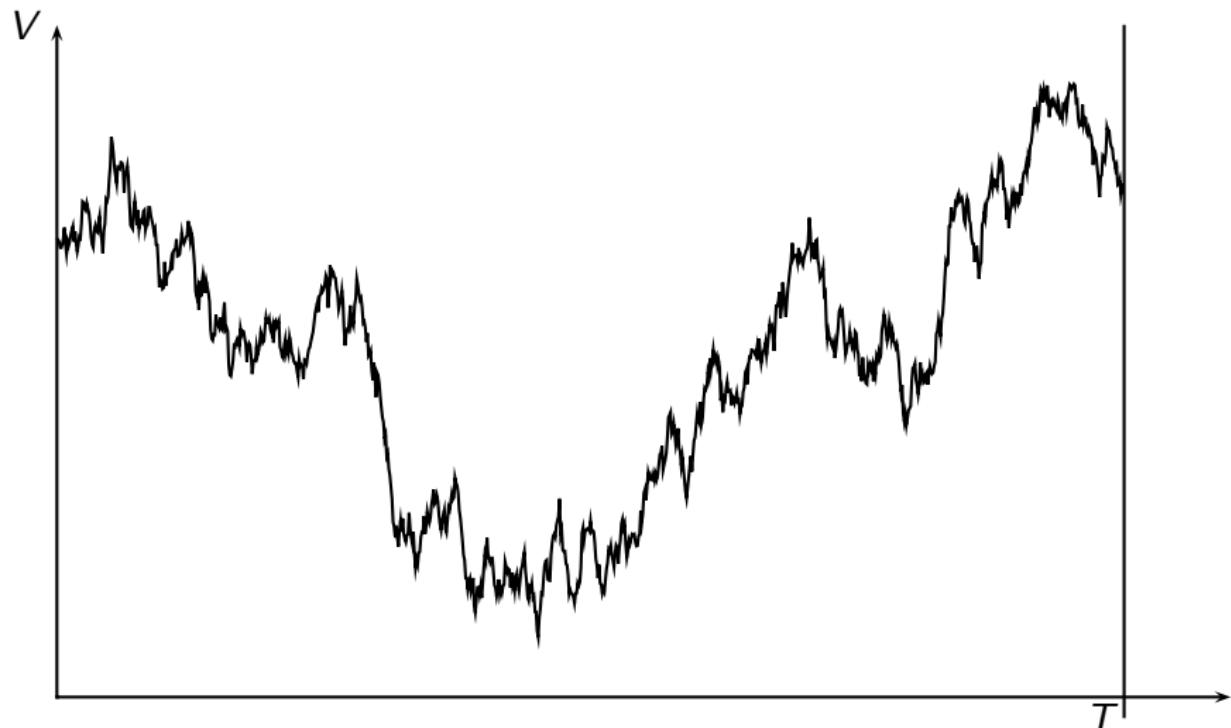
$$LD = \bar{L} D e^{\lambda Z - \lambda^2/2}.$$

# Graphical Representation I

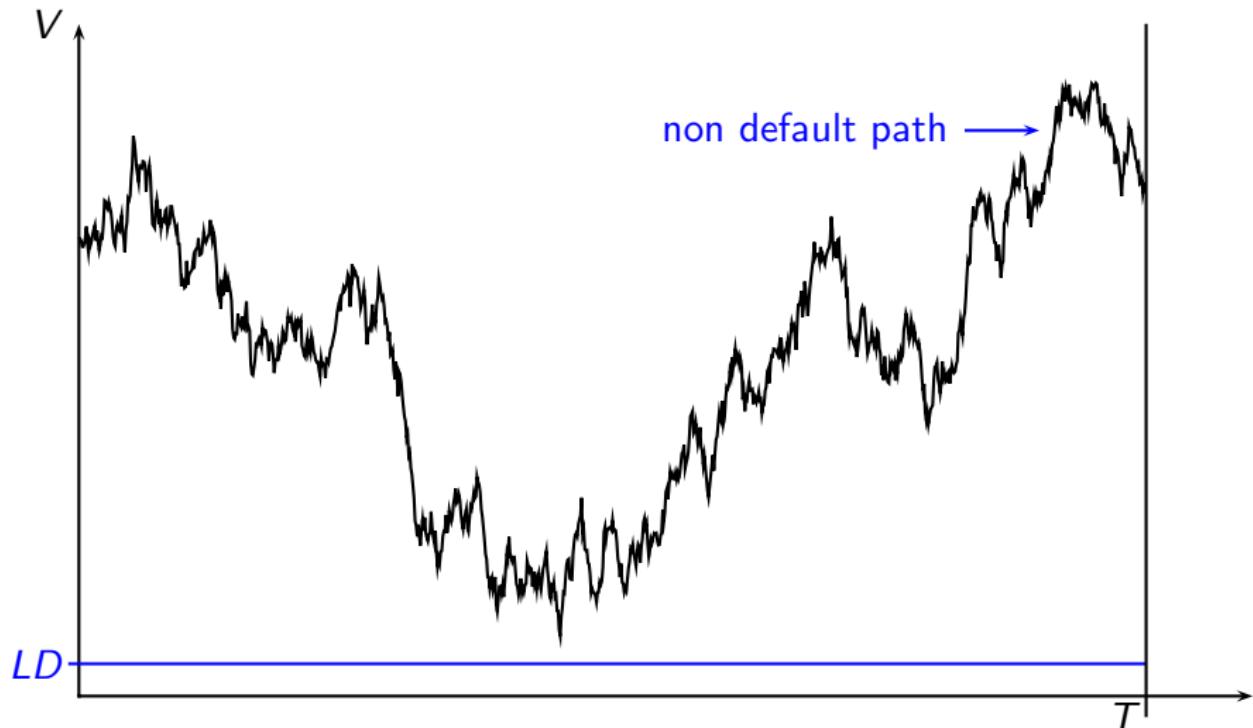


density of default boundary  $LD$

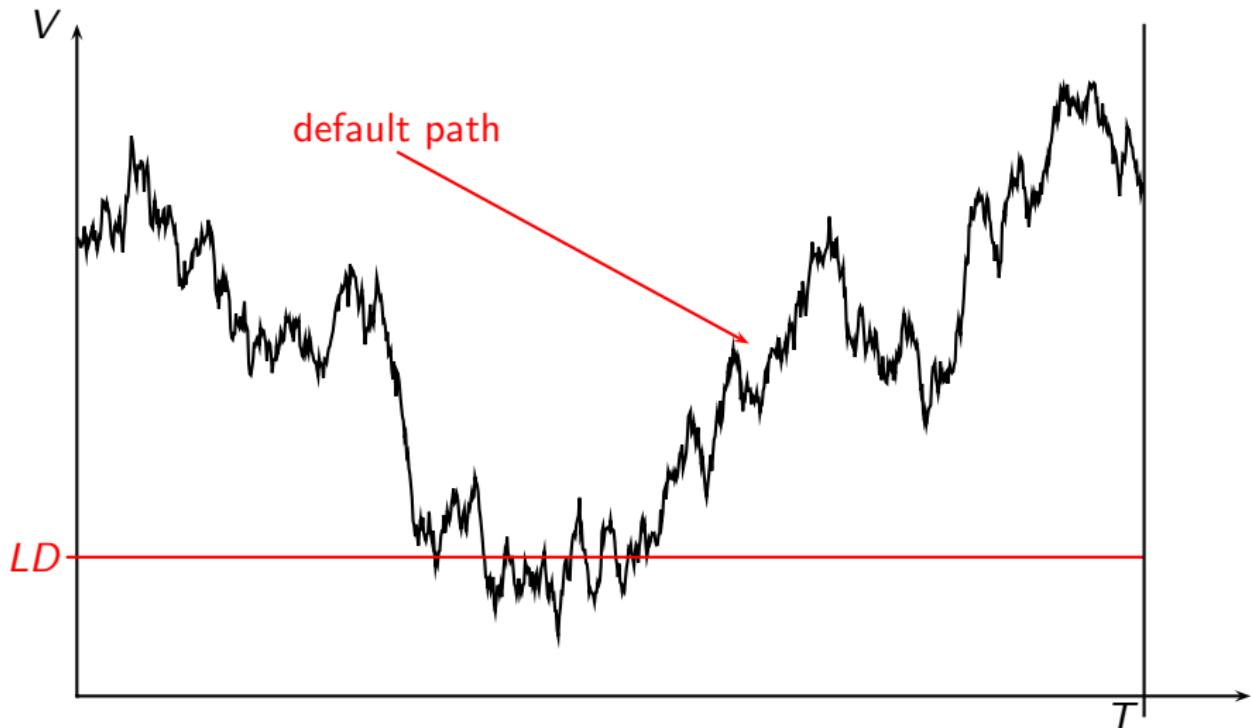
## Graphical Representation II



## Graphical Representation II



## Graphical Representation II



# Survival Probability

Default does not occur as long as

$$V_t > LD \\ \iff V_0 e^{\sigma W_t - \sigma^2 t / 2} > L D e^{\lambda Z - \lambda^2 / 2},$$

where  $V_0$  initial asset value per share.

The probability that the asset value does not reach the barrier before time  $t$  is the survival probability of the company up to time  $t$ .

This survival probability is the quantity of interest in CreditGrades.

# Computing the Survival Probability

Define a process

$$X_t = \sigma W_t - \lambda Z - \frac{\sigma^2 t}{2} - \frac{\lambda^2}{2}.$$

Then default does not occur as long as

$$X_t > \log\left(\frac{\bar{L}D}{V_0}\right) - \lambda^2.$$

We need to compute

$$\mathbb{P}(V_s > LD, \forall s < t) = \mathbb{P}(X_s > \log\left(\frac{\bar{L}D}{V_0}\right) - \lambda^2, \forall s < t).$$

# Classical CreditGrades vs. "exact" CreditGrades

## Determining the survival probability

Two possibilities to determine the survival probability:

1. (Finger et al. 2002) use an approximation  $\hat{X}$  of  $X$  and determine a closed-form formula for the survival probability.
2. However, an explicit formula for the exact survival probability can also be determined (Veraart 2004).

# Approximating the Survival Probability I

Idea in (Finger et al. 2002): Approximate the process  $X$  with a process  $\hat{X}$  which has the same expected value and variance as  $X$ .  
But  $\hat{X}$  starts in the past!

Then

$$X_t = -\frac{\sigma^2}{2} \left( t + \frac{\lambda^2}{\sigma^2} \right) + \sigma \left( W_t + \left( -\frac{\lambda}{\sigma} Z \right) \right) \quad \text{for } t \geq 0,$$

$$\hat{X}_t = -\frac{\sigma^2}{2} \left( t + \frac{\lambda^2}{\sigma^2} \right) + \sigma W_{t + \frac{\lambda^2}{\sigma^2}} \quad \text{for } t \geq -\frac{\lambda^2}{\sigma^2}.$$

Randomness of default barrier is captured via this time shift. The problem is reduced to only one random component.

## Approximating the Survival Probability II

Standard results from first-passage times of Brownian motion with drift give the (approximated) survival probability up to time  $t$

$$P(t) = \Phi\left(-\frac{A_t}{2} + \frac{\log(d)}{A_t}\right) - d\Phi\left(-\frac{A_t}{2} - \frac{\log(d)}{A_t}\right) \quad (3)$$

where

$$A_t^2 = \sigma^2 t + \lambda^2,$$

$$d = \frac{V_0 e^{\lambda^2}}{LD}.$$

The survival probability as given by (3) includes the possibility of default in the period  $(-\frac{\lambda^2}{\sigma^2}, 0]$ !

## Theorem (Veraart 2004): Exact Formula

The exact survival probability up to time  $t$  is given by

$$PE(t) = \mathbb{P}(V_s > LD, \forall s < t) = \mathbb{P}(X_s > \log\left(\frac{LD}{V_0}\right) - \lambda^2, \forall s < t)$$

$$\begin{aligned} &= \Phi_2\left(-\frac{\lambda}{2} + \frac{\log(d)}{\lambda}, -\frac{A_t}{2} + \frac{\log(d)}{A_t}; \frac{\lambda}{A_t}\right) \\ &\quad - d \Phi_2\left(\frac{\lambda}{2} + \frac{\log(d)}{\lambda}, -\frac{A_t}{2} - \frac{\log(d)}{A_t}; -\frac{\lambda}{A_t}\right), \end{aligned}$$

where  $A_t^2 = \sigma^2 t + \lambda^2$  and  $d = \frac{V_0 e^{\lambda^2}}{LD}$  and

$$\Phi_2(a, b; \rho) = \int_{-\infty}^a \int_{-\infty}^b \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} \left( \frac{x^2 - 2\rho xy + y^2}{1-\rho^2} \right)\right) dx dy$$

## Comments

- ▶ The approximated survival probability (CreditGrades formula) includes the possibility of default in the period  $(-\frac{\lambda^2}{\sigma^2}, 0]$ !
- ▶ The exact formula in (Veraart 2004) is not the formula given in (Finger et al. 2002).
- ▶ “For practical purposes, the numerical differences between the survival probabilities given by the two approaches are marginal” (Finger et al. 2002).  
**Is that still true for banks???**

## A First Example:

Comparison of the approximated five-year survival probability  $P(5)$  and our exact formula  $PE(5)$  for  $\lambda = 0.3$  and  $\bar{L} = 0.5$ .

Firm	$S_0 = S^*$	$D$	$\sigma_S^*$	$P(5)$	$PE(5)$
1	39.6	16.28	0.5	0.8688	0.8688
2	24	20.11	0.6	0.6668	0.6668
3	25.4	22.38	0.7	0.5538	0.5538
4	10.5	9.53	0.94	0.3473	0.3473
5	37.3	554.70	0.33	0.4579	0.6385

Example 1.-4. are given in (Finger et al. 2002).

Example 5: A highly leveraged firm, e.g. a bank.

# The Data<sup>1</sup>

Three banks: Commerzbank AG, Deutsche Bank AG and Bayerische HypoVereinsbank AG.

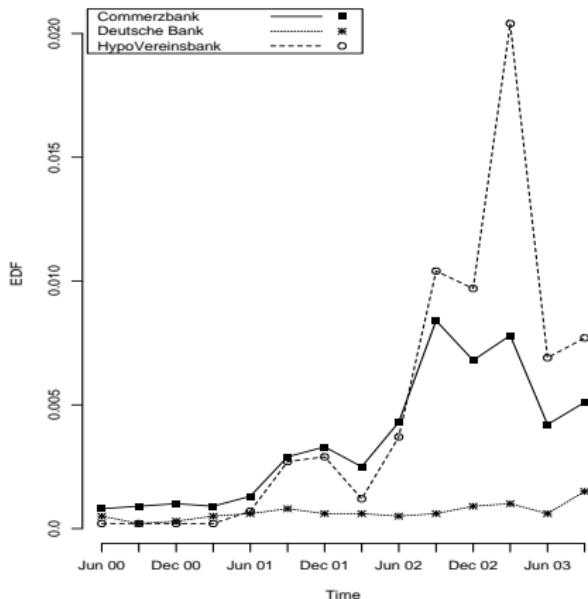
- ▶ KMV-Data: Monthly data of EDF for three banks from June 2000 until September 2003.
- ▶ Data for CreditGrades:  
Market and balance sheet data from Bloomberg.
  - ▶ Commerzbank and HypoVereinsbank: data range June 2000 until December 2003.
  - ▶ Deutsche Bank: data range March 2002 until September 2003.

Balance sheet data are available quarterly, stock price data daily (trading days).

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<sup>1</sup>The data have been kindly supplied by Deutsche Bundesbank.

# KMV-EDF

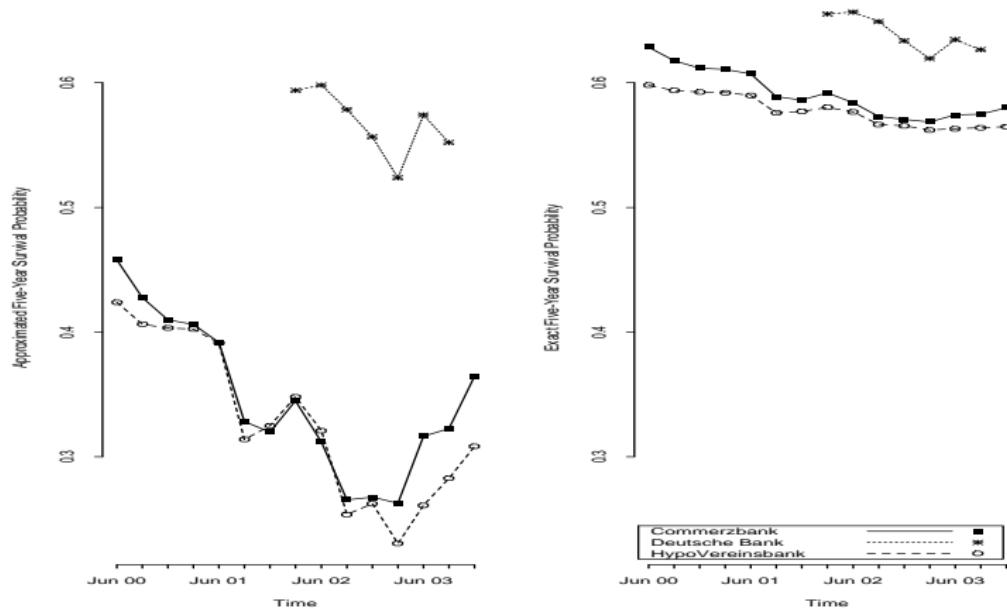


- DB lowest EDF ([0.0002, 0.0015]),
- CB highest EDF until June 2002, afterwards HVB highest EDF.
- EDF of DB quite stable in contrast to HVB and CB

# CreditGrades - Empirical Results

- ▶ Approximated vs. “exact” CreditGrades
- ▶ Choice of model parameters, in particular for  $\lambda = \mathbb{V}(\log(L))$  and computation of the debt-per-share.

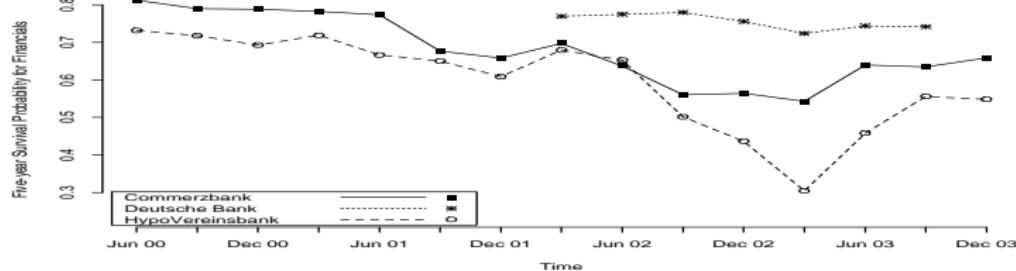
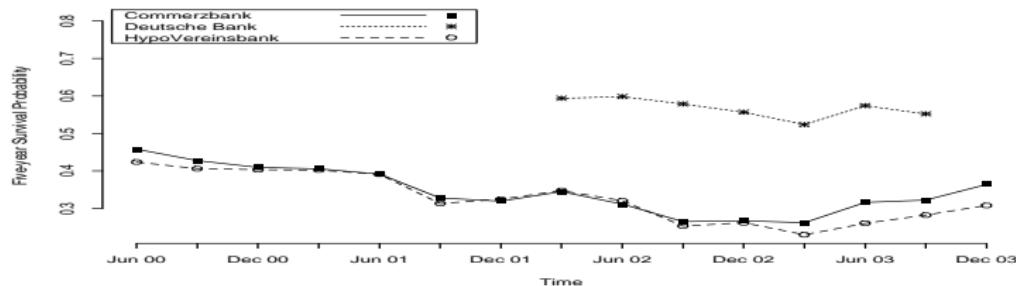
# Approximated vs. “exact” 5-year Survival Probability



## Choice of Model Parameters

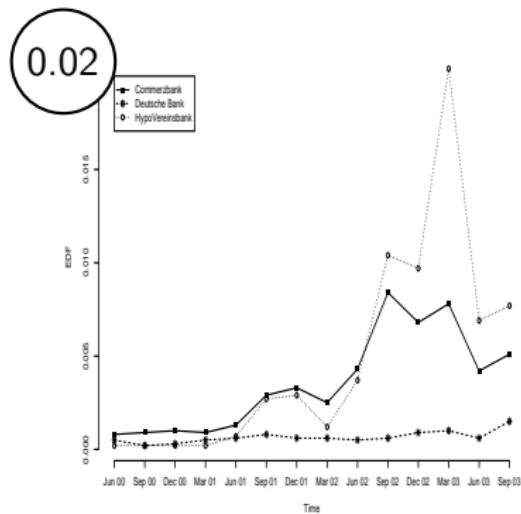
- ▶ We modify the computation of the debt per share by eliminating the short term liabilities.
- ▶ For industrials, CreditGrades uses  $\lambda = 0.3$ . Since the financial sector is strongly regulated, we expect  $\lambda$  to be lower for the financial sector. We choose  $\lambda = 0.1$ .
- ▶ Both modifications seem to improve the model.

# Choice of Model Parameters

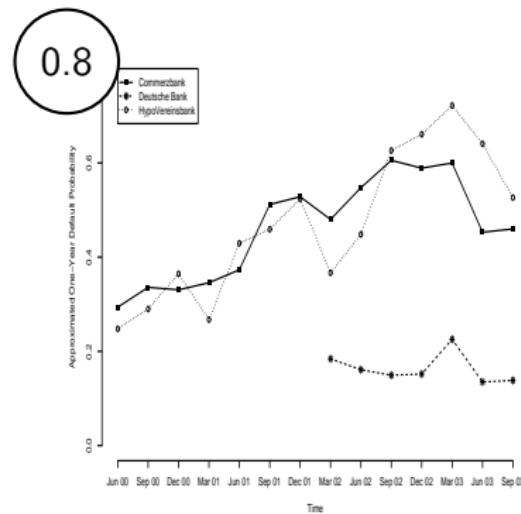


# Where we started ...

KMV

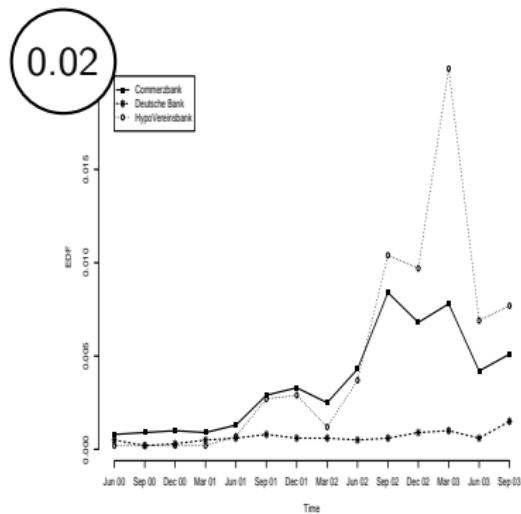


Standard CreditGrades

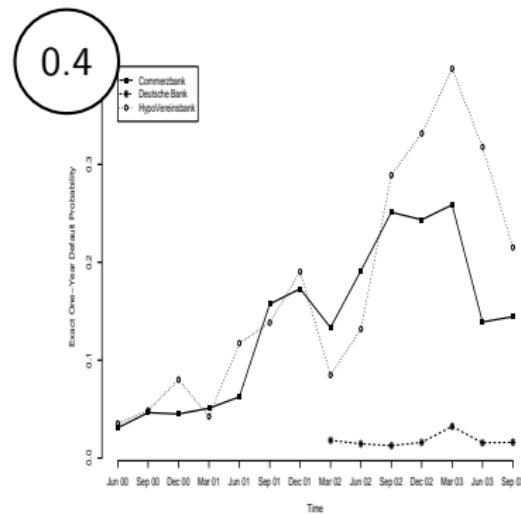


# Where we are ...

KMV



Modified exact model



# Conclusions

- ▶ Significant difference between the standard CreditGrades default probabilities and our exact formula.
- ▶ Results from our exact formula coincide much better with the KMV results than the standard CreditGrades results.
- ▶ Absolute default probabilities are still quite high.
- ▶ However, modified “exact” CreditGrades gives a good relative description of the default risk of the three banks and their behaviour over time.

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