Numbers and Numerals

… and how to represent unsigned integers with n bits

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Numbers and Numerals

- Numbers: Mathematical objects

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

- Numerals: Symbolic representation of a number

- Example for representing “fifteen”

  𝍸𝍸𝍸 (tally marks)

  XV (roman numeral)

  15 (decimal numeral)

  0xF (hexadecimal numeral)

  017 (octal numeral)

  b1111 (binary numeral)

Unsigned Decimal Numerals

- Example with four digits

$$\begin{aligned}1234 &:= 1 \cdot 1000 + 2 \cdot 100 + 3 \cdot 10 + 4 \cdot 1 \\ &= 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0\end{aligned}$$

- Digits 0, 1, 2, .., 9

- Positional notation with base 10

Unsigned Decimal Numerals with n Digits

- General notation with 4 digits

$$u(a_3, a_2, a_1, a_0)_{10} := \sum_{k=0}^{3} a_k 10^k \in \{0, \ldots, 10^4 - 1\}$$

- General notation with n digits

$$u(a_{n-1}, \ldots, a_0) = \sum_{k=0}^{n-1} a_k 10^k \quad \in \{0, \ldots, 10^n - 1\}$$

Unsigned n-Digit Numerals with Base b

- General notation with n digits

$$u(a_{n-1},\ldots,a_0)_b := \sum_{k=0}^{n-1} a_k b^k \in \{0,\ldots,b^n-1\}$$

- Example with n = 4 and b = 2

Bit = Binary Digit

$$u(a_3,a_2,a_1,a_0)_b := \sum_{k=0}^{3} a_k 2^k \in \{0,\ldots,15\}$$

most significant bit

least significant bit

Hexadecimal: Base 16

- We have 16 digits

`0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F`

`0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f`

- Convention: “Prefix 0x indicates hexadecimal”

`0x1B` $= 1 \cdot 16^1 + 11 \cdot 16^0 = 27$

`0x12` $= 1 \cdot 16^1 + 2 \cdot 16^0 = 18$

Representing Bit Patterns with Hex-Digits

- 4 bits can be represented with 1 hex-digit

```
0000 = 0   0100 = 4   1000 = 8    1100 = C
0001 = 1   0101 = 5   1001 = 9    1101 = D
0010 = 2   0110 = 6   1010 = A    1110 = E
0011 = 3   0111 = 7   1011 = B    1111 = F
```

- Preserves the value of unsigned integers

$$u(00111111_2) = u(0x3F)$$

Octal: Base 8

- We have 8 digits

```
0, 1, 2, 3, 4, 5, 6, 7
```

- Convention: “Leading zero indicates octal”

$$u(017) = 1 \cdot 8^1 + 7 \cdot 8^0 = 15$$

- 3 bits can be represented with 1 oct-digit

Bit Pattern	*Decimal*	*Hexadecimal*	*Octal*
0000	0	0x0	00
0001	1	0x1	01
0010	2	0x2	02
0011	3	0x3	03
0100	4	0x4	04
0101	5	0x5	05
0110	6	0x6	06
0111	7	0x7	07
1000	8	0x8	010
1001	9	0x9	011
1010	10	0xA	012
1011	11	0xB	013
1100	12	0xC	014
1101	13	0xD	015
1110	14	0xE	016
1111	15	0xF	017