ON THE ASSOCIATED ZONOID AND ASYMPTOTIC COVARIANCE MATRICES OF STATIONARY POISSON HYPERPLANE PROCESSES

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Outline of the talk

1. Stationary Poisson hyperplane processes in R^d and their k–flat intersection processes for $k = 0, 1, ..., d-1$
2. CLTs for the number of of intersection k–flats and their total k–volume in B_r^d
3. Basic facts on CLTs for (random) U-statistics
4. Zonoids and distributions on S_+^{d-1}
   Basic notions from convex geomtry
5. Formulae for asymptotic variances and covariances in terms of the associated zonoid
6. Some Applications and properties of the asymptotic covariance matrices

1. Stationary Poisson hyperplane processes in $\mathbb{R}^d$ and their k-flat intersection processes for $k = 0, 1, ..., d-1$

$[\Omega, \sigma(\Omega), \mathsf{P}]$: common prob.spc., E, Var = expectation, variance

$\mathcal{A}_k^d$: space of all affine k–dim. subspaces in $\mathbb{R}^d$, $k = 0, 1, ..., d-1$

Parametrization of an undirected hyperplane $H \in \mathcal{A}_{d-1}^d$:

$$H(p, v) = \{x \in \mathbb{R}^d : \langle v, x \rangle = p\} \quad p \in \mathbb{R}^1 \, , \; v \in \mathrm{S}_+^{d-1}$$

with *orientation vector* $v \in \mathrm{S}_+^{d-1}$ and signed perpendicular distance $p \in \mathbb{R}^1$ from the origin.

$d = 2$ **Poisson process** Ψ

Fig.1: Straight line in Hessian normal form

Fig. 2: Poisson points P_i on the line with marks V_i

Stationary Poisson Hyperplane Processes Φ

with *intensity* λ and *orientation distribution* $\Theta(\cdot)$ (on S_+^{d-1})

$\stackrel{\triangle}{=}$ a random point process on $[\mathcal{A}_{d-1}^d, \mathcal{B}(\mathcal{A}_{d-1}^d)]$, i.e. $\Phi : \Omega \to N(\mathcal{A}_{d-1}^d)$ is a $\Big(\sigma(\Omega), \mathcal{N}(\mathcal{A}_{d-1}^d)\Big)$–measurable mapping defined by

$$\Phi(\cdot) = \sum_{i \geq 1} \delta_{H(P_i, V_i)}(\cdot) \, , \quad \text{where} \quad \Psi(\cdot) = \sum_{i \geq 1} \delta_{[P_i, V_i]}(\cdot)$$

is a stationary and independently marked Poisson point process on the real line $\mathbb{R}^1$ with intensity λ and mark distribution $\Theta(\cdot)$ on the mark space S_+^{d-1} .

Stationary k–flat intersection process Φ_k induced by Φ

$\stackrel{\triangle}{=}$ a random point process on $[\mathcal{A}_k^d, \mathcal{B}(\mathcal{A}_k^d)]$, i.e.

$\Phi_k : \Omega \to N(\mathcal{A}_k^d)$ is a $\Big(\sigma(\Omega), \mathcal{N}(\mathcal{A}_k^d)\Big)$–measurable mapping, defined by intersections of $d-k$ distinct hyperplanes (=atoms of Φ) leading to the representation as multiple sum over $d-k$ pairwise distinct indices:

$$\Phi_k(\cdot) = \frac{1}{(d-k)!} \sum_{i_1,\ldots,i_{d-k}\geq 1}^{*} \delta_{\cap_{j=1}^{d-k} H(P_{i_j},V_{i_j})}(\cdot)$$

for $k = 0, 1, ..., d-1$, where Φ_{d-1} is identified with Φ .

Stationarity of Φ (and thus of all Φ_k , i.e. $\mathrm{supp}(\Phi_k) + x \stackrel{d}{=} \mathrm{supp}(\Phi_k) \;\; \forall x \in \mathbb{R}^d$) follows by the stationarity and the Poisson property of $\Psi(\cdot)$.

Isotropy of Φ (and thus of all Φ_k , i.e. $O(\mathrm{supp}(\Phi_k)) \stackrel{d}{=} \mathrm{supp}(\Phi_k)$ $\forall$ orthogonal O with $\det(O) = 1$) holds iff $\Theta(\cdot)$ is the uniform distribution on S_+^{d-1} .

The intensities λ_k , $k = 0, 1, ..., d-1$:

$$\lambda_k = \frac{\mathsf{E}\Phi_k(\{L \in \mathcal{A}_k^d \, : \, L \cap B_r^d \neq \emptyset\})}{\nu_{d-k}(B_r^{d-k})} \quad \text{for any} \quad r > 0 \; ,$$

where $B_r^d = \{x \in \mathbb{R}^d : \|x\| \leq r\}$ or, alternatively,

$$\lambda_k = \frac{1}{\nu_d(B)} \mathsf{E}\Big(\sum_{L \in \mathrm{supp}(\Phi_k)} \nu_k\big(\, L \cap B \, \big) \Big) \quad \text{for any} \quad B \in \mathcal{B}(\mathbb{R}^d)$$

$=$ mean total k–volume of all k–flats in the unit cube $[0,1)^d$.

2. CLTs for the number of of intersection k–flats and their total k–volume in B_r^d

$$\Psi_k^{(d)}(B_r^d) = \Phi_k(\{L \in \mathcal{A}_k^d : L \cap B_r^d \neq \emptyset\})$$

$$\stackrel{d}{=} \frac{1}{(d-k)\,!} \sum_{1 \le i_1,\ldots,i_{d-k} \le N_r}^{*} \chi(\cap_{j=1}^{d-k} H(P_{i_j}, V_{i_j}) \cap B_r^d)\,,$$

where $\chi(K) = 1$ for $K \neq \emptyset$ and $\chi(\emptyset) = 0$, and

$$\zeta_k^{(d)}(B_r^d) = \sum_{L \in \mathrm{supp}(\Phi_k)} \nu_k(L \cap B_r^d)$$

$$\stackrel{d}{=} \frac{1}{(d-k)\,!} \sum_{1 \le i_1,\ldots,i_{d-k} \le N_r}^{*} \nu_k(\cap_{j=1}^{d-k} H(P_{i_j}, V_{i_j}) \cap B_r^d)\,,$$

where $N_r = \Psi([-r,r] \times \mathrm{S}_+^{d-1}) \sim \mathrm{Poi}(2\lambda\, r)$ is independent of the i.i.d. random vectors $(P_i, V_i), i = 0, 1, \ldots$ with independent component such that $P_0 \sim U[-r,r]$ and $V_0 \sim \Theta$.

Mean values

$$\mathsf{E}\Psi_k^{(d)}(B_r^d) = \lambda_k\, \nu_{d-k}(B_r^{d-k}) = \lambda_k\, \kappa_{d-k}\, r^{d-k}$$

$$= \mathsf{E}\binom{N_r}{d-k} \mathsf{E}\chi(\cap_{i=1}^{d-k} H(P_i, V_i) \cap B_r^d \neq \emptyset)$$

$$= \frac{(2\lambda r)^{d-k}}{(d-k)!} \mathsf{P}(\cap_{i=1}^{d-k} H(P_i, V_i) \cap B_r^d \neq \emptyset)\,,$$

$$\mathsf{E}\zeta_k^{(d)}(B_r^d) = \lambda_k\, \nu_d(B_r^d) = \lambda_k\, \kappa_d\, r^d$$

$$= \frac{(2\lambda r)^{d-k}}{(d-k)!} \mathsf{E}\nu_k(\cap_{i=1}^{d-k} H(P_i, V_i) \cap B_r^d)$$

Note that always $\lambda_{d-1} = \lambda$ and $\lambda_k > 0$ for $k = 0, 1, ..., d-2$ if $\Theta(H(0,v) \cap \mathrm{S}_+^{d-1}) < 1$ for all $p \in \mathbb{R}^1$ (Non-degeneracy of Φ!)

Centered and normalized total number / k-volume in B_r^d

$$Z_{k,r}^{(d)}(\chi) := \frac{(d-k-1)!}{(\,2\lambda\, r\,)^{d-k-1/2}} \Big(\Psi_k(B_r^d) - \mathsf{E}\,\Psi_k(B_r^d) \Big)$$

and

$$Z_{k,r}^{(d)}(\nu) := \frac{(d-k-1)!\, r^{-k}}{(\,2\lambda\, r\,)^{d-k-1/2}} \Big(\zeta_k(B_r^d) - \mathsf{E}\,\zeta_k(B_r^d) \Big)$$

Theorem 1.(Heinrich, H. & V. Schmidt, to appear in 2006) Let Φ be a non-degenerate PHP in $\mathbb{R}^d$. Then, for $k = 0, \ldots, d-1$,

$$Z_{k,r}^{(d)}(\chi) \xrightarrow[r\to\infty]{\mathrm{d}} \mathcal{N}\Big(0, \sigma_{kd}^2(\chi)\Big) \quad \text{and} \quad Z_{k,r}^{(d)}(\nu) \xrightarrow[r\to\infty]{\mathrm{d}} \mathcal{N}\Big(0, \sigma_{kd}^2(\nu)\Big)\ ,$$

where

$$\begin{aligned}
\sigma_{kd}^2(\chi) &:= \mathsf{E}\chi(\cap_{i=1}^{d-k} H(P_i, V_i) \cap B_r^d)\, \chi(\cap_{i=d-k}^{2d-2k-1} H(P_i, V_i) \cap B_r^d) \\
&= \mathsf{E}\chi(\cap_{i=1}^{d-k} H(Q_i, V_i) \cap B_1^d)\, \chi(\cap_{i=d-k}^{2d-2k-1} H(Q_i, V_i) \cap B_1^d) \\
&= \mathsf{E}\,\big(\, g_{kd}^{(\chi)}(Q_0, V_0)\,\big)^2\ , \\
\sigma_{kd}^2(\nu) &:= r^{-2k}\, \mathsf{E}\nu_k(\cap_{i=1}^{d-k} H(P_i, V_i) \cap B_r^d)\, \nu_k(\cap_{i=d-k}^{2d-2k-1} H(P_i, V_i) \cap B_r^d) \\
&= \mathsf{E}\nu_k(\cap_{i=1}^{d-k} H(Q_i, V_i) \cap B_1^d)\, \nu_k(\cap_{i=d-k}^{2d-2k-1} H(Q_i, V_i) \cap B_1^d) \\
&= \mathsf{E}\big(\, g_{kd}^{(\nu)}(Q_0, V_0)\,\big)^2\ .
\end{aligned}$$

Here, Q_i , $i = 1, 2, \ldots$, are independent copies of $Q_0 \sim U[-1,1]$ and the functions $g_{kd}^{(\chi)}(p,v)$, $g_{kd}^{(\nu)}(p,v)$ defined on $[-1,1] \times \mathrm{S}_+^{d-1}$ are the *conditional expectations*

$$
\begin{aligned}
g_{kd}^{(\chi)}(p,v) &= \mathsf{E}\chi(\cap_{i=1}^{d-k-1} H(Q_i,V_i) \cap H(p,v) \cap B_1^d) \\
&= \mathsf{P}(\cap_{i=1}^{d-k-1} H(Q_i,V_i) \cap H(p,v) \cap B_1^d \neq \emptyset)
\end{aligned}
$$

and

$$
g_{kd}^{(\nu)}(p,v) = \mathsf{E}\nu_k(\cap_{i=1}^{d-k} H(Q_i,V_i) \cap H(p,v) \cap B_1^d) \, .
$$

The proof of Thm. 1 is based on techniques from the theory of U-statistics (Hoeffding's decomposition) and the following

Lemma 1. For each $k = 0, \ldots, d-1$,

$$
\begin{aligned}
\lim_{r\to\infty} \frac{\mathsf{Var}\, \Psi_k^{(d)}(B_r^d)}{r^{2d-2k-1}} &= \frac{(2\lambda)^{2d-2k-1}}{((d-k-1)!)^2}\, \sigma_{kd}^2(\chi) \, , \\
\lim_{r\to\infty} \frac{\mathsf{Var}\, \zeta_k^{(d)}(B_r^d)}{r^{2d-1}} &= \frac{(2\lambda)^{2d-2k-1}}{((d-k-1)!)^2}\, \sigma_{kd}^2(\nu) \, .
\end{aligned}
$$

PROOF.

$$
\begin{aligned}
\mathsf{Var}\, \Psi_k^{(d)}(B_r^d) &= \sum_{j=1}^{d-k} \frac{(2\lambda r)^{2d-2k-j}}{j!((d-k-j)!)^2}\, \mathsf{E}\chi(\cap_{s=1}^{d-k} H(P_s,V_s) \cap B_r^d) \\
&\qquad \times \chi(\cap_{t=d-k-j+1}^{2(d-k)-j} H(P_t,V_t) \cap B_r^d)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathsf{Var}\, \zeta_k^{(d)}(B_r^d) &= \sum_{j=1}^{d-k} \frac{(2\lambda r)^{2d-2k-j}}{j!((d-k-j)!)^2}\, \mathsf{E}\nu_k(\cap_{s=1}^{d-k} H(P_s,V_s) \cap B_r^d) \\
&\qquad \times \nu_k(\cap_{t=d-k-j+1}^{2(d-k)-j} H(P_t,V_t) \cap B_r^d) \, .
\end{aligned}
$$

Furthermore, the mean values $\mathsf{E}\Psi_k^{(d)}(B_r^d)$ and $\mathsf{E}\zeta_k^{(d)}(B_r^d)$ are expressible by $g_{kd}^{(\chi)}(p,v)$ and $g_{kd}^{(\nu)}(p,v)$, respectively:

$$\mathsf{E}\Psi_k^{(d)}(B_r^d) = \frac{(2\,\lambda)^{d-k}}{(d-k)!}\, r^{d-k}\ \mathsf{E}g_{kd}^{(\chi)}(Q_0,V_0)$$

$$\mathsf{E}\zeta_k^{(d)}(B_r^d) = \frac{(2\,\lambda)^{d-k}}{(d-k)!}\, r^{d}\ \mathsf{E}g_{kd}^{(\nu)}(Q_0,V_0)$$

Aim: Finding expression for $g_{kd}^{(\chi)}(p,v)$ and $g_{kd}^{(\nu)}(p,v)$ in dependence of Θ (Use Crofton's formula if Θ is uniform!)

$$\begin{aligned} g_{kd}^{(\chi)}(p,v) &= \mathsf{P}(\,\cap_{i=1}^{d-k-1} H(Q_i,V_i)\cap H(p,v)\cap B_1^d \neq \emptyset\,) \\ &= \frac{1}{2^{d-k-1}}\int_{\mathrm{S}_+^{d-1}}\cdots\int_{\mathrm{S}_+^{d-1}}\int_{-1}^{1}\cdots\int_{-1}^{1}\chi(\cap_{i=1}^{d-k} H(q_i,v_i) \\ &\times\ \cap H(p,v)\ \cap\ B_1^d)\,\mathrm{d}q_1\ldots\mathrm{d}q_{d-k-1}\,\Theta(\mathrm{d}v_1)\ldots\Theta(\mathrm{d}v_{d-k-1}) \end{aligned}$$

$$\begin{aligned} g_{kd}^{(\nu)}(p,v) &= \mathsf{E}\nu_k(\cap_{i=1}^{d-k} H(Q_i,V_i)\cap H(p,v)\cap B_1^d) \\ &= \frac{1}{2^{d-k-1}}\int_{\mathrm{S}_+^{d-1}}\cdots\int_{\mathrm{S}_+^{d-1}}\int_{-1}^{1}\cdots\int_{-1}^{1}\nu_k(\cap_{i=1}^{d-k} H(q_i,v_i) \\ &\times\ \cap H(p,v)\ \cap\ B_1^d)\,\mathrm{d}q_1\ldots\mathrm{d}q_{d-k-1}\,\Theta(\mathrm{d}v_1)\ldots\Theta(\mathrm{d}v_{d-k-1})\ . \end{aligned}$$

3. Basic facts on (random) U–statistics

$X_1, X_2, \ldots$ be a sequence of i.i.d. random elements in some meas. space $[E, \sigma(E)]$ and, for fixed $m \geq 2$, let $f : E^m \to \mathbb{R}^1$ be a meas. symmetric function such that $\mathsf{E}|f(X_1, \ldots, X_m)| < \infty$.

Def.: A U-statistic $U_n^{(m)}(f)$ of order m with *kernel function* f is defined by

$$U_n^{(m)}(f) = \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} f(X_{i_1}, \ldots, X_{i_m}) \quad \text{for} \quad n \geq m .$$

Note that $U_n^{(m)}(f)$ is an unbiased estimator for $\mu = \mathsf{E}f(X_1, \ldots, X_m)$.

By *Hoeffding's decomposition*,

$$U_n^{(m)}(f) - \mu \; = \; \frac{m}{n} \sum_{i=1}^{n} (\, g(X_i) - \mu \,) \; + \; R_n^{(m)}(f) \; ,$$

where $g(x) = \mathsf{E}(\, f(X_1, X_2, ..., X_m) \,|\, X_1 = x \,) = \mathsf{E}f(x, X_2, ..., X_m)$.

and (as cucial outcome of Hoeffding's decomposition)

$$\mathsf{E}\big(R_n^{(m)}(f)\big)^2 \leq \frac{c_m}{n^2} \, \mathsf{E}\, f^2(X_1, \ldots, X_m) \quad \text{for} \quad n \geq m$$

and for some constant $c_m < \infty$ only depending on m.

Hoeffding's CLT for U–statistics

If $\mathsf{E}f^2(X_1, \ldots, X_m) < \infty$, then

$$\sqrt{n} \, \big(\, U_n^{(m)}(f) - \mu \, \big) \underset{n \to \infty}{\overset{\mathrm{d}}{\longrightarrow}} \mathcal{N}\big(0, m^2(\mathsf{E}\, g^2(X_1) - \mu^2)\big) \, ,$$

where $\overset{\mathrm{d}}{\longrightarrow}$ denotes convergence in distribution.

4. Zoonoids and distributions on S_+^{d-1}

Basic notions from convex geometry

1. Support function of a convex body $K \subset \mathbb{R}^d$:

$$h(K,u) = \max_{x \in K} \langle u, x \rangle \quad \text{for} \quad u \in \mathbb{R}^d$$

Geometric meaning:

$h(K,u)$ = signed perpendicular distance of the support plane to K with exterior normal vector u from the origin (for $u \in \mathrm{S}^{d-1}$)

Important properties:

$h(\alpha\, K \oplus \beta\, L, \cdot) = \alpha\, h(K,\cdot) + \beta\, h(L,\cdot)$ for $\alpha, \beta \geq 0$

$d_H(K,L) = \max_{u \in \mathrm{S}^{d-1}} |\, h(K,u) - h(L,u)\,|\;,\; h(K,\cdot) \in C(\mathrm{S}^{d-1})$

$h(K, \gamma\, u) = \gamma\, h(K,u)$ and $h(K, u+v) \leq h(K,u) + h(K,v)$ for $\gamma \geq 0\,;\, u,v \in \mathbb{R}^d$

i.e. support functions of convex bodies are *sublinear* on $\mathbb{R}^d$. Conversely, if $h : \mathbb{R}^d \mapsto \mathbb{R}^1$ is a sublinear function, then there exists a unique convex body in $\mathbb{R}^d$ with support function $h(\cdot)$.

Examples:

1. Ellipse E : $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1 \Rightarrow h(E, u_1, u_2) = \sqrt{a^2\, u_1^2 + b^2\, u_2^2}$
2. Diamond D : $|x_1| + |x_2| \leq 1 \Rightarrow h(D, u_1, u_2) = \max\{|u_1|, |u_2|\}$
3. Square S : $\max\{|x_1|, |x_2|\} \leq 1 \Rightarrow h(S, u_1, u_2) = |u_1| + |u_2|$

Fig. 3: Illustration of the support function of a planar ovoid

2. Centrally symmetric zonotopes in $\mathbb{R}^d$:

A *zonotope* in $\mathbb{R}^d$ is a polytope of the form $Z = S_1 \oplus \cdots \oplus S_n$ (= Minkowski sum of the segments $S_1, ..., S_n$);
Z is *centrally symmetric*, if it generated by segments of the form $S_i = \text{conv}\{-\alpha_i u_i, \alpha_i u_i\}$ with $u_i \in \mathrm{S}_+^{d-1}$, $\alpha_i > 0$ for $i = 1, ..., n$.

$$h(Z, u) = \sum_{i=1}^{n} \alpha_i \, |\langle u, u_i \rangle| \quad \text{for} \quad u \in \mathbb{R}^d$$

Fig. 4: Illustration of a planar zonotope generated by $n = 3$ segments

$S_1 = \text{conv}\{-(\alpha_1, 0), (\alpha_1, 0)\}$, $S_2 = \text{conv}\{-(0, \alpha_2), (0, \alpha_2)\}$, $S_3 = \text{conv}\{-(\alpha_3, \alpha_3), (\alpha_3, \alpha_3)\} \Rightarrow Z = [-\alpha_1, \alpha_1] \times [-\alpha_2, \alpha_2] \oplus S_3$.

Note that any centrally symmetric polygon is a zonotope in $\mathbb{R}^2$. In $\mathbb{R}^d$, $d \geq 3$, further conditions are needed.

3. Zoonoids in $\mathbb{R}^d$:

= limits of sequences of centrally symmetric zonotopes in $\mathbb{R}^d$ (in the sense of the Hausdorff metric)

or equivalently

A centrally symmetric convex body Z_ϱ in $\mathbb{R}^d$ is said to be a *zonoid* with generating measure $\varrho(\cdot)$ if its support function can represented in the form

$$h(Z_\varrho, u) = \int_{\mathrm{S}_+^{d-1}} |\langle u, v \rangle| \, \varrho(\mathrm{d}v) \quad \text{for} \quad u \in \mathbb{R}^d$$

with some (finite) measure $\varrho(\cdot)$ on S^{d-1}.

Note that there is a ono-to-one correspondence between zonoids and their generating measures, i.e.

$$Z_{\varrho_1} = Z_{\varrho_2} \Longleftrightarrow h(Z_{\varrho_1}, \cdot) = h(Z_{\varrho_2}, \cdot) \Longleftrightarrow \varrho_1(\cdot) = \varrho_2(\cdot)$$

Hence, to any given orientation distribution $\Theta(\cdot)$ of a stationary PHP there exists a zonoid Z_Θ such that

$$h(Z_\Theta, u) = \int\limits_{\mathrm{S}_+^{d-1}} |\langle u, v\rangle|\, \Theta(\mathrm{d}v) \quad \text{for} \quad u \in \mathrm{I\!R}^d \,.$$

$$\Theta(\cdot) = \frac{\nu_{d-1}(\cdot)}{d\,\kappa_d/2} \quad \text{uniform distribution} \quad \Longleftrightarrow Z_\Theta = \frac{\kappa_{d-1}}{d\,\kappa_d}\, B_1^d$$

4. Intrinsic volumes $V_k^{(d)}(K)$ of a convex body $K \subset \mathrm{I\!R}^d$

Definition by means of Steiner's formula:

$$\nu_d(K \oplus B_r^d) = \sum_{k=0}^{d} r^{d-k}\, \kappa_{d-k}\, V_k^{(d)}(K) \quad \text{for} \quad r \geq 0$$

$$V_0^{(d)}(K) = 1 \;,\; V_d^{(d)}(K) = \nu_d(K) \;,\; V_{d-1}^{(d)}(K) = \frac{1}{2}\, \nu_{d-1}(\partial K)$$

For the zonoid Z_Θ associated with Θ it holds (cf. W. Weil (1979))

$$V_k^{(d)}(Z_\Theta) = \frac{1}{k} \int\limits_{\mathrm{S}_+^{d-1}} V_{k-1}^{(d-1)}(Z_\Theta^v)\, \Theta(\mathrm{d}v) \quad \text{for} \quad k = 1, ..., d\,,$$

where K^v denotes the image of the convex body K under orthogonal projection onto the hyperplane $H(0, v)$.

5. Formulae for asymptotic variances and covariances in terms of the associated zonoid

Theorem 2. Let Φ be a non-degenerate PHP in $\mathbb{R}^d$.
Then, for $k = 0, \ldots, d-1$ and $(p, v) \in [-1, 1] \times \mathrm{S}_+^{d-1}$,

$$g_{kd}^{(\chi)}(p, v) = \frac{(d-k-1)!\,\kappa_{d-k-1}}{2^{d-k-1}} \left(1 - p^2\right)^{(d-k-1)/2} V_{d-k-1}^{(d-1)}(Z_\Theta^v)$$

$$g_{kd}^{(\nu)}(p, v) = \frac{(d-k-1)!\,\kappa_{d-1}}{2^{d-k-1}} \left(1 - p^2\right)^{(d-1)/2} V_{d-k-1}^{(d-1)}(Z_\Theta^v) \,.$$

$\Longrightarrow$ (together with $\int_{-1}^{1}(1-p^2)^{j/2}\,\mathrm{d}p = \kappa_{j+1}/\kappa_j$)

$$\mathsf{E}g_{kd}^{(\chi)}(Q_0, V_0) = \frac{(d-k-1)!\,\kappa_{d-k}}{2^{d-k}} \int_{\mathrm{S}_+^{d-1}} V_{d-k-1}^{(d-1)}(Z_\Theta^v)\,\Theta(\mathrm{d}v)$$

$$= \frac{(d-k)!\,\kappa_{d-k}}{2^{d-k}} V_{d-k}^{(d)}(Z_\Theta)$$

$$\Longrightarrow \quad \lambda_k = \lambda^{d-k}\, V_{d-k}^{(d)}(Z_\Theta)$$

$$\mathsf{E}g_{kd}^{(\nu)}(Q_0, V_0) = \frac{(d-k)!\,\kappa_d}{2^{d-k}} V_{d-k}^{(d)}(Z_\Theta)$$

$$\sigma_{kd}^2(\chi) = \frac{\left(((d-k-1)!)^2\,\kappa_{d-k-1}\right)^2}{(2d-2k-1)!} \int_{\mathrm{S}_+^{d-1}} \left(V_{d-k-1}^{(d-1)}(Z_\Theta^v)\right)^2 \Theta(\mathrm{d}v)$$

$$\sigma_{kd}^2(\nu) = \frac{\left(2^k\,(d-k-1)!\,(d-1)!\,\kappa_{d-1}\right)^2}{(2d-1)!} \int_{\mathrm{S}_+^{d-1}} \left(V_{d-k-1}^{(d-1)}(Z_\Theta^v)\right)^2 \Theta(\mathrm{d}v) \,.$$

Isotropic case:

$$Z_\Theta = \frac{\kappa_{d-1}}{d\,\kappa_d}\,B_1^d \Longrightarrow V_{d-k}^{(d)}(Z_\Theta) = \left(\frac{\kappa_{d-1}}{d\,\kappa_d}\right)^{d-k}\binom{d}{k}\frac{\kappa_d}{\kappa_k}$$

$\Longrightarrow$

$$Z_\Theta^v = \frac{\kappa_{d-1}}{d\,\kappa_d}\,B_1^{d-1}\,,\; V_{d-k-1}^{(d-1)}(Z_\Theta^v) = \left(\frac{\kappa_{d-1}}{d\,\kappa_d}\right)^{d-k-1}\binom{d-1}{k}\frac{\kappa_{d-1}}{\kappa_k}\,,$$

where

$$\kappa_d := \nu_d(B_1^d) = \pi^{\frac{d}{2}}/\Gamma\Big(\frac{d}{2}+1\Big)\,.$$

Solution of the *isoperimetric* problem yields

$$\max_\Theta \lambda_0 = \lambda_0\Big|_{\Theta=\text{unif.dist.}} = \lambda^d\,\kappa_d\left(\frac{\kappa_{d-1}}{d\,\kappa_d}\right)^d \quad \text{for fixed} \quad \lambda_{d-1}=\lambda$$

Lemma 2. Let Θ be the uniform distribution on S_+^{d-1} and $k\in\{0,1,\ldots,d-1\}$. Then, for $k=0,1,\ldots,d-1$,

$$\lambda_k = \binom{d}{k}\frac{\kappa_d}{\kappa_k}\left(\frac{\kappa_{d-1}}{d\,\kappa_d}\right)^{d-k}\lambda^{d-k}\,,$$

$$\sigma_{kd}^{(\chi)} = \frac{(\kappa_{d-k-1}\,(d-k-1)!)^2}{(2d-2k-1)!}\left(\frac{d!\,\kappa_d}{k!\,\kappa_k}\right)^2\left(\frac{\kappa_{d-1}}{d\,\kappa_d}\right)^{2(d-k)}$$

and

$$\sigma_{kd}^{\nu)} = \frac{\big(2^k\,\kappa_{d-1}\,(d-1)!\,\big)^2}{(2d-1)!}\left(\frac{d!\,\kappa_d}{k!\,\kappa_k}\right)^2\left(\frac{\kappa_{d-1}}{d\,\kappa_d}\right)^{2(d-k)}\,.$$

Multivariate CLTs : Joint asymptotic behaviour (as $r \longrightarrow \infty$) of the closely correlated random variables $Z_{0,r}^{(d)}(\chi)$, ..., $Z_{d-1,r}^{(d)}(\chi)$ as well as of $Z_{0,r}^{(d)}(\nu)$, ..., $Z_{d-1,r}^{(d)}(\nu)$

Theorem 3. (Heinrich, H. & V. Schmidt, to appear in 2006) Let Φ be a non-degenerate PHP in $\mathbb{R}^d$. Then,

$$\big(Z_{k,r}^{(d)}(\chi)\big)_{k=0}^{d-1} \xrightarrow[r\to\infty]{\mathrm{d}} \mathcal{N}(0, \Sigma^{(d)}(\chi)) \quad \text{and} \quad \big(Z_{k,r}^{(d)}(\nu)\big)_{k=0}^{d-1} \xrightarrow[r\to\infty]{\mathrm{d}} \mathcal{N}(0, \Sigma^{(d)}(\nu))$$

with (asymptotic) covariance matrices

$$\Sigma^{(d)}(\chi) = \big(\sigma_{kl}^{(d)}(\chi)\big)_{k,l=0}^{d-1} \quad \text{and} \quad \Sigma^{(d)}(\nu) = \big(\sigma_{kl}^{(d)}(\nu)\big)_{k,l=0}^{d-1},$$

where the mixed second moments are given by (cf. Heinrich & Schertz (2006))

$$\begin{aligned} \sigma_{kl}^{(d)}(\chi) &= \mathsf{E}\big(g_{kd}^{(\chi)}(Q_0, V_0)\, g_{ld}^{(\chi)}(Q_0, V_0)\big) \\ &= a_{kl}^{(d)} \int_{\mathrm{S}_+^{d-1}} V_{d-k-1}^{(d-1)}(Z_\Theta^v)\, V_{d-l-1}^{(d-1)}(Z_\Theta^v)\, \Theta(\mathrm{d}v) \end{aligned}$$

with $a_{kl}^{(d)} = \kappa_{d-k-1}\, \kappa_{d-l-1} \Big(\Gamma\big(d - \frac{k+l}{2}\big)\Big)^2 \frac{(d-k-1)!\,(d-l-1)!}{(2d-k-l-1)!}$

and

$$\begin{aligned} \sigma_{kl}^{(d)}(\nu) &= \mathsf{E}\big(g_{kd}^{(\nu)}(Q_0, V_0)\, g_{ld}^{(\nu)}(Q_0, V_0)\big) \\ &= b_{kl}^{(d)} \int_{\mathrm{S}_+^{d-1}} V_{d-k-1}^{(d-1)}(Z_\Theta^v)\, V_{d-l-1}^{(d-1)}(Z_\Theta^v)\, \Theta(\mathrm{d}v) \end{aligned}$$

with $b_{kl}^{(d)} = (d-k-1)!\,(d-l-1)!\,2^{k+l}\,\kappa_{d-1}^2 \frac{((d-1)!)^2}{(2d-1)!}$.

Computation of the mixed second-order moments when Φ isotropic

Lemma 3. Let Θ be the uniform distribution on S_{+}^{d-1} and let $k, l \in \{0, \ldots, d-1\}$. Then,

$$\sigma_{kl}^{(d)}(\chi) = \frac{(\,d!\,\kappa_d\,)^2\,\kappa_{d-k-1}\,\kappa_{d-l-1}}{k!\,l!\,\kappa_k\,\kappa_l\,2^{2d-k-l-1}}\,\frac{\kappa_{2d-k-l-1}}{\kappa_{2d-k-l-2}}\left(\frac{\kappa_{d-1}}{d\,\kappa_d}\right)^{2d-k-l}$$

$$= \sqrt{\sigma_{kk}^{(d)}(\chi)\,\sigma_{ll}^{(d)}(\chi)}\,\frac{\mathrm{B}\Bigl(\frac{2d-k-l}{2}, \frac{2d-k-l}{2}\Bigr)}{\sqrt{\mathrm{B}(d-k, d-k)\,\mathrm{B}(d-l, d-l)}}\,,$$

where $\mathrm{B}(s,t) := \int_0^1 x^{s-1}\,(1-x)^{t-1}\,dx$ denotes Euler's Beta function, and

$$\sigma_{kl}^{(d)}(\nu) = \frac{(\kappa_d\,d!\,(d-1)!)^2\,2^{k+l}}{k!\,l!\,\kappa_k\,\kappa_l\,(2d-1)!}\left(\frac{\kappa_{d-1}}{d\,\kappa_d}\right)^{2d-k-l} = \sqrt{\sigma_{kk}^{(d)}(\nu)\,\sigma_{ll}^{(d)}(\nu)}\,.$$

Corollary. The rank of the covariance matrix $\Sigma^{(d)}(\nu)$ equals 1 for any dimension $d \geq 1$, whereas $\Sigma^{(d)}(\chi)$ possesses always full rank d.

Furthermore,

$$\frac{Z_{l,r}^{(d)}(\nu)}{\sqrt{\sigma_{ll}^{(d)}(\nu)}} - \frac{Z_{k,r}^{(d)}(\nu)}{\sqrt{\sigma_{kk}^{(d)}(\nu)}} \underset{r\to\infty}{\overset{\mathsf{P}}{\longrightarrow}} 0 \qquad \text{for} \quad 0 \leq k < l \leq d-1\,.$$

6. Some Applications and properties of the asymptotic covariance matrices

Let Φ be a stationary PHP with intensity $\lambda > 0$ and non-degenerate orientation distribution Θ.

Then

$$\widehat{\lambda}^{(d)}_{k,r} := \frac{\Psi^{(d)}_k(B^d_r)}{\nu_{d-k}(B^{d-k}_r)} \quad \text{and} \quad \widetilde{\lambda}^{(d)}_{k,r} := \frac{\zeta^{(d)}_k(B^d_r)}{\nu_d(B^d_r)}$$

are unbiased, strongly consistent estimators for the intensity λ_k of the stationary k–flat intersection process generated by Φ.
(! Φ is ergodic, even mixing, cf. Schneider & Weil (2000))

Lemma 1 and Theorem 2 yield

$$\lim_{r\to\infty} r\,\mathsf{Var}\left(\widehat{\lambda}^{(d)}_{k,r}\right) = \lambda^{2d-2k-1}\, c_{d-k} \int_{S^{d-1}_+} \left(V^{(d-1)}_{d-k-1}(Z^v_\Theta)\right)^2 \Theta(\mathrm{d}v)$$

$$\lim_{r\to\infty} r\,\mathsf{Var}\left(\widetilde{\lambda}^{(d)}_{k,r}\right) = \lambda^{2d-2k-1}\, c_{d} \int_{S^{d-1}_+} \left(V^{(d-1)}_{d-k-1}(Z^v_\Theta)\right)^2 \Theta(\mathrm{d}v)$$

for $k = 0, ..., d-1$, where $c_j = \frac{2^{2j-1}}{(2j-1)!}\left(\frac{(j-1)!\,\kappa_{j-1}}{\kappa_j}\right)^2$ for $j \geq 1$.
This implies that

$$\lim_{r\to\infty} \frac{\mathsf{Var}\left(\widehat{\lambda}^{(d)}_{k,r}\right)}{\mathsf{Var}\left(\widetilde{\lambda}^{(d)}_{k,r}\right)} = \frac{c_{d-k}}{c_d} < 1\,,$$

since $c_j < c_{j+1}$ for $j \geq 1$. Thus, it holds that

$$\lim_{r\to\infty} r\,\mathsf{Var}\left(\widehat{\lambda}^{(d)}_{k,r}\right) < \lim_{r\to\infty} r\,\mathsf{Var}\left(\widetilde{\lambda}^{(d)}_{k,r}\right) \quad \text{for} \quad k = 1, ..., d-1\,.$$

The rank of the covariance matrices $\Sigma^{(d)}(\chi)$ and $\Sigma^{(d)}(\nu)$

Theorem 4.(Heinrich & Schertz (2006)) We have

1. $\operatorname{rank}\big(\Sigma^{(d)}(\chi)\big) = d \Longleftrightarrow \nu_d(Z_\Theta) = V_d^{(d)}(Z_\Theta) > 0$
   (iff $\Theta(\cdot)$ is non-degenerate)

2. $\operatorname{rank}\big(\Sigma^{(d)}(\nu)\big) = d \Longleftrightarrow \Theta(\cdot)$ is non-degenerate and there is no vector $(t_0, \ldots, t_{d-1}) \neq 0$ such that

$$\Theta\Big(\{u \in \mathrm{S}_+^{d-1} : \sum_{k=0}^{d-1} t_k \, V_k^{(d-1)}(Z_\Theta^u) = 0\}\Big) = 1$$

3. $\operatorname{rank}\big(\Sigma^{(d)}(\nu)\big) = 1 \Longleftrightarrow$ For $k = 2, \ldots, d\,$,

$$\Theta\Big(\{u \in \mathrm{S}_+^{d-1} : V_{k-1}^{(d-1)}(Z_\Theta^u) = \text{const} \ \ (= k\, V_k^{(d)}(Z_\Theta)\,)\}\Big) = 1\ .$$

Special case $d = 2$:

$$\Sigma^{(2)}(\chi) = \begin{pmatrix} \frac{2}{3}\int \big(V_1^{(1)}(Z_\Theta^u)\big)^2 \Theta(\mathrm{d}u) & \frac{\pi}{2}\, V_2^{(2)}(Z_\theta) \\ \frac{\pi}{2}\, V_2^{(2)}(Z_\theta) & 1 \end{pmatrix}$$

$$\Sigma^{(2)}(\nu) = \begin{pmatrix} \frac{2}{3}\int \big(V_1^{(1)}(Z_\Theta^u)\big)^2 \Theta(\mathrm{d}u) & \frac{8}{3}\, V_2^{(2)}(Z_\theta) \\ \frac{8}{3}\, V_2^{(2)}(Z_\theta) & \frac{8}{3} \end{pmatrix}$$

$$\operatorname{rank}\big(\Sigma^{(2)}(\nu)\big) = 1 \Leftrightarrow \det\big(\Sigma^{(2)}(\nu)\big) = 0 \ \Leftrightarrow\ \Theta(\{u : V_1^{(1)}(Z_\Theta^u) = c\}) = 1$$

Example: Θ concentrated on $u_i = (\cos\alpha_i, \sin\alpha_i)$ for $i = 1, 2$
with $0 \le \alpha_1 < \alpha_2 < \pi$ and $\Theta(\{u_1\}) = \Theta(\{u_2\}) = 1/2$
$\Longrightarrow V_1^{(1)}(Z_\Theta^{u_1}) = V_1^{(1)}(Z_\Theta^{u_2}) = \sin(\alpha_2 - \alpha_1)$

Note that, by definition of the support function and $Z_\Theta = -Z_\Theta$,

$$V_1^{(d-1)}(Z_\Theta^u) = 2\, h(Z_\Theta, u^\perp) = 2 \int\limits_{S_+^{d-1}} |\langle u^\perp, v\rangle|\, \Theta(\mathrm{d}v) \quad \text{for} \quad u \in \mathbb{R}^d$$

so that

$$V_2^{(d)}(Z_\Theta) = \frac{1}{2} \int\limits_{S_+^{d-1}} V_1^{(d-1)}(Z_\Theta^u)\Theta(\mathrm{d}u) = \int\limits_{S_+^{d-1}} \int\limits_{S_+^{d-1}} |\langle u^\perp, v\rangle|\, \Theta(\mathrm{d}v)\, \Theta(\mathrm{d}u)\,.$$

Example (continued): Θ concentrated on $u_i = (\cos\alpha_i, \sin\alpha_i)$ for $i = 1, 2$, $0 \le \alpha_1 < \alpha_2 < \pi$, with $\Theta(\{u_1\}) = 1 - \Theta(\{u_2\}) = p$

$\Longrightarrow$

$$\begin{aligned} V_1^{(1)}(Z_\Theta^{u_1}) &= 2\,(1-p)\, \sin(\alpha_2 - \alpha_1)\,, \\ V_1^{(1)}(Z_\Theta^{u_2}) &= 2\,p\, \sin(\alpha_2 - \alpha_1)\,, \\ V_2^{(2)}(Z_\Theta) &= 4\,p\,(1-p)\, \sin(\alpha_2 - \alpha_1) \end{aligned}$$

Variational problem for the planar case: (Heinrich & Schertz (2006))

$$\int \big(V_1^{(1)}(Z_\Theta^u)\big)^2 \Theta(\mathrm{d}u) = 4 \int_0^\pi \Big(\int_0^\pi |\sin(\beta - \alpha)|\, \Theta(\mathrm{d}\alpha)\Big)^2 \Theta(\mathrm{d}\beta) \mapsto \text{min!}$$

under the condition

$$V_2^{(2)}(Z_\Theta) = \int_0^\pi \int_0^\pi |\sin(\beta - \alpha)|\, \Theta(\mathrm{d}\alpha)\Theta(\mathrm{d}\beta) = \text{const} \,\Big(\le \frac{1}{\pi}\Big)$$

In the above case this means to minimize the integral

$$\int \big(V_1^{(1)}(Z_\Theta^u)\big)^2 \Theta(\mathrm{d}u) = 4\,p\,(1-p)\, \sin^2(\alpha_2 - \alpha_1) = \frac{\big(V_2^{(2)}(Z_\Theta)\big)^2}{p\,(1-p)}\,.$$

Obviously, this minimum is attained for $p = 1/2$.

References:

[1] R.Schneider and W. Weil (2000). *Stochastische Geometrie*, Teubner, Stuttgart.

[2] W. Weil (1979). Centrally symmetric convex bodies and distributions II. *Israel J. Math.* **32**, 173-182.

[3] L. Heinrich, H. Schmidt, and V. Schmidt (2006). Central limit theorems for Poisson hyperplane tessellations. *Ann. Appl. Probab.* (in press).

[4] L. Heinrich and S. Schertz (2006). On asymptotic covariance matrices and the associated zonoid of Poisson hyperplane processes. Report of the Institute of Mathematics, University of Augsburg (in preparation).