Estimation of Specific Intrinsic Volumes and Asymptotical Tests

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Overview

1. Introduction
   - Germ–grain models
   - Associated random fields
2. Estimation of the mean
   - Unbiasedness and Consistency
   - Asymptotic normality
3. Estimation of the asymptotic variance
   - Weighted covariance estimator
   - Consistency
   - Empirical covariance estimator

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Germ–grain models

- $X = \{X_i\}$ point process of germs
- $M = \{M_i\}$ process of grains, $M_i \overset{d}{=} M_0$ i.i.d. RACS
- If $\mathbb{E}\,|M_0 \oplus \check{K}| < \infty, K \subset \mathbb{R}^d$ compact, then $\Xi = \bigcup_{i=1}^{\infty}(M_i + X_i)$ is well defined.

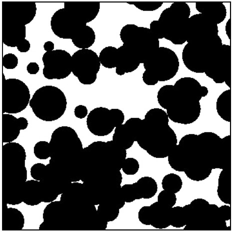
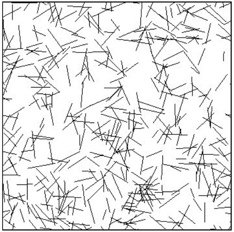

Figure: Germ–grain models of discs and line segments

Random fields associated with GGM

Stationary germ–grain model Ξ such that $\Xi \cap K \in \mathcal{R}$, $K \in \mathcal{K}$.

- $\varphi : \mathcal{R} \to \mathbb{R}$ additive, i.e., $\varphi(\emptyset) = 0$,
$$\varphi(K_1 \cup K_2) = \varphi(K_1) + \varphi(K_2) - \varphi(K_1 \cap K_2)$$
- conditionally bounded:
$$\sup\{|\varphi(K')| : K' \subseteq K, K' \in \mathcal{K}\} < \infty$$
- Random field $Y = \{Y(x), x \in \mathbb{R}^d\}$, test set $K \in \mathcal{K}$:
$$Y(x) = \varphi((\Xi - x) \cap K), \quad x \in \mathbb{R}^d .$$

$$\boxed{\text{Objective: } \quad \text{Estimation of } \mu = \mathbb{E}Y(x).}$$

1 Introduction

Examples

Stationary GGM Ξ, random field $Y(x) = \varphi((\Xi - x) \cap K), x \in \mathbb{R}^d$, test set $K \in \mathcal{K}$.

- $Y_1(x) = \mathbb{1}((\Xi - x) \cap \{o\}) = \mathbb{1}(x \in \Xi)$
  with $\mathbb{E} Y_1(x) = \mathbb{E}(|\Xi \cap [0,1]^d|) = p$ volume fraction
- $Y_2(x) = \mathbb{1}((\Xi - x) \cap K) = \mathbb{1}(x \in \Xi \oplus \check{K})$
  with $\mathbb{E} Y_2(x) = P(\Xi \cap K \neq \emptyset) = T_\Xi(K)$ capacity functional
- $Y_3(x) = V_0((\Xi - x) \cap B_r(o)) = V_0(\Xi \cap B_r(x))$, $r > 0$
  with $\mathbb{E} Y_3(x) = \mathbb{E} V_0(\Xi \cap B_r(o))$ local Euler charakteristic
  $= \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} \overline{V}_j(\Xi)$ spec. intrinsic volumes

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Overview

Preliminaries:

- Stationary random field $Y(x) = \varphi\left((\Xi - x) \cap K\right)$, $x \in \mathbb{R}^d$, $K \in \mathcal{K}$ with $\mu = \mathbb{E}Y(x)$.
- Observation window $W_n = nW_0$ with $W_0 \in \mathcal{K}$, $o \in \operatorname{int}(W_0)$.
- Weighting functional $G : \mathcal{B}_0^d \times \mathbb{R}^d \to [0, \infty)$ with

$$G(W_n, x) = 0,\ x \in \mathbb{R}^d \setminus W_n \ominus \check{K}, \quad \int_{W_n} G(W_n, x)\, dx = 1\,.$$

Unbiased estimator of μ: $\mathbb{E}\left(\widehat{\mu}_n\right) = \mu$ for

$$\boxed{\widehat{\mu}_n = \int_{W_n} Y(x)\ G(W_n, x) dx}$$

2.1 Consistent estimation of the mean

Conditions:

- $\mathrm{Cov}_Y(x) = \mathrm{Cov}\,(Y(o), Y(x))$ satisfies $\int_{\mathbb{R}^d} |\mathrm{Cov}_Y(x)|\, dx < \infty$.
- Weighting functional G satisfying:

$$\sup_{x \in \mathbb{R}^d} G(W_n, x) \leq \frac{c_0}{|W_n|} \quad \text{and} \quad \lim_{n\to\infty} |W_n|\, \Gamma_n(x) = \theta\,.$$

for $\Gamma_n(x) = \int G(W_n, y) G(W_n, x+y)\, dy$ and $c_0, \theta < \infty$.

Mean–square consistency of $\widehat{\mu}_n$: $\lim_{n\to\infty} \mathbb{E}\,(\widehat{\mu}_n - \mu)^2 = 0$, since

$$\boxed{\lim_{n\to\infty} |W_n| \mathrm{Var}(\widehat{\mu}_n) = \theta \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx\,.}$$

2.2 Asymptotic normality

Conditions:

Germ–grain model $\Xi = \bigcup_{i \geq 1}(M_i + X_i)$ is

- a Boolean model with $\mathbb{E}(|M_0 \oplus \check{K}|^2) < \infty$
- or $\mathbb{E}\, 2^{(2+\delta)N(\Xi \cap K)} < \infty$, point process $X = \{X_i\}$ is 'strongly mixing' and $\mathbb{E}(\,||M_0 \oplus \check{K}||^{2d(1+1/\delta)+\varepsilon}) < \infty$, $\delta, \varepsilon > 0$;$||A|| = \sup\{|x| : x \in A\}$.

Then it holds that

$$\boxed{\sqrt{|W_n|}\,(\widehat{\mu}_n - \mu) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right)\ ,}$$

for $\sigma^2 = \theta \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx$.

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Germ–grain model $\Xi = \bigcup_{i \geq 1}(M_i + X_i)$ is

- a Boolean model with $\mathbb{E}(|M_0 \oplus \check{K}|^2) < \infty$

$$\Longrightarrow \quad \int_{\mathbb{R}^d} |\mathrm{Cov}_Y(x)| \, dx < \infty$$

Then it holds that

$$\boxed{\sqrt{|W_n|}\,(\widehat{\mu}_n - \mu) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right) \ ,}$$

for $\sigma^2 = \theta \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx$.

Overview

3 Estimation of the asymptotic variance

Define

$$\boxed{\widehat{\sigma}_n^2 = \int_{U_n} \widehat{\mathrm{Cov}}_n(x)\ \gamma(W_n, x)\, dx}$$

for

- asymp. unbiased estimator $\widehat{\mathrm{Cov}}_n(x)$ of $\mathrm{Cov}_Y(x)$,
- weight $\gamma(W_n, x) = |W_n|\Gamma_n(x), x \in \mathbb{R}^d$,
- averaging window $U_n \subseteq (W_n \ominus \check{K}) \oplus (-W_n \ominus \check{K})$, $o \in U_n$ and

$$\lim_{n\to\infty} \frac{|U_n|^2}{|W_n|} = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{x \in U_n} |\theta - \gamma(W_n, x)| = 0\,.$$

Put $W_n^K = W_n \ominus \check{K}$.

Uniform weights: $G(W_n, x) = \mathbb{1}(x \in W_n^K) / |W_n^K|$.

Define

$$\widehat{\mathrm{Cov}}_n(x) = \int\limits_{W_n^K \cap (W_n^K - x)} \frac{\mathbf{Y}(\mathbf{y})\,\mathbf{Y}(\mathbf{y}+\mathbf{x}) - \widehat{\boldsymbol{\mu}}_\mathbf{n}^\mathbf{2}}{|W_n^K \cap (W_n^K - x)|}\, dy$$

and let

$$\widehat{\sigma}_n^2 \approx \int\limits_{U_n} \widehat{\mathrm{Cov}}_n(x)\, \frac{|W_n^K \cap (W_n^K - x)|}{|W_n^K|}\, dx\,.$$

Boolean model $\Xi = \bigcup_{i\geq 1}(M_i + X_i)$

If $\mathbb{E}\left(|M_0 \oplus \check{K}|^2\right) < \infty$ then

$$\lim_{n\to\infty} \mathbb{E}\,\widehat{\sigma}^2 = \sigma^2\,, \quad \text{asymptotically unbiased}$$

and

$$\lim_{n\to\infty} \mathbb{E}\left(\widehat{\sigma}^2 - \sigma^2\right)^2 = 0\,, \quad \text{mean–square consistent}$$

for $\sigma^2 = \theta \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\,dx$.

Germ–grain model $\Xi = \bigcup_{i \geq 1}(M_i + X_i)$

- If $\int_{\mathbb{R}^d} |\mathrm{Cov}_Y(x)| \, dx < \infty$ then

$$\lim_{n \to \infty} \mathbb{E}\,\widehat{\sigma}^2 = \sigma^2 \,, \quad \text{asymptotically unbiased,}$$

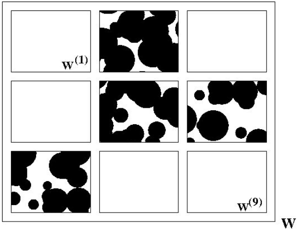
- If $\sup_{x_1,x_2} \int_{\mathbb{R}^d} |\mathrm{Cov}(Y(o)Y(x_1), Y(y)Y(x_2+y))| \, dy < \infty$ and $\sup_{x_1,x_2} \int_{\mathbb{R}^d} |\mathbb{E}\,((Y(o)-\mu)(Y(y)-\mu)Y(x_1)Y(x_2))| \, dy < \infty$ (or Y is uniformly bounded) then

$$\lim_{n \to \infty} \mathbb{E}\left(\widehat{\sigma}^2 - \sigma^2\right)^2 = 0 \,, \quad \text{mean–square consistent.}$$

3.3 Empirical covariance estimator

Subdivision of the sampling window:

Let $W_n^{(1)}, \ldots, W_n^{(m)}$ for $m = m(n)$ such that $\bigcup_{k=1}^m W_n^{(k)} \subseteq W_n$ and $\mathrm{int}(W_n^{(k)}) \cap \mathrm{int}(W_n^{(\ell)}) = \emptyset, k \neq \ell$



and define

$$\widetilde{\sigma}_n^2 = \frac{1}{m-1} \sum_{k=1}^{m} \left(\widehat{\mu}_n^{(k)} - \overline{\mu}_n \right)^2$$

for $\widehat{\mu}_n^{(k)}$ estimate of μ on $W_n^{(k)}$ and $\overline{\mu}_n = \frac{1}{m} \sum_{k=1}^m \widehat{\mu}_n^{(k)}$.

3.3 Empirical covariance estimator

Germ–grain model $\Xi = \bigcup_{i \geq 1}(M_i + X_i)$

- If $\int_{\mathbb{R}^d} |\mathrm{Cov}_Y(x)| \, dx < \infty$ then

$$\lim_{n \to \infty} \mathbb{E}\, \widetilde{\sigma}^2 = \sigma^2 \,, \quad \text{asymptotically unbiased,}$$

- If $\int_{\mathbb{R}^{3d}} \left| s^{(4)}(o, x_1, x_2, x_3) \right| \, d(x_1, x_2, x_3) < \infty$ and $m(n) \to \infty$ then

$$\lim_{n \to \infty} \mathbb{E} \left(\widetilde{\sigma}^2 - \sigma^2 \right)^2 = 0 \,, \quad \text{mean–square consistent.}$$

Example:

Ξ Boolean model and $Y(x) = \mathbb{1}((\Xi - x) \cap K)$, then assume that

$$\mathbb{E} \left(|M_0 \oplus \check{K}|^4 \right) < \infty \,.$$

4 Resume

- Germ–grain model $\Xi = \bigcup_{i \geq 1}(M_i + X_i)$,
- Random field $Y(x) = \varphi((\Xi - x) \cap K), x \in \mathbb{R}^d$ with $\mu = \mathbb{E}\, Y(x)$ and $\mathrm{Cov}_Y(x)$ unknown, φ bounded valuation, $K \in \mathcal{K}$ test set,
- weighted average $\widehat{\mu}_n = \int_{W_n} Y(x) G(W_n, x)\, dx$ as unbiased and mean-square consistent estimator of μ,
- asymptotic normality of $\sqrt{|W_n|}\,(\widehat{\mu}_n - \mu)$ with asympt. variance $\sigma^2 = \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx$,
- weighted average $\widehat{\sigma}_n = \int_{U_n} \widehat{\mathrm{Cov}}_n(x) \gamma(W_n, x)\, dx$ as asymptotically unbiased and mean-square consistent estimator of σ^2.

5 References

1. Böhm S., Heinrich L. and Schmidt V. (2004), Asymptotic properties of estimators for the volume fraction of jointly stationary random sets. *Statistica Neerlandica* **58**, 388–406.

2. Heinrich L., Molchanov I. (1999), Central limit theorem for a class of random measures associated with germ–grain models. *Advances in Applied Probability* **31**, 283–314.

3. Heinrich L. (2005), Large deviations of the empirical volume fraction for stationary Poisson grain models. *Annals of Applied Probability* **15**, 1A, 392–420.

4. Schmidt, V., Spodarev, E. (2005), Joint estimators for the specific intrinsic volumes of stationary random sets, *Stochastic Processes and their Applications*, **115**, 959 - 981

5. Pantle, U., Schmidt, V., Spodarev, E. (2006), Central limit theorems for functionals of stationary germ-grain models, *Advances in Applied Probability* **38**, (to appear)

6. Pantle, U., Schmidt, V., Spodarev, E. (2006), On the estimation of the integrated covariance function of stationary random fields, *Working paper* (in preparation)

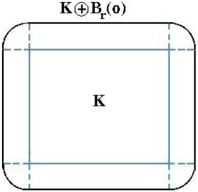
Steiner formula

Let $K \in \mathcal{K}$ and $r > 0$, then

$$\boxed{|K \oplus B_r(o)| = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K)}$$

for functionals $V_j : \mathcal{K} \to [0, \infty)$, called the intrinsic volumes.

Figure: Steiner formula in $\mathbb{R}^2$



$$\begin{aligned} |K \oplus B_r(o)| &= \pi r^2 V_0(K) + 2r V_1(K) + V_2(K). \\ &= \pi r^2 + r S(K) + A(K). \end{aligned}$$

A: area, S: boundary length

5 Introduction

Let Ξ be a stationary RACS in $\mathbb{R}^d$ with $P((\Xi \cap K) \in \mathcal{R}) = 1$.

Let $\{W_n\}$ be a sequence of compact and convex observation windows $W_n = nW_0$ with $|W_0| > 0$ and $o \in \operatorname{int}(W_0)$.

If $\mathbb{E}\, 2^{N(\Xi \cap [0,1]^d)} < \infty$ then the limit

$$\boxed{\overline{V}_j(\Xi) = \lim_{n\to\infty} \frac{\mathbb{E} V_j(\Xi \cap W_n)}{|W_n|}}$$

exists for each $j = 0, \ldots, d$ and is called the

j-th specific intrinsic volume.