# Random Point Processes

*Models, Characteristics and Structural Properties*


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# Introduction

*Aims*

1. Explain some basic ideas for stochastic modelling of spatial point patterns
   - stationarity (homogeneity) vs. spatial trends
   - isotropy (rotational invariance)
   - complete spatial randomness
   - interaction between points (clustering, repulsion)

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1. Explain some basic ideas for stochastic modelling of spatial point patterns
   - stationarity (homogeneity) vs. spatial trends
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   - complete spatial randomness
   - interaction between points (clustering, repulsion)
2. Describe some basic characteristics of point–process models
   - Intensity measure, conditional intensities
   - Pair correlation function, Ripley's K-function, etc.

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2. Describe some basic characteristics of point–process models
   - Intensity measure, conditional intensities
   - Pair correlation function, Ripley's K-function, etc.
3. Present techniques for the statistical analysis of spatial point patterns

1. Examples of spatial point patterns

# Introduction
## *Overview*

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   - point patterns in networks

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   - Nonparametric estimation of model characteristics

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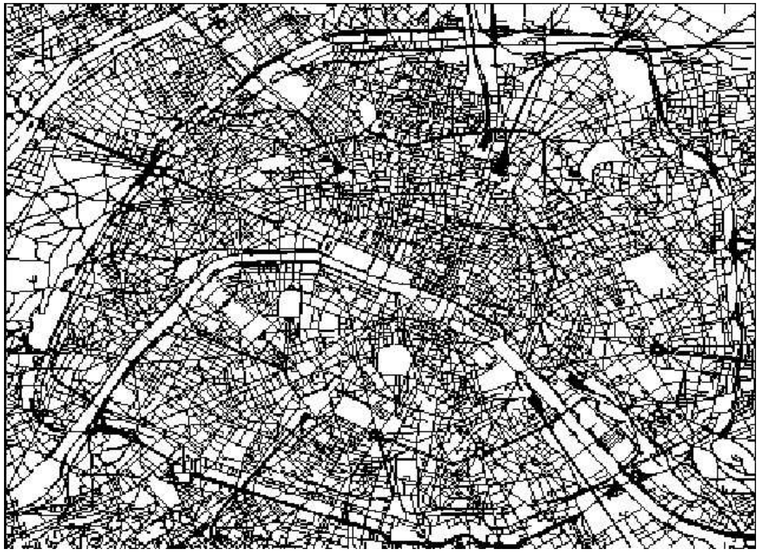
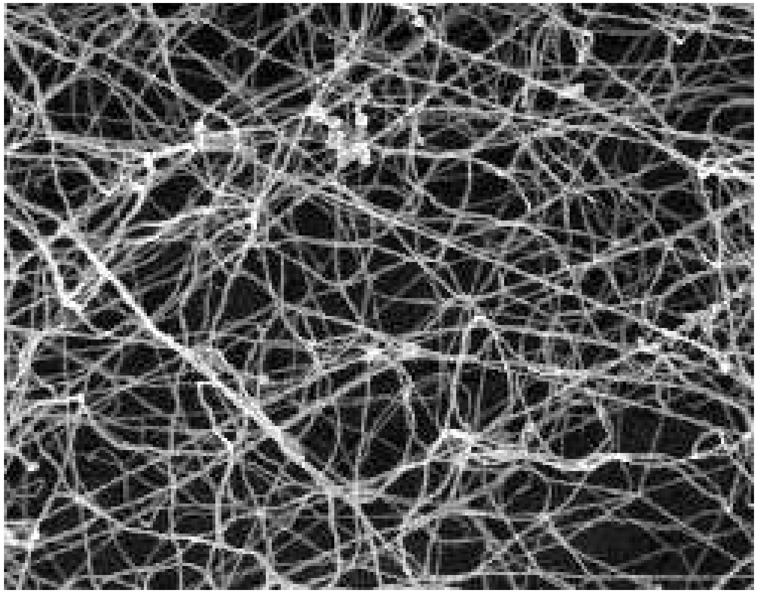
   - Nonparametric estimation of model characteristics
   - Maximum pseudolikelihood estimation

# Examples

*Point patterns in networks*



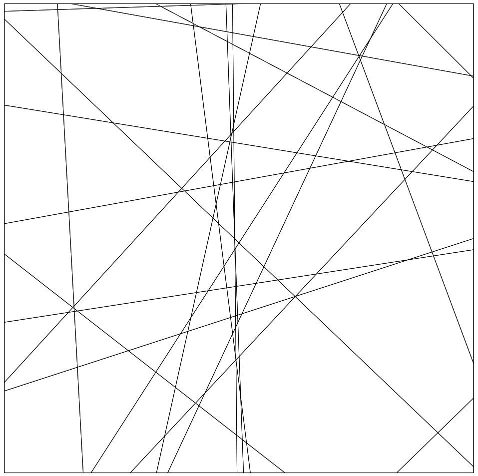
*Street system of Paris*



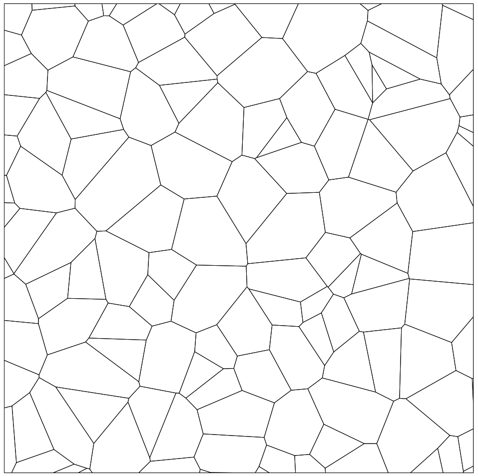
*Cytoskeleton of a leukemia cell*

# Examples

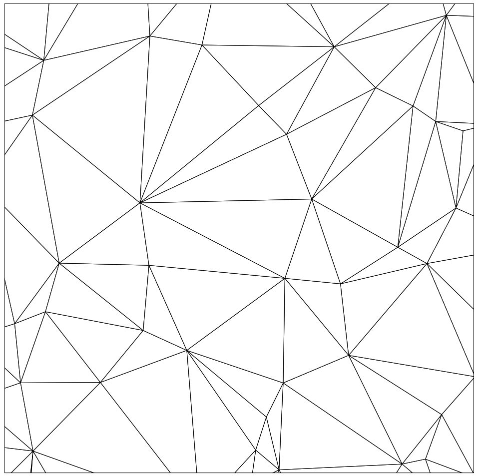
*Random tessellations*



*Poisson line (PLT)*

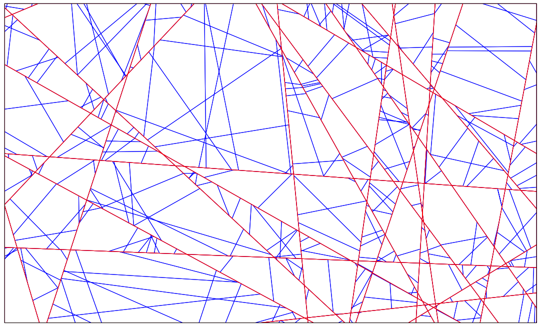


*Poisson-Voronoi (PVT)*

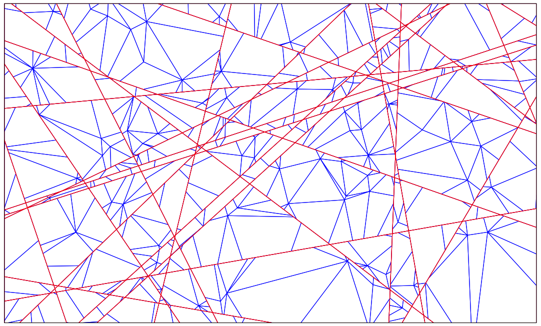


*Poisson-Delaunay (PDT)*

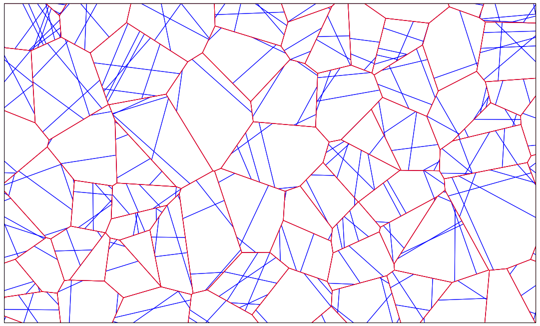
## Simple and iterated tessellation models



*PLT/PLT*

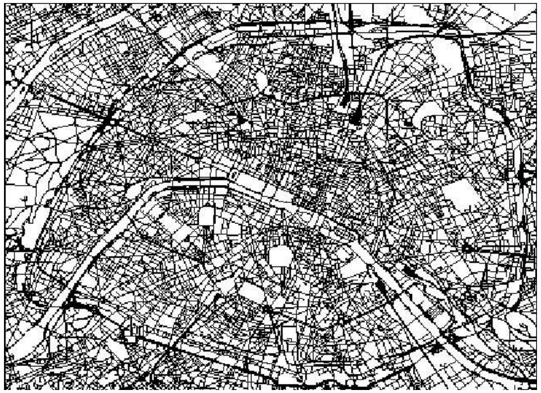


*PLT/PDT*



*PVT/PLT*

# Examples
*Model fitting for network data*



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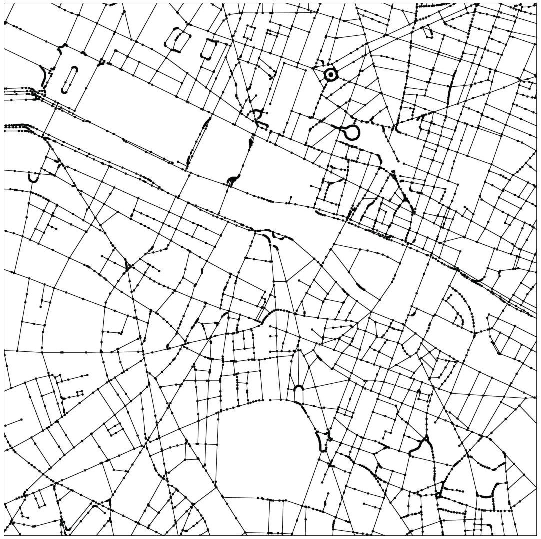
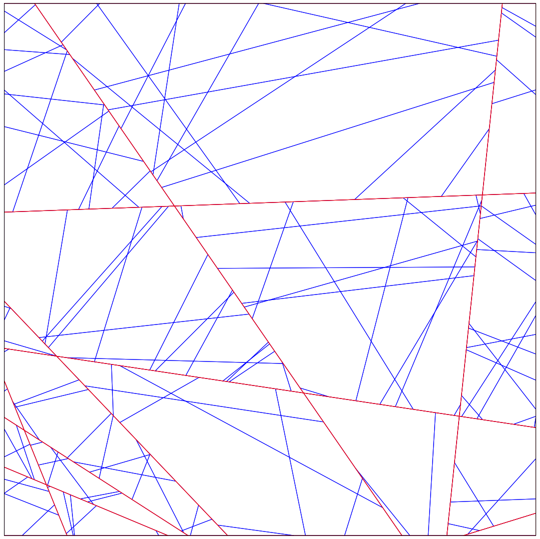


*Cutout*

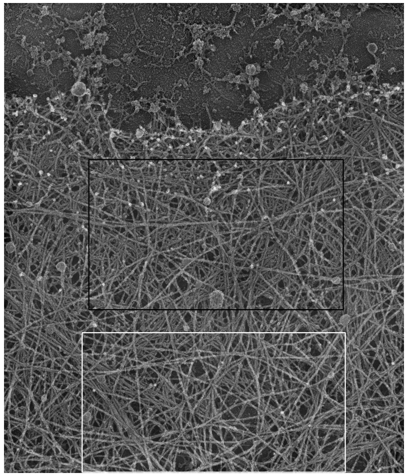


*Tessellation model (PLT/PLT)*

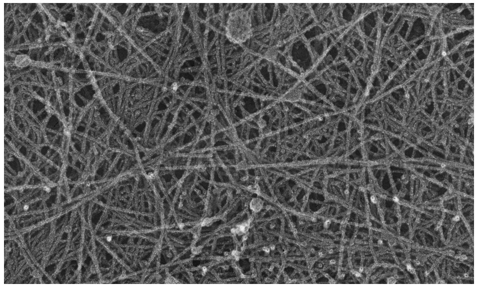
Model fitting for telecommunication networks

# Examples

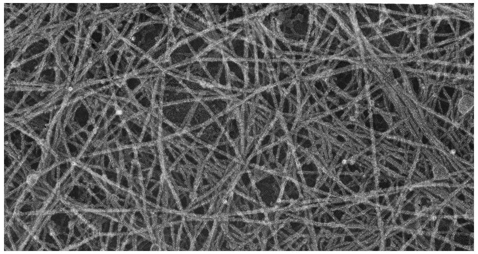
*Analysis of biological networks*



a) Actin network at the cell periphery



b) Lamellipodium



c) Region behind lamellipodia

Actin filament networks

# Examples

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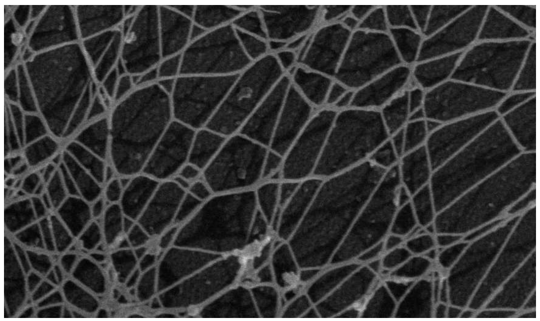
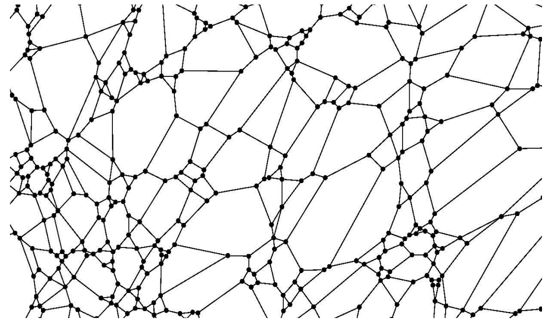
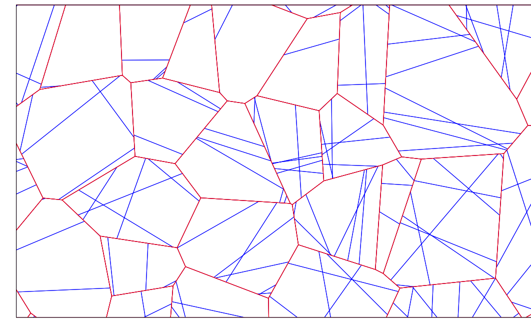


*SEM image (keratin)*

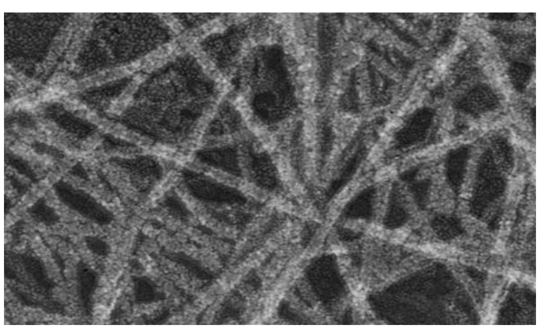


*Graph structure*

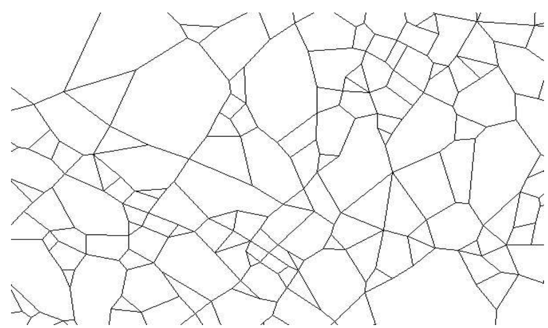


*Tessellation model (PVT/PLT)*

## Model fitting for keratin and actin networks



*SEM image (actin)*



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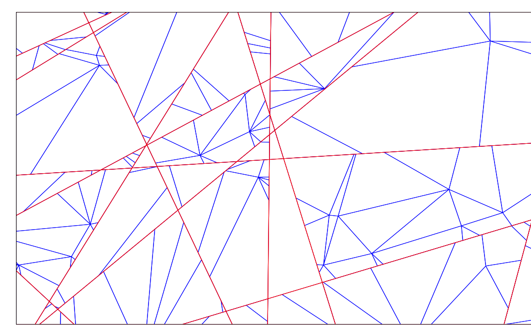
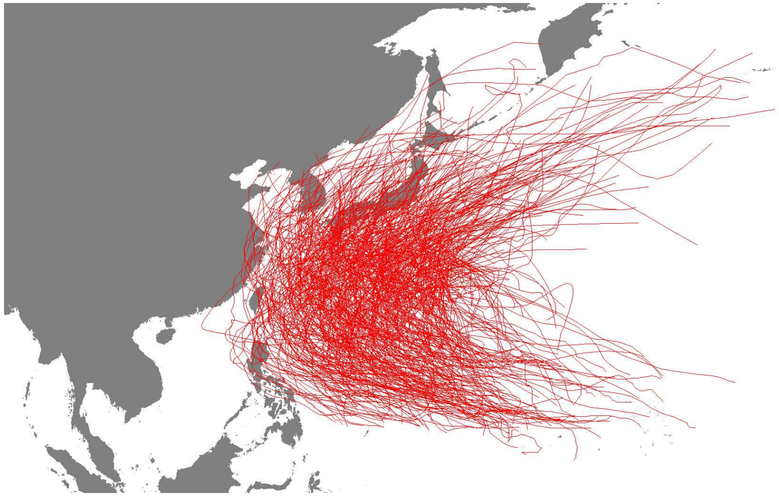
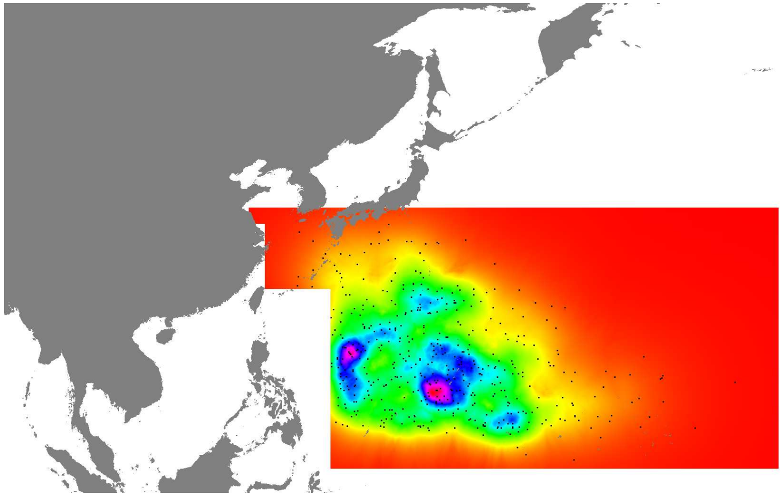


*Tessellation model (PLT/PDT)*

# Other examples
## *Modelling of tropical storm tracks*



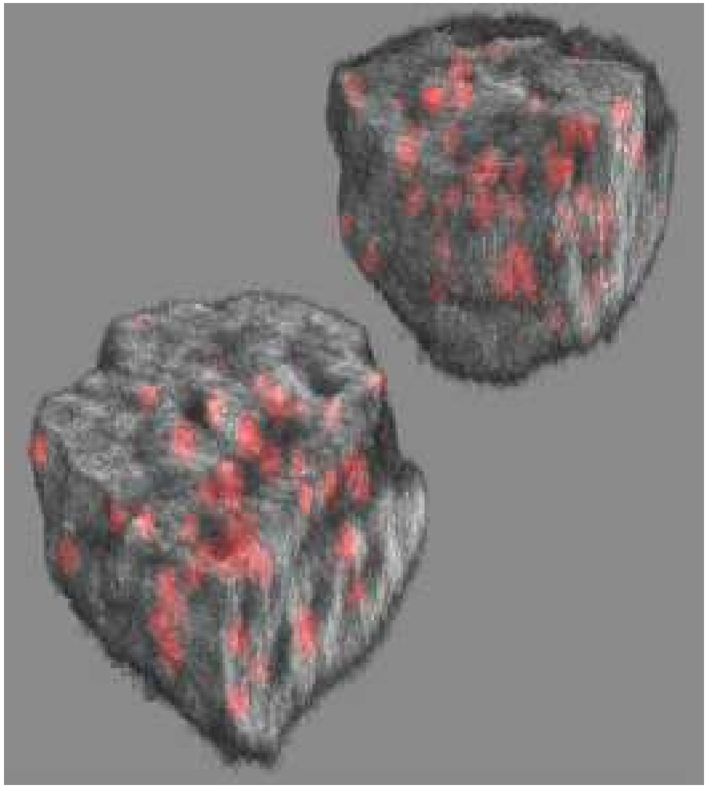
*Storm tracks of cyclons over Japan 1945-2004*



*Estimated intensity field for initial points of storm tracks over Japan*

# Other examples
*Point patterns in biological cell nuclei*

Heterochromatin structures in interphase nuclei



3D-reconstruction of NB4 cell nuclei (DNA shown in gray levels) and centromere distributions (shown in red)

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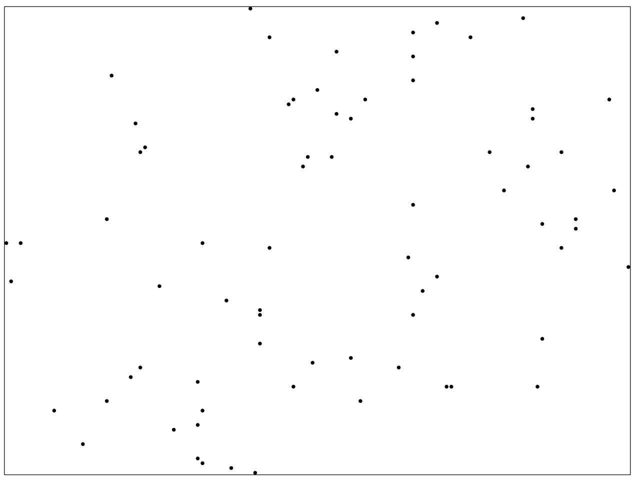

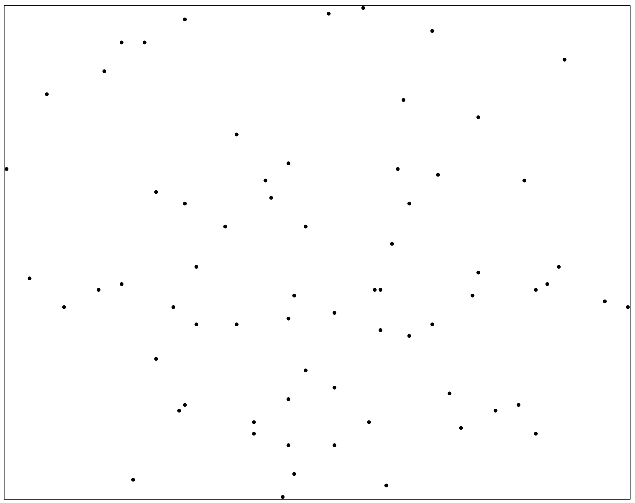

Projections of the 3D chromocenter distributions of an undifferentiated (left) NB4 cell and a differentiated (right) NB4 cell onto the xy-plane

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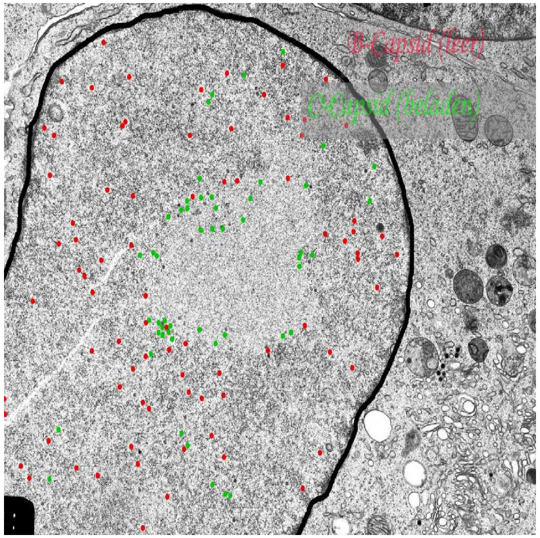
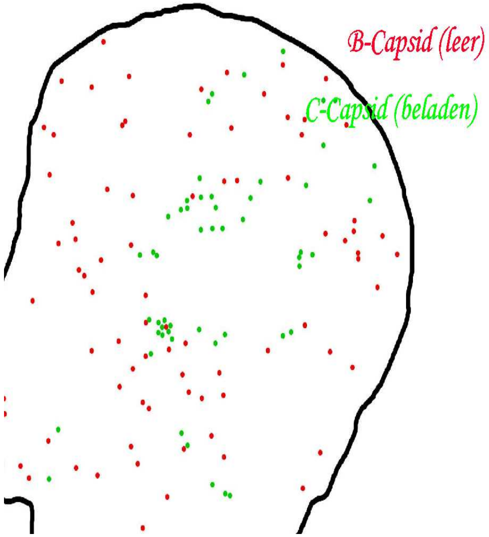
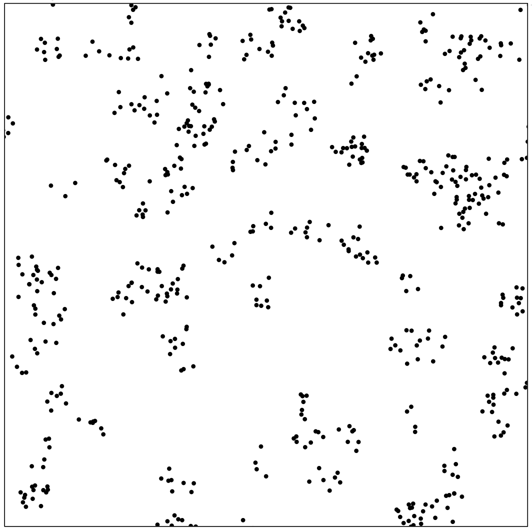



*Capsides of cytomegalovirus*




*Extracted point pattern*



*Model for a point field (cluster)*

1. Mathematical definition of spatial point processes
   - Let $\{X_1, X_2, ...\}$ be a sequence of random vectors with values in $\mathbb{R}^2$ and
   - let $X(B) = \#\{n : X_n \in B\}$ denote the number of „points" $X_n$ located in a set $B \subset \mathbb{R}^2$.

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   - isotropic if the distribution of $\{X_n\}$ is invariant w.r.t rotations around the origin

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     point field instead of spatial point process
   - Point processes are not necessarily dynamic
   - Dynamics (w.r.t. time/space) can be added
     => spatial birth-and-death processes

# Models
## *Stationary Poisson process*

1. Definition of stationary Poisson processes
   - Poisson distribution of point counts: $X(B) \sim \mathrm{Poi}(\lambda|B|)$ for any bounded $B \subset \mathbb{R}^2$ and some $\lambda > 0$

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   - Void–probabilities: $\mathbb{P}(X(B) = 0) = \exp(-\lambda|B|)$
   - Conditional uniformity: Given $X(B) = n$, the locations of the $n$ points in $B$ are independent and uniformly distributed random variables

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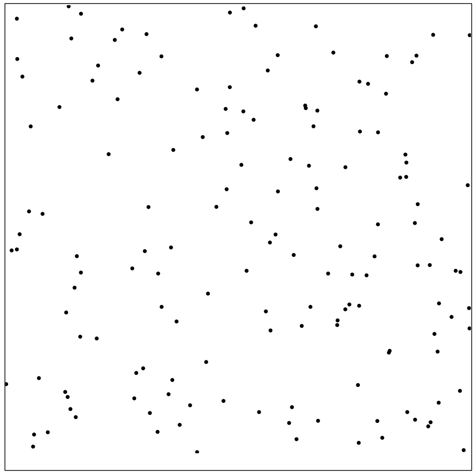

Simulated realization of a stationary Poisson process

# Models

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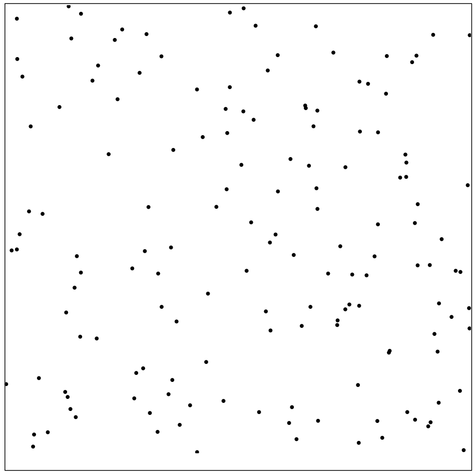

Simulated realization of a stationary Poisson process

convincingly shows

- Complete spatial randomness

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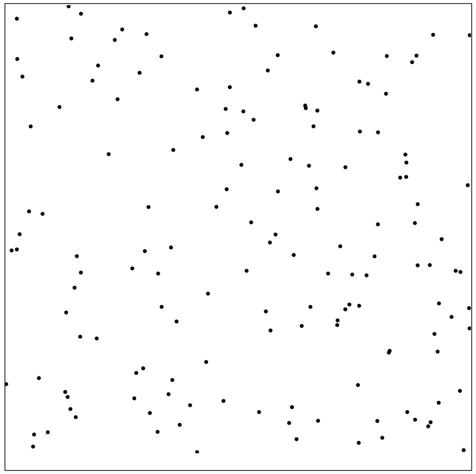

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# Models

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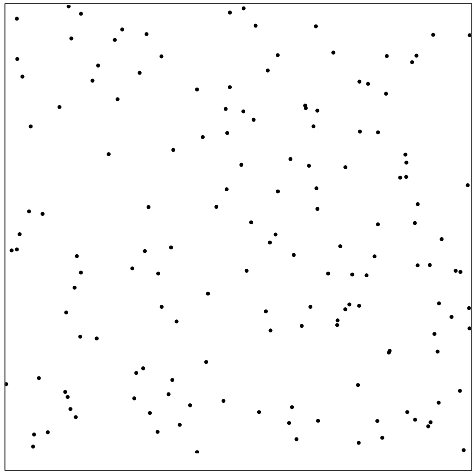

Simulated realization of a stationary Poisson process

convincingly shows

- Complete spatial randomness
- No spatial trend
- No interaction between the points

# Models

## *General Poisson process*

1. Definition of general (non–homogeneous) Poisson processes
   - Poisson distribution of point counts: $X(B) \sim \mathrm{Poi}(\Lambda(B))$ for any bounded $B \subset \mathbb{R}^2$ and some measure $\Lambda : \mathcal{B}(\mathbb{R}^2) \to [0, \infty]$

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   - $\mathbb{E}X(B) = \Lambda(B) \Longrightarrow \Lambda$ = intensity measure
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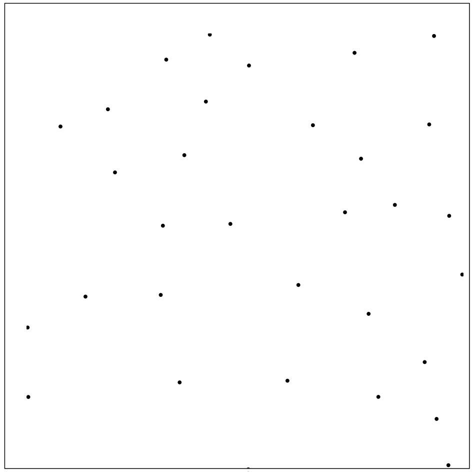
   - Intensity function $\lambda(x) \geq 0$ if $\Lambda(B) = \int_B \lambda(x)\, dx$

# Further basic models
## *Poisson hardcore process*



Realisation of a Poisson hardcore process

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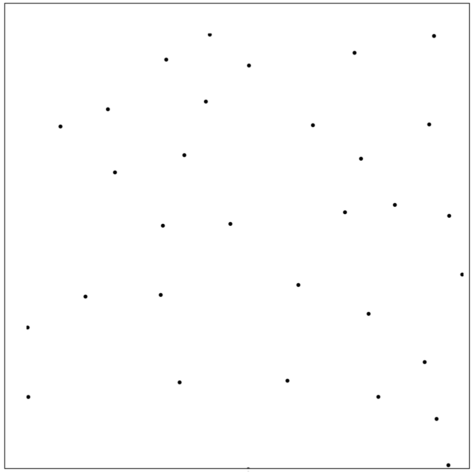

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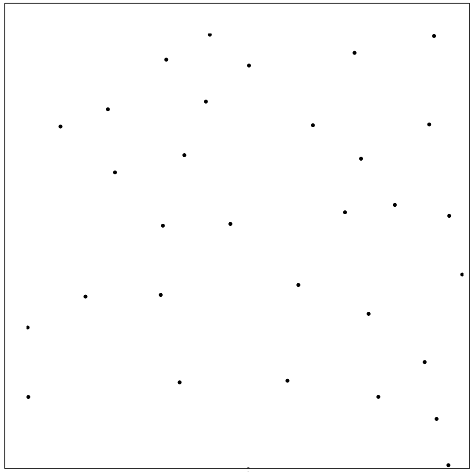

Realisation of a Poisson hardcore process

- Constructed from stationary Poisson processes (by random deletion of points)
- Realizations are relatively regular point patterns (with smaller spatial variability than in the Poisson case)

# Further basic models
## *Poisson hardcore process*

Description of the model

- Start from a stationary Poisson process (with some intensity $\lambda > 0$)

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- Start from a stationary Poisson process (with some intensity $\lambda > 0$)
- Cancel all those points whose distance to their nearest neighbor is smaller than some $R > 0$

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Description of the model

- Start from a stationary Poisson process (with some intensity $\lambda > 0$)
- Cancel all those points whose distance to their nearest neighbor is smaller than some $R > 0$
- Minimal (hardcore) distance between point–pairs => $R/2$ = hardcore radius

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## Further basic models
### *Poisson hardcore process*

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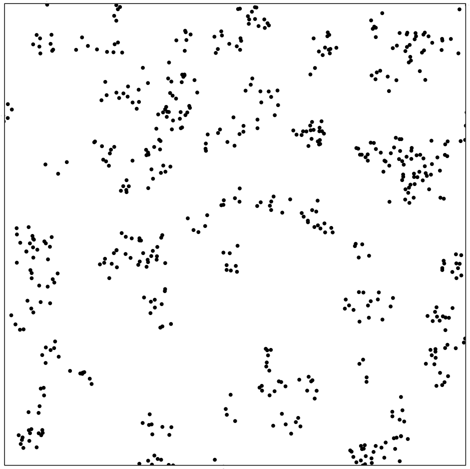
# Further basic models
## *Matern–cluster process*



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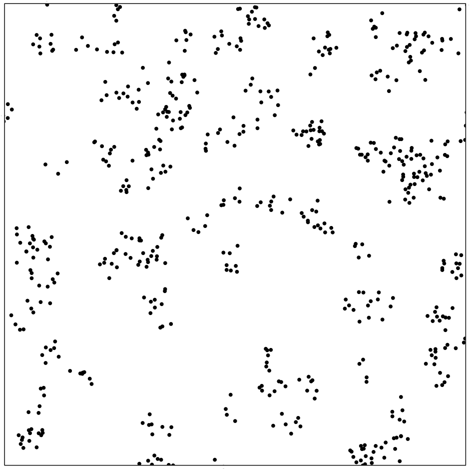

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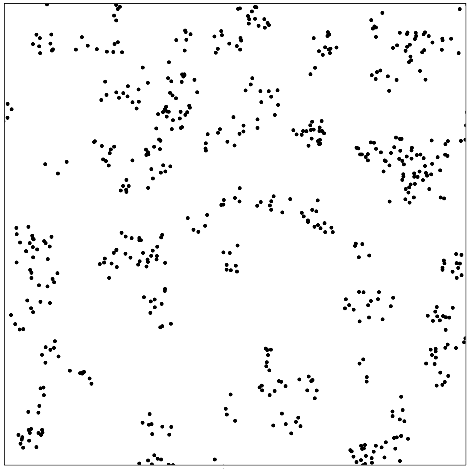

Realization of a Matern-cluster process

- Constructed from stationary Poisson processes (of so–called cluster centers)
- Realizations are clustered point patterns (with higher spatial variability than in the Poisson case)

# Further basic models
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Description of the model

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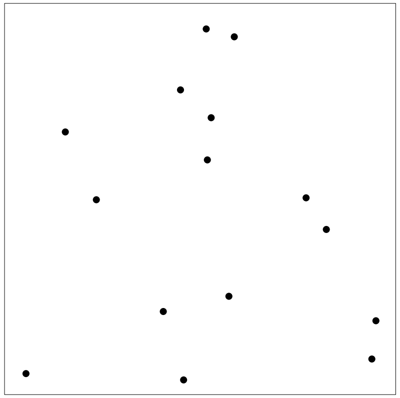
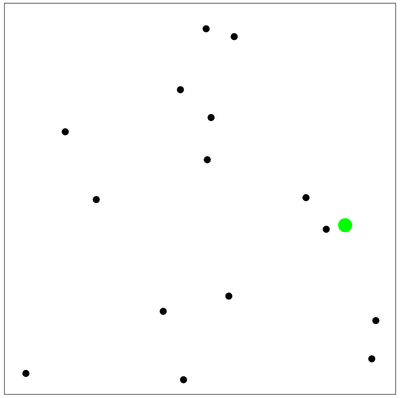
a)



Initial configuration

b)



Birth of a point

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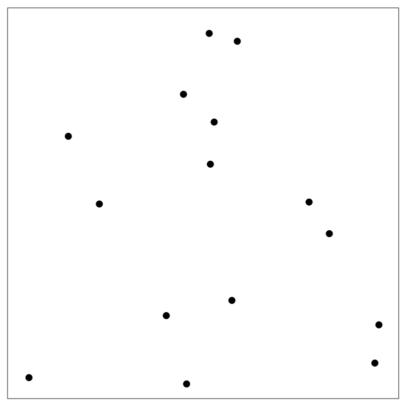

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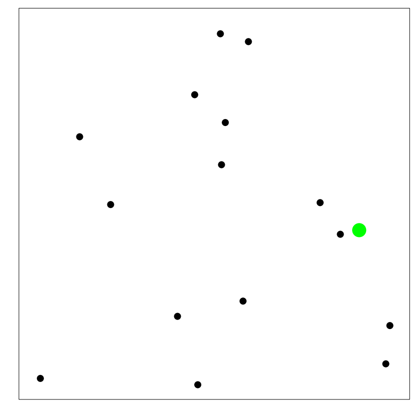

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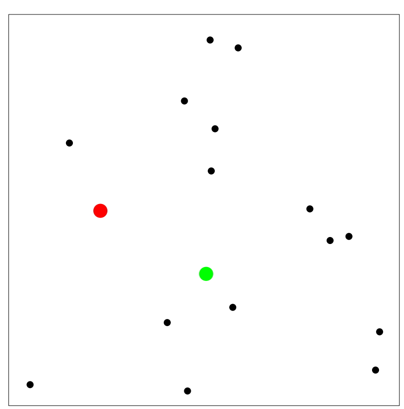
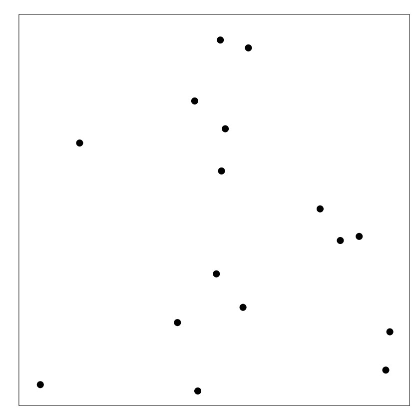


Birth of another point

d)



Death of a point

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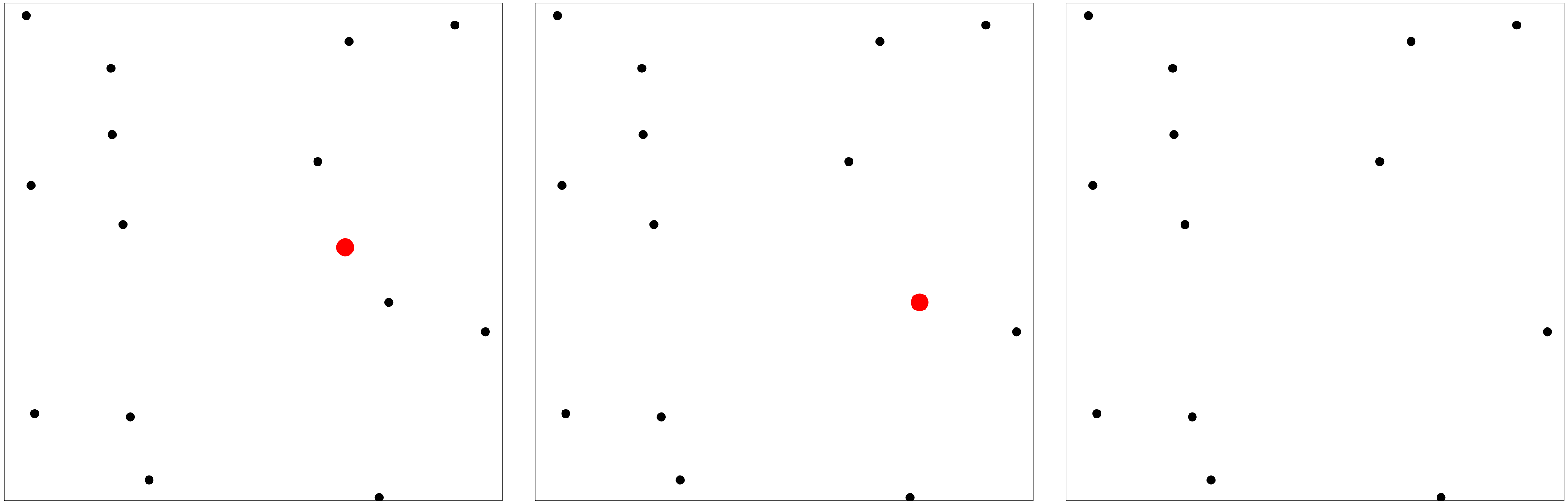

*Initial configuration* *Death of a point* *Death of another point*

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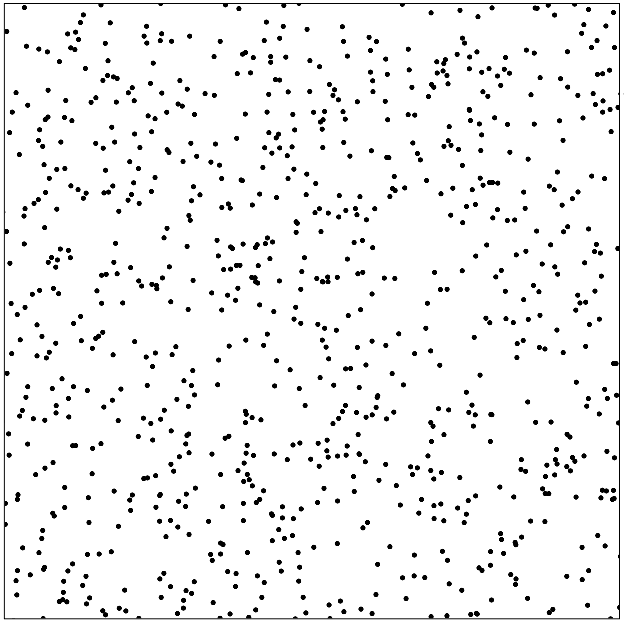
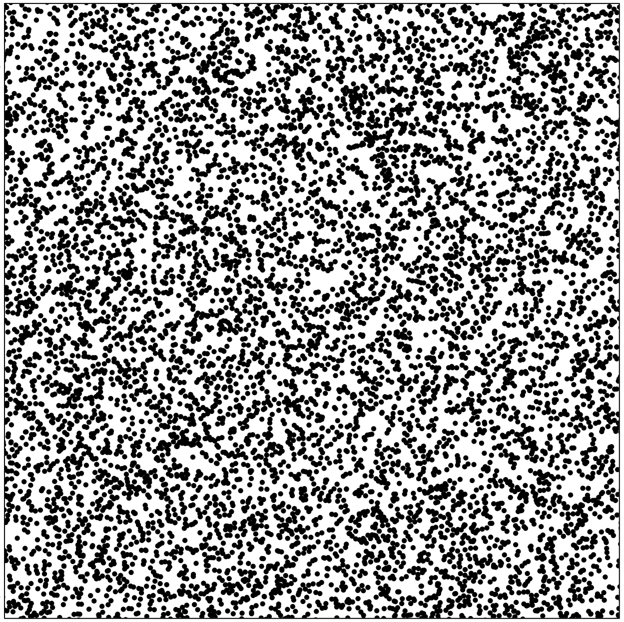


$\lambda = 0.01$



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- $\rho^{(2)}(u_1, u_2)\, du_1\, du_2 \approx$ probability that in each of the sets $du_1,\; du_2$ there is at least one point of $\{X_n\}$

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- Often $\alpha^{(2)}(B_1 \times B_2) = \int_{B_1} \int_{B_2} \rho^{(2)}(u_1, u_2)\, du_1\, du_2$
  $\rho^{(2)}(u_1, u_2)$ = product density
- $\rho^{(2)}(u_1, u_2)\, du_1\, du_2 \approx$ probability that in each of the sets $du_1,\; du_2$ there is at least one point of $\{X_n\}$
- In the stationary case: $\rho^{(2)}(u_1, u_2) = \rho^{(2)}(u_1 - u_2)$

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## *Other functions*

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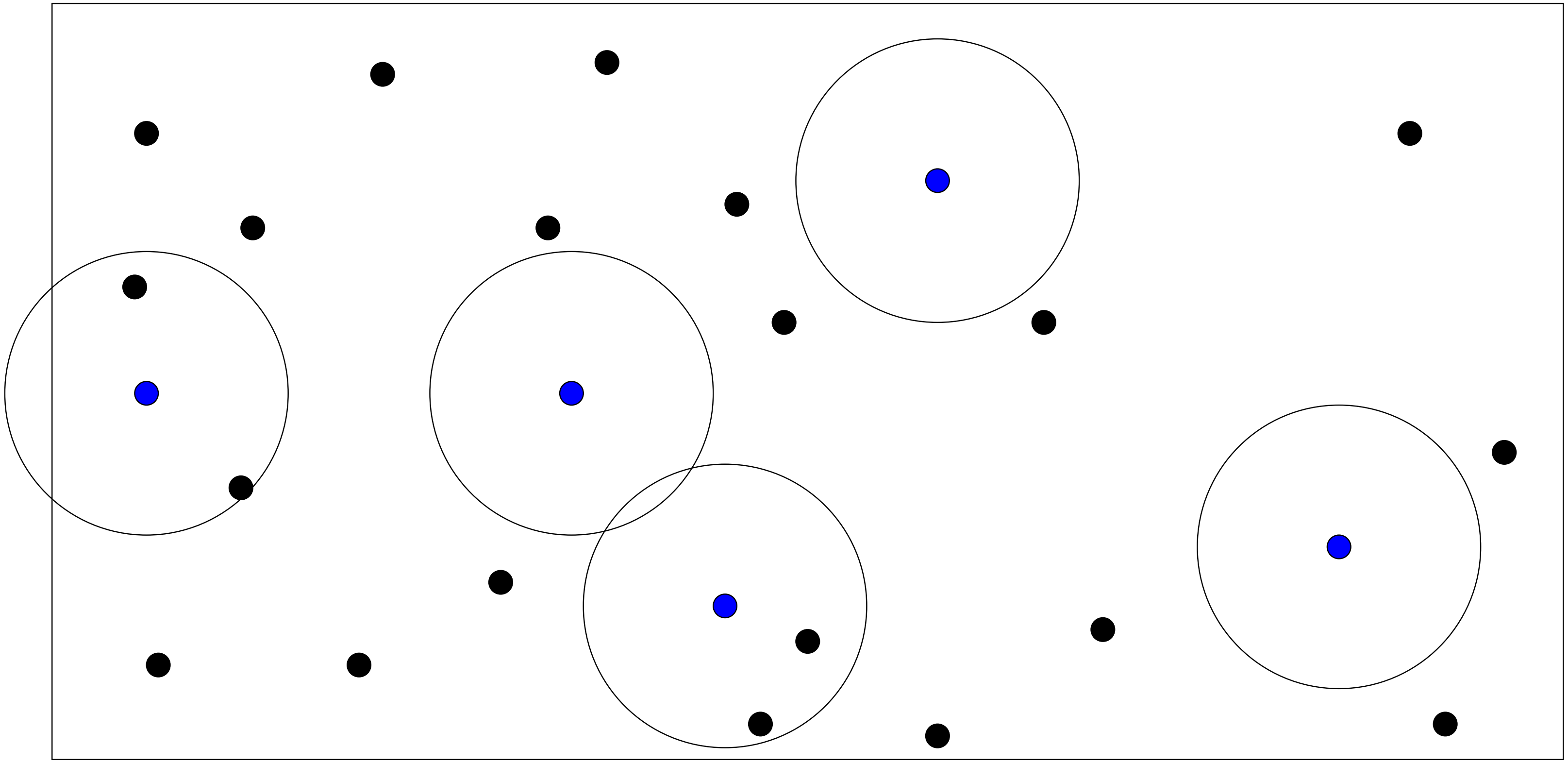

Edge effects occurring in the estimation of $\lambda K(r)$

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where $\widehat{\lambda^2} = X(W)(X(W) - 1)/|W|$

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## *Further edge-corrected estimators*

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where $d(X_n)$ is the distance from $X_n$ to its nearest neighbor

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- The maximum pseudolikelihood estimate $\widehat{\theta}$ of $\theta$ is the value which maximizes $\mathrm{PL}(\theta; \mathbf{x})$

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$$\log \mathrm{PL}(\theta; \mathbf{x}) = \sum_{j=1}^{m} \Big( z_j \log \lambda_\theta(u_j, \mathbf{x}) - w_j \lambda_\theta(u_j, \mathbf{x}) \Big) = \sum_{j=1}^{m} w_j (y_j \log \lambda_j - \lambda_j)$$

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- Thus, standard statistical software for fitting generalized linear models can be used to compute $\widehat{\theta}$

# References


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